Applied Mathematical Sciences, Vol. 6, 2012, no. 33, 1637 – 1641

Diagram Groups Classes over the Semigroup Presentations by Adding New Letters to Set of Generators

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Abstract

Given any semigroup presentation S, we may obtain diagram group $D(S,U)$ with base point U, where U is a word in the set of generators. In this paper we investigate the relationship of diagram groups if new letters are added to the set of generators. These diagram groups are found to be isomorphic and closed under finite free product.

Mathematics Subject Classification: 20M05, 14L40

Keywords: Diagram groups, Semigroup presentation, Set of generators, Letters, Isomorphism

1. Introduction

Diagram groups are groups can be defined as fundamental groups of the Squier complexes associated with monoid presentations. The class of diagram group was introduced by Meakin and Sapir in 1993. Kilibarda obtained the first result about diagram group in (1994, 1997) (refer to [4], [5]). For further details about diagram groups and semigroup presentations (see [1], [2], or [3]).
Let $S = \langle X \mid R \rangle$ be a semigroup presentation. Add a new generator x_1 to the set of generators X, and consider the new obtained semigroup presentation $S_1 = \langle X \cup \{x_1\} \mid R \rangle$. The purpose of this paper, is to determine the relationship between diagram groups over the general case of semigroup presentation $S_n = \langle X \cup \{x_1, x_2, \ldots, x_n\} \mid R \rangle$. Then we will prove that for every word U over the set of generators X, the diagram groups $D(S,U)$, $D(S_1, x_1Ux_1)$, $D(S_2, x_2x_1Ux_1x_2)$, $D(S_n, x_n \ldots x_2x_1Ux_1x_2 \ldots x_n)$ are isomorphic. We will also prove that the class of these diagram groups over the semigroup presentations is closed under finite free product.

In this paper we proved the following theorems.

Main Theorem 1: Let $S = \langle X \mid R \rangle$ be a semigroup presentation, add new letters $x_1, x_2, \ldots, x_n$ to set of generators X, and consider the new semigroup presentation $S_n = \langle X \cup \{x_1, x_2, \ldots, x_n\} \mid R \rangle$. Then for every word U over X the diagram groups $D(S,U)$, $D(S_1, x_1Ux_1)$, $D(S_2, x_2x_1Ux_1x_2)$, $\ldots, D(S_n, x_n \ldots x_2x_1Ux_1x_2 \ldots x_n)$ are isomorphic.

Main Theorem 2: Consider diagram groups $D(S_i, U_i)$, $(i = 1,2,\ldots,n+1)$ and consider the presentation S_i obtained by adding a new letter a_i to the set of generators of the semigroup presentation S_{i-1}, then the diagram group $D(S, a_nU_na_n)$ is isomorphic to $D(S_1,U_1) * D(S_2,U_2) * \cdots * D(S_{n+1},U_{n+1})$.

In our proofs we used techniques of some lemmas and theorems from Guba and Sapir (see [1], [6], and [8]).

First we recall some theoretical results, necessary in the proof of Main Theorems(refer to [1] and [7]).

Lemma 1: Let $S_i = \langle X_i \,|\, R_i \rangle$, $(i=1,2)$ be semigroup presentations with $X_1 \cap X_2 = \varnothing$. Let $U_i \in X_i^{+}$. Let $S = \langle X_1 \cup X_2 \,|\, R_1 \cup R_2 \rangle$ be the free product of S_1 and S_2. Then the diagram group $D(S, U_1 U_2)$ is isomorphic to the direct product of the diagram groups $D(S_1, U_1)$ and $D(S_2, U_2)$·

Lemma 2: Let $S_i = \langle X_i \,|\, R_i \rangle$, $(i=1,2)$ be semigroup presentations and let $U_i, (i=1,2)$ be a word over the alphabet X_i^{+}. Let X_1 and X_2 be disjoint sets. Suppose that the congruence class of $U_i, (i=1,2)$ modulo S_i does not contain words of the form YU_iZ, where Y, Z are words over X_1, X_2 and YZ are not empty. Consider the following semigroup presentation $S = \langle X_1 \cup X_2 \,|\, R_1 \cup R_2 \cup \{U_1 = U_2\} \rangle$.Then $D(S, U_1 U_2) \cong D(S_1, U_1) * D(S_2, U_2)$·

2. Results

Proof of the Main Theorem 1.

Let $S = \langle X \,|\, R \rangle$ be a semigroup presentation. Add a new generator $x_1 \in X$, and consider $S_1 = \langle X \cup \{x_1\} \,|\, R \rangle$. Then the diagram groups $D(S,U)$ and $D(S_1, x_1 U x_1)$ are isomorphic. The isomorphism between these two diagram groups is determined by the map $\Delta \to \varepsilon(x_1) + \Delta + \varepsilon(x_1)$ where Δ is a diagram group over S.

Every word in the congruence class $x_1 U x_1$ of modulo S_1 has the form $x_1 V x_1$ where V is a word from the congruence class of U modulo S . Thus $x_1 U x_1$ satisfies the condition of the theorem, that is if $x_1 U x_1 = \delta\, x_1 U x_1 \gamma$ modulo S_1 then $\delta\gamma = \phi$.

Now if we add a new generator $x_2 \in X \cup \{x_1\}$ and consider the presentation $S_2 = \langle X \cup \{x_1, x_2\} \,|\, R \rangle$. Then for every word S over $X \cup \{x_1\}$, we have the diagram groups $D(S_1, x_1 U x_1)$, $D(S_2, x_2 S x_2)$. If $S \in X \cup \{x_1\}$ then there exists some word V over X such that $S = x_1Vx_1$, $U \equiv V$ (modulo S_1), then $D(S_1, x_1Ux_1) \cong D(S_2, x_2x_1Ux_1x_2)$. The isomorphism between these two diagram groups is determined by the map $\Delta_1 \to \varepsilon(x_2) + \Delta_1 + \varepsilon(x_2)$ where Δ_1 is a diagram group over S_1.

Thus as above method we may obtain $D(S_{n-1}, x_{n-1} \ldots x_2x_1Ux_1x_2 \ldots x_{n-1}) \cong D(S_n, x_n \ldots x_2x_1Ux_1x_2 \ldots x_n)$. The isomorphism between these two diagram groups is determined by the map $\Delta_{n-1} \to \varepsilon(x_n) + \Delta_{n-1} + \varepsilon(x_n)$ where Δ_{n-1} is a diagram group over S_{n-1}.

Proof of the Main Theorem 2.
In the Main Theorem 1, we have $D(S_i, a_iU_ia_i) \cong D(S_{i-1}, U_{i-1})$, and the word $a_iU_ia_i$ satisfies the conditions of Lemma 2. By using the lemma 2, we can obtain that the diagram group $D(S, a_1U_1a_1)$ is isomorphic to $D(S_1, U_1) * D(S_2, U_2)$, thus the statement is true for $n = 1$. Now we assume that the theorem for n is true. Next we must show that the diagram group $D(S, a_{n+1}U_{n+1}a_{n+1})$ is isomorphic to $D(S_1, U_1) * D(S_2, U_2) * \ldots * D(S_{n+1}, U_{n+1}) * D(S_{n+2}, U_{n+2})$. By using the assumption of the Main Theorem 2 and applying the Lemma 2, we have $D(S, a_{n+1}U_{n+1}a_{n+1}) \cong D(S, a_nU_na_n) * D(S_{n+2}, U_{n+2}) \cong D(S_1, U_1) * D(S_2, U_2) * \ldots * D(S_{n+1}, U_{n+1}) * D(S_{n+2}, U_{n+2})$. □

3. Conclusion

In this study, we proved that if we add the new letters to the set of generators, then the diagram groups $D(S,U)$, $D(S_1, x_1Ux_1)$, $D(S_2, x_2x_1Ux_1x_2)$, $\ldots, D(S_n, x_n \ldots x_2x_1Ux_1x_2 \ldots x_n)$ are isomorphic. We also proved that these diagram groups over the semigroup presentations is closed under finite free product.

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Received: October, 2011