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On Some New Variations of Integral Inequalities

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Abstract. The aim of the present paper is to establish some new integral inequalities of Gronwall type involving functions of two independent variables which provide explicit bounds on unknown functions.The inequalities given here can be used as tools in the qualitative theory of certain partial differential and integral equations.

Keywords: Integral inequalities, two independent variables, partial differential equations, nondecreasing, nonincreasing.

Mathematics Subject Classification: 26D15, 35A05

1. Introduction

The Gronwall type integral inequalities provide a necessary tool for the study of the theory of differential equations,integral equations and inequalities of the various types (please, see Gronwall [7] and Guiliano [8]).Some applications of this result to the study of stability of the solution of linear and nonlinear differential equations may be found in Bellman [3]. Numerous applications to the existence and uniqueness theory of differential equations may be found in Nemyckii-Stepanov [11], Bihari [4], and Langenhop [9]. During the past few years several authors(see references below and some of the references cited therein) have established several Gronwall type integral inequalities in one or two independent real variables [1-13]. Of course, such results have application in the theory of partial differential equations and Volterra integral equations.

In [13],Pachpatte investigated the following inequality:

Let $u : E \to R_1$, $p : E \to R_+$ be continuous functions and $c \geq 1$ be a constant,where $E = [0,\alpha] \times [0,\beta]$ $\alpha > 0$, $\beta > 0$, $R_1 = [1,\infty)$ and $R_+ = [0,\infty)$.If

$$u(x,y) \leq c + \int_0^x \int_0^y p(s,t) u(s,t) \log u(s,t)\, ds dt,$$

for $x, y \in E$, then

$$u(x,y) \leq C^{Q_1(x,y)}, \quad \text{for } x, y \in E$$

where $$Q_1(x,y) \leq \exp\left(\int_0^x \int_0^y p(s,t)\, ds dt \right), \quad \text{for } x, y \in E$$

2. Main Results

Theorem 2.1. Let $u : E \to R_1$, $p, q : E \to R_+$, $f : E \to (0,\infty)$ be continuous functions and $c \geq 1$ be a constant,where $E = [0,\alpha] \times [0,\beta]$, $\alpha > 0$, $\beta > 0$, $R_1 = [1,\infty)$ and $R_+ = [0,\infty)$.If

$$u(x,y) \leq c + \int_0^x \int_0^y f(s,t) u(s,t) \left[\int_0^s \int_0^t p(\sigma,\eta) u(\sigma,\eta) \left[\log u(\sigma,\eta) + \int_0^\sigma \int_0^\eta q(s_1,t_1) \log u(s_1,t_1)\, ds_1 dt_1 \right] d\sigma d\eta\, ds dt \right], \tag{2.1}$$

for $x, y \in E$, then

$$u(x,y) \leq c^{Q(x,y)}, \quad \text{for } x, y \in E$$

where

$$Q(x,y) = \exp\left(\int_0^x \int_0^y f(s,t) A_*(s,t)\, ds dt \right), \quad \text{for } x, y \in E \tag{2.2}$$

and $$A(x,y) = \exp\left(\int_0^x \int_0^y [p(s,t) + q(s,t)]\, ds dt \right), \quad \text{for } x, y \in E \tag{2.3}$$

$$A_*(x,y) = \exp\left(1 + \int_0^x \int_0^y p(s,t) A(s,t)\, ds dt \right), \quad \text{for } x, y \in E \tag{2.4}$$

Proof: Define a function

$$z(x,y)=c+\int_0^x\int_0^y f(s,t)u(s,t)\left[\int_0^s\int_0^t p(\sigma,\eta)u(\sigma,\eta)\left[\log u(\sigma,\eta)+\int_0^\sigma\int_0^\eta q(s_1,t_1)\log u(s_1,t_1)ds_1dt_1\right]d\sigma d\eta dsdt\right], \tag{2.5}$$

$$z(x,0)=z(0,y)=c\,,\qquad z_x(x,0)=z_y(0,y)=0 \tag{2.6}$$

By substituting (2.5) in (2.1),we get

$$u(x,y)\le z(x,y) \tag{2.7}$$

By differentiating (2.5) with respect to x and y,we get

$$z_{xy}(x,y)=f(x,y)u(x,y)\left[\int_0^x\int_0^y p(s,t)u(s,t)\left[\log u(s,t)+\int_0^s\int_0^t q(\sigma,\eta)\log u(\sigma,\eta)d\sigma d\eta\right]dsdt\right],$$

By using (2.7) in the above equation.we observe that

$$z_{xy}(x,y)\le f(x,y)z(x,y)m(x,y), \tag{2.8}$$

Where $$m(x,y)=\int_0^x\int_0^y p(s,t)z(s,t)\left[\log z(s,t)+\int_0^s\int_0^t q(\sigma,\eta)\log z(\sigma,\eta)d\sigma d\eta\right]dsdt \tag{2.9}$$

$$m(x,0)=m(0,y)=0\,, \tag{2.10}$$

By differentiating (2.9) with respect to x and y and since $z(x,y)\le m(x,y)$ from (2.9)we have

$$m_{xy}(x,y)\le p(x,y)m(x,y)\left[\log m(x,y)+\int_0^x\int_0^y q(s,t)\log m(s,t)dsdt\right]$$

Which can be rewritten as

$$m_{xy}(x,y)\le p(x,y)m(x,y)r(x,y) \tag{2.11}$$

Where $$r(x,y)=\log m(x,y)+\int_0^x\int_0^y q(s,t)\log m(s,t)dsdt \tag{2.12}$$

$$r(x,0)=r(0,y)=\log 0=1,\quad r_x(x,0)=r_y(0,y)=0 \tag{2.13}$$

By differentiating (2.12) with respect to x and y and since from (2.12) it is obvious that $\log m(x,y)\le r(x,y)$ and using (2.13),we have

$$\begin{aligned} r_{xy}(x,y)&=\frac{1}{m(x,y)}m_{xy}(x,y)+q(x,y)\log m(x,y)\\ &\le p(x,y)r(x,y)+q(x,y)r(x,y)\\ &\le r(x,y)[p(x,y)+q(x,y)] \end{aligned}$$

From the above inequality and the facts that $r(x,y) > 0, r_x(x,y) \ge 0, r_y(x,y) \ge 0$ for $x, y \in E$, we observe that

$$\frac{r_{xy}(x,y)}{r(x,y)} \le p(x,y) + q(x,y) \tag{2.14}$$

By keeping first x fixed in (2.14) and set $y = t$ and integrate from 0 to y then again keeping y fixed, set $x = s$ and integrate from 0 to x and using (2.13),we get

$$r(x,y) \le \exp\left(\int_0^x \int_0^y [p(s,t) + q(s,t)]dtds \right) = A(x,y) \tag{2.15}$$

From (2.11) and (2.15) and the facts that $m(x,y) > 0, m_x(x,y) \ge 0, m_y(x,y) \ge 0$ for $x, y \in E$, we observe that

$$\frac{m_{xy}(x,y)}{m(x,y)} \le p(x,y)A(x,y) \tag{2.16}$$

By keeping first x fixed in (2.16) and set $y = t$ and integrate from 0 to y then again keeping y fixed, set $x = s$ and integrate from 0 to x and using (2.10),we get

$$m(x,y) \le \exp\left[1 + \int_0^x \int_0^y p(s,t)A(s,t)dsdt \right] = A_*(x,y) \tag{2.17}$$

By substituting (2.17) in (2.8) the facts that $z(x,y) > 0, z_x(x,y) \ge 0, z_y(x,y) \ge 0$ for $x, y \in E$, we observe that

$$\frac{z_{xy}(x,y)}{z(x,y)} \le f(x,y)A_*(x,y) \tag{2.18}$$

By keeping first x fixed in (2.18) and set $y = t$ and integrate from 0 to y then again keeping y fixed, set $x = s$ and integrate from 0 to x and using (2.6),we get

$$z(x,y) \le c^{Q(x,y)} \tag{2.19}$$

where $$Q(x,y) = \exp\left(\int_0^x \int_0^y f(s,t)A_*(s,t)dsdt \right),$$

From (2.19) and (2.7) we get the required inequality.

Theorem 2.2. Let $u : E \to R_1$, $f, p, q : E \to R_+$, be continuous functions and $c \ge 1$ be a constant,where $E = [0,\alpha] \times [0,\beta]$, $\alpha > 0, \beta > 0, R_1 = [1,\infty)$ and $R_+ = [0,\infty)$.If

$$u(x,y) \le c + \int_0^x\int_0^y f(s,t)u(s,t)\left[\log u(s,t) + \int_0^s\int_0^t p(\sigma,\eta)\log u(\sigma,\eta)\left[\int_0^\sigma\int_0^\eta q(s_1,t_1)\log u(s_1,t_1)ds_1dt_1\right]d\sigma d\eta\right]dsdt \tag{2.20}$$

for $x, y \in E$, then

$$u(x,y) \le c^{Q_2(x,y)}, \quad \text{for } x, y \in E \tag{2.21}$$

where

$$Q_2(x,y) = \exp\left(\int_0^x\int_0^y [f(s,t) + p(s,t)B(s,t)]dsdt\right) \tag{2.22}$$

$$B(x,y) = \exp\left(1 + \int_0^x\int_0^y q(s,t)dsdt\right) \tag{2.23}$$

Proof: Define a function

$$z(x,y) = c + \int_0^x\int_0^y f(s,t)u(s,t)\left[\log u(s,t) + \int_0^s\int_0^t p(\sigma,\eta)\log u(\sigma,\eta)\left[\int_0^\sigma\int_0^\eta q(s_1,t_1)\log u(s_1,t_1)ds_1dt_1\right]d\sigma d\eta\right]dsdt \tag{2.24}$$

$$z(x,0) = z(0,y) = c, \qquad z_x(x,0) = z_y(0,y) = 0 \tag{2.25}$$

By substituting (2.24) in (2.20),we get

$$u(x,y) \le z(x,y) \tag{2.26}$$

By differentiating (2.24) with respect to x and y ,we get

$$z_{xy}(x,y) = f(x,y)u(x,y)\left[\log u(x,y) + \int_0^x\int_0^y p(s,t)\log u(s,t)\left[\int_0^s\int_0^t q(\sigma,\eta)\log u(\sigma,\eta)d\sigma d\eta\right]dsdt\right],$$

By using (2.26) in the above equation.we observe that

$$z_{xy}(x,y) \le f(x,y)z(x,y)m(x,y), \tag{2.27}$$

Where $$m(x,y) = \log z(x,y) + \int_0^x\int_0^y p(s,t)\log z(s,t)\left[\int_0^s\int_0^t q(\sigma,\eta)\log z(\sigma,\eta)d\sigma d\eta\right]dsdt \tag{2.28}$$

$$m(x,0) = m(0,y) = \log c, \tag{2.29}$$

By differentiating (2.28) with respect to x and y and since $\log z(x,y) \le m(x,y)$ from (2.28)we have

$$m_{xy}(x,y) \le f(x,y)m(x,y) + p(x,y)m(x,y)\left[\int_0^x\int_0^y q(s,t)m(s,t)dsdt\right]$$

Which can be rewritten as

$$m_{xy}(x,y) \le f(x,y)m(x,y) + p(x,y)m(x,y)r(x,y) \tag{2.30}$$

Where

$$r(x,y) = \int_0^x \int_0^y q(s,t)m(s,t)dsdt \tag{2.31}$$

$$r(x,0) = r(0,y) = 0\,, \tag{2.32}$$

By differentiating (2.31) with respect to x and y and since from (2.31) it is obvious that $m(x,y) \le r(x,y)$ and using (2.32),we have

$$r_{xy}(x,y) = q(x,y)m(x,y)$$

$$r_{xy}(x,y) \le q(x,y)r(x,y)$$

From the above inequality and the facts that $r(x,y) > 0,\, r_x(x,y) \ge 0,\, r_y(x,y) \ge 0$ for $x, y \in E,$ we observe that

$$\frac{r_{xy}(x,y)}{r(x,y)} \le q(x,y) \tag{2.33}$$

By keeping first x fixed in (2.33) and set $y = t$ and integrate from 0 to y then again keeping y fixed, set $x = s$ and integrate from 0 to x and using (2.32),we get

$$r(x,y) \le \exp\left(\int_0^x \int_0^y [1 + q(s,t)]dtds\right) = B(x,y) \tag{2.34}$$

From (2.30) and (2.34) and the facts that $m(x,y) > 0,\, m_x(x,y) \ge 0,\, m_y(x,y) \ge 0$ for $x, y \in E,$ we observe that

$$\frac{m_{xy}(x,y)}{m(x,y)} \le f(x,y) + p(x,y)B(x,y) \tag{2.35}$$

By keeping first x fixed in (2.35) and set $y = t$ and integrate from 0 to y then again keeping y fixed, set $x = s$ and integrate from 0 to x and using (2.29),we get

$$m(x,y) \le (\log c)\exp \int_0^x \int_0^y [f(s,t) + p(s,t)B(s,t)]dsdt \tag{2.36}$$

Since $\log z(x,y) \le m(x,y)$ and from (2.36), we observe that

$$\log z(x,y) \le m(x,y)$$

$$\le (\log c)\exp \int_0^x \int_0^y [f(s,t) + p(s,t)B(s,t)]dsdt$$

which can be rewritten as

$$z(x,y) \le c \exp \int_0^x \int_0^y [f(s,t) + p(s,t)B(s,t)]dsdt$$

or

$$z(x,y) \le c^{Q_2(x,y)} \tag{2.37}$$

where $Q_2(x,y)$ is defined as in (2.22).From (2.26) and (2.37) we get the desired inequality .

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