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Exact and Numerical Solutions for Coupled Nonlinear Klein-Gordon-Schrödinger Equations

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Abstract

In this paper we give a derivation of the exact solution for coupled nonlinear Klein-Gordon-Schrödinger equation by the ansatz method. We present a numerical solution by implicit finite difference scheme which is fourth order accurate in space and second order in time. We discuss the accuracy and stability of the scheme. Numerical results are given to show that this method is quite accurate.

Keywords: Klein-Gordon-Schrödinger equations, ansatz method, exact solution, Solitary wave solution

1 Introduction

In this paper we consider the coupled nonlinear Klein-Gordon-Schrödinger equation of the form

$$i\frac{\partial \emptyset}{\partial t}+\frac{1}{2}\frac{\partial^2 \emptyset}{\partial x^2}+\phi\psi=0,$$
$$\frac{\partial^2 \psi}{\partial t^2}-\frac{\partial^2 \psi}{\partial x^2}+M^2\psi-|\phi|^2=0, \quad -\infty<x<\infty,\ \ t>0 \tag{1}$$
$$\emptyset(\mathrm{x},0)=\emptyset_0(\mathrm{x}), \quad \psi(x,0)=\psi_0 \quad \frac{\partial}{\partial t}\psi(x,0)=\psi_1(x),$$
$$\phi(x,t)=0, \quad \psi(x,t)=0, \quad |x|\to\infty \tag{2}$$

which describes a classical model of interaction of a nucleon field with the meson field [1] and play important role in modern physics? Here $\emptyset(x,t)$ is a complex field, $\psi(x,t)$ a real field and M the mass of a meson. It posses two conserved quantities.

$$Q(t)=\|\emptyset\|^2=\mathrm{Q}(0)\,, \tag{3}$$
$$E(t)=\frac{1}{2}\left(M^2\|\psi\|^2+\|\partial_t\psi\|^2+\|\partial_x\psi\|^2+\|\partial_x\emptyset\|^2\right)-2(|\emptyset|^2,\psi) \tag{4}$$

where $Q(0)$ and $E(0)$ are constants depending only on initial values, and $\|.\|, (.\ ,\ .)$ denote the norm and inner product in any Hilbert space respectively.

Much work has been done on the existence of global solutions, asymptotic behavior and stability of discussed problem [1, 2, 8, 9]. For the exact solutions, many methods have been implemented to solve this problem, such as F-expansion method [5], the domain decomposition method [10], mapping method [14], homogeneous balance method [13] and Jacobi elliptic function expansion method [6].

In this paper we use the ansatz method to get the exact solution of (1) in section 2. In section 3 we derive the numerical solution by the fourth order finite difference scheme. In section 4 we discuss the accuracy of the scheme. In section 5 we show that the scheme is unconditionally stable. Numerical results are given to show the validity of the method in section 6.

2 Mathematical analysis

In order to get the exact solution we use the ansatz method [4, 5, 11] we suppose that

$$\phi(x,t)=A_1 sech^{p_1}\tau\, e^{i\eta} \tag{5}$$
$$\psi(x,t)=A_2 sech^{p_2}\tau \tag{6}$$

where A_1 and A_2 are the amplitudes of ϕ and ψ solution respectively, and

$$\tau=B(x-qt)$$

where B is the inverse width of the solution and q is the soliton velocity. In (5) η represents the phase of the solution that is defined as

$$\eta=-\kappa x+\omega t+\theta$$

where κ is the frequency while ω is the soliton wave number and θ is the phase constant

From (5) and (6) we get the following expressions

$$\phi_t = (p_1 q A_1 B sech^{p_1}\tau \, tanh\tau + i\omega sech^{p_1}\tau)e^{i\eta} \quad (7)$$

$$\phi_{xx} = (p_1^2 A_1 B^2 sech^{p_1}\tau - p_1(p_1+1)A_1 B^2 sech^{(p_1+2)}\tau$$
$$+2i\kappa p_1 A_1 sech^{p_1}\tau \, tanh\tau - \kappa^2 A_1 sech^{p_1}\tau)e^{i\eta} \quad (8)$$

$$\psi_{tt} = p_2^2 q^2 A_2 B^2 sech^{p_2}\tau - p_2(p_2+1)q^2 A_2 B^2 sech^{(p_2+2)}\tau \quad (9)$$

$$\psi_{xx} = p_2^2 A_2 B^2 sech^{p_2}\tau - p_2(p_2+1)A_2 B^2 sech^{(p_2+2)}\tau \quad (10)$$

Substituting from (7) – (10) into (1) we get

$$ip_1 qB tanh\tau - \omega + \frac{1}{2}(B^2 p_1^2 - p_1(p_1+1)B^2 sech^2\tau + 2i\kappa p_1 \, tanh\tau - \kappa^2)$$
$$+ A_2 sech^{p_2}\tau = 0 \quad (11)$$

$$p_2^2 q^2 A_2 B^2 sech^{p_2}\tau - p_2(p_2+1)q^2 A_2 B^2 sech^{(p_2+2)}\tau - p_2^2 A_2 B^2 sech^{p_2}\tau$$
$$+p_2(p_2+1)A_2 B^2 sech^{(p_2+2)}\tau + M^2 A_2 sech^{p_2}\tau - A_1^2 sech^{2p_1}\tau = 0 \quad (12)$$

By setting the imaginary part to zero in eq. (11) we have

$$\kappa = -q \quad (13)$$

From (11) we get

$$\omega = \frac{1}{2}(B^2 p_1^2 - q^2) \quad (14)$$

By balancing the power of $sech^{p_2}\tau$ and $sech^2\tau$ in (11) we get

$$p_2 = 2 \quad (15)$$

Also, from (11) by equating the coefficient of $sech^2\tau$ to zero we have

$$A_2 = \frac{1}{2}p_1(p_1+1)B^2 \quad (16)$$

By balancing the power of $sech^{(p_2+2)}\tau$ and $sech^{2p_1}\tau$ in (12) we get

$$p_1 = 2 \quad (17)$$

By equating the coefficients of $sech^{p_2}\tau$ to zero in (12) we get

By $\quad B = \frac{M}{2\sqrt{(1-q^2)}} \quad (18)$

equating the coefficients of $sech^{(p_2+2)}\tau$ and $sech^{2p_1}\tau$ in (12) we get

$$A_1 = B\sqrt{6(1-q^2)A_2} \quad (19)$$

From (16)

$$A_2 = \frac{3M^2}{4(1-q^2)} \quad (20)$$

From (19)

$$A_1 = \frac{3\sqrt{2}M^2}{4\sqrt{(1-q^2)}} \quad (21)$$

From (14)

$$\omega = \frac{1}{2}\left(\frac{M^2 - q^2 + q^4}{1-q^2}\right) \quad (22)$$

Substituting in (5), (6) we get the general solution

$$\phi(x,t)=\frac{3\sqrt{2}\mathrm{M}^2}{4\sqrt{(1-\mathrm{q}^2)}}sech^2\left(\frac{\mathrm{M}}{2\sqrt{(1-\mathrm{q}^2}}(\mathrm{x}-\mathrm{qt})\right)\exp\left\{\mathrm{i}\left[\mathrm{qx}+\frac{1}{2}\left(\frac{\mathrm{M}^2-\mathrm{q}^2+\mathrm{q}^4}{1-\mathrm{q}^2}\right)\mathrm{t}\right]+\theta\right\}, \tag{23}$$

$$\psi(x,t)=\frac{3\mathrm{M}^2}{4(1-\mathrm{q}^2)}sech^2\left(\frac{\mathrm{M}}{2\sqrt{(1-\mathrm{q}^2}}(\mathrm{x}-\mathrm{qt})\right) \tag{24}$$

Which is the exact solution obtained in [12] by F-expansion function.

3 Numerical method

We assume that the solution of (1) is negligibly small outside the interval $[x_L, x_R]$, and so we consider the coupled nonlinear Klein-Gordon-Schrödinger equations of the form

$$i\frac{\partial\phi}{\partial t}+\frac{1}{2}\frac{\partial^2\phi}{\partial x^2}+\phi\psi=0,\quad x_{\mathrm{L}}<x<x_R,\ ,\ \ t>0\,, \tag{25}$$
$$\frac{\partial^2\psi}{\partial t^2}-\frac{\partial^2\psi}{\partial x^2}+M^2\psi-|\phi|^2=0,\quad x_{\mathrm{L}}<x<x_R,\ \ t>0 \tag{26}$$

we compose the function $\phi(x,t)$ in real parts and imaginary parts by writing

$$\phi(x,t)=u_1+iu_2$$

Also we set

$$\psi(x,t)=u_3,\qquad \frac{\partial\psi}{\partial t}=u_4$$

Substituting in (1) we get

$$\frac{\partial\mathbf{u}}{\partial\mathrm{t}}+\frac{1}{2}\mathrm{A}\frac{\partial^2\mathbf{u}}{\partial\mathrm{x}^2}+\mathrm{F}(\mathbf{u})\mathbf{u}=0 \tag{27}$$

$$\mathbf{u}=\begin{pmatrix}u_1\\u_2\\u_3\\u_4\end{pmatrix},\quad A=\begin{pmatrix}0&1&0&0\\-1&0&0&0\\0&0&0&0\\0&0&-2&0\end{pmatrix},\ \mathbf{F}(\mathbf{u})=\begin{pmatrix}0&u_3&0&0\\-u_3&0&0&0\\0&0&0&-1\\-u_1&-u_2&M^2&0\end{pmatrix}$$

3.1 Finite difference method

The x and t coordinates are discretized by a grid spacing h and a time step k. This gives the grid points *(mh,nk)=(m,n)* with *m,n*=1, 2, …, $\mathbf{U}_{\mathrm{m}}^{\mathrm{n}}$ is used to denote an approximation solution and $\mathbf{u}_{\mathrm{m}}^{\mathrm{n}}$ is used to the exact solution. We approximate the differential equation (27) by the finite difference scheme

$$\frac{1}{k}\delta_t \mathbf{U}_m^n + \frac{1}{2h^2} A\left(1+\frac{1}{12}\delta_x^2\right)^{-1}\delta_x^2 \mathbf{U}_m^{n+\frac{1}{2}} + F\left(\mathbf{U}_m^{n+\frac{1}{2}}\right)\mathbf{U}_m^{n+\frac{1}{2}} = 0 \tag{28}$$

where

$$\mathbf{U}_m^{n+\frac{1}{2}} = \frac{\mathbf{U}_m^{n+1} + \mathbf{U}_m^n}{2}$$

$$\delta_t \mathbf{U}_m^n = \mathbf{U}_{\mathrm{m}}^{\mathrm{n+1}} - \mathbf{U}_{\mathrm{m}}^{\mathrm{n}}$$

$$\delta_x^2 \mathbf{U}_m^n = \mathbf{U}_{m+1}^n - 2\mathbf{U}_{\mathrm{m}}^{\mathrm{n}} + \mathbf{U}_{\mathrm{m-1}}^{\mathrm{n}}$$

4 Accuracy of the Scheme

To study the accuracy of the scheme we replace the numerical solution $\mathbf{U}_{\mathrm{m}}^{\mathrm{n}}$ by the exact solution $\mathbf{u}_m^n$ in (28). Doing this the proposed scheme will be of the form

$$\frac{1}{k}\left(1+\frac{1}{12}\delta_x^2\right)(\mathbf{u}_m^{n+1} - \mathbf{u}_m^n) + \frac{1}{4h^2}A\delta_x^2(\mathbf{u}_m^{n+1} - \mathbf{u}_m^n) + \left(1+\frac{1}{12}\delta_x^2\right)G\left(\frac{\mathbf{u}_m^{n+1}+\mathbf{u}_m^n}{2}\right) = 0 \tag{29}$$

where G(u)=F(u)u

Taylor expansion of all terms in (29) can be displayed as follows

$$\frac{1}{k}\left(1+\frac{1}{12}\delta_x^2\right)(\mathbf{u}_m^{n+1} - \mathbf{u}_m^n) = \frac{\partial \mathbf{u}}{\partial \mathrm{t}} + \frac{1}{2}\frac{\partial^2 \mathbf{u}}{\partial \mathrm{t}^2} + \frac{\mathrm{h}^2}{12}\frac{\partial^3 \mathbf{u}}{\partial \mathrm{t}\,\partial \mathrm{x}^2} + \frac{\mathrm{h}^2\mathrm{k}}{24}\frac{\partial^4 \mathbf{u}}{\partial^2\,\partial^2 \mathrm{u}} + \mathrm{O}(\mathrm{h}^4, \mathrm{k}^2, \mathrm{h}^2\mathrm{k}^2) \tag{30}$$

$$\frac{1}{4\mathrm{h}^2}\delta_{\mathrm{x}}^2(\mathbf{u}_m^{n+1} + \mathbf{u}_m^n) = \frac{1}{2}\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} + \frac{k}{4}\frac{\partial^3 \mathrm{u}}{\partial \mathrm{x}^2\partial \mathrm{t}} + \frac{\mathrm{h}^2}{24}\frac{\partial^4 \mathrm{u}}{\partial \mathrm{x}^4} + \frac{\mathrm{kh}^2}{48}\frac{\partial^5 \mathrm{u}}{\partial \mathrm{x}^4\,\partial \mathrm{t}} + \mathrm{O}(\mathrm{h}^4, \mathrm{k}^2, \mathrm{h}^2\mathrm{k}^2) \tag{31}$$

$$\left(1+\frac{1}{12}\delta_x^2\right)G\left(\frac{\mathbf{u}_m^{n+1}+\mathbf{u}_m^n}{2}\right) = G + \frac{k}{2}\frac{\partial \mathbf{G}}{\partial t} + \frac{\mathrm{h}^2}{12}\frac{\partial^2 \mathrm{G}}{\partial \mathrm{x}^2} + \frac{h^2\mathrm{k}}{24}\frac{\partial^3 \mathrm{G}}{\partial \mathrm{x}^2\,\partial \mathrm{t}} + \mathrm{O}(\mathrm{h}^4, \mathrm{k}^2, \mathrm{h}^2\mathrm{k}^2) \tag{32}$$

Substituting from (30)-(32) in (29) we get

$$\begin{aligned}\mathrm{T}_m^n = {} & \left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathrm{u}}{\partial \mathrm{x}^2} + G\right)_m^n + \frac{k}{2}\frac{\partial}{\partial t}\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathbf{u}}{\partial \mathrm{x}^2} + G\right)_m^n \\ & + \frac{h^2}{12}\frac{\partial^2}{\partial x^2}\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathbf{u}}{\partial \mathrm{x}^2} + G\right)_m^n + \frac{h^2 k}{24}\frac{\partial^3}{\partial x^2 \partial t}\left(\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2}A\frac{\partial^2 \mathbf{u}}{\partial \mathrm{x}^2} + G\right)_m^n \\ & + \mathrm{O}(\mathrm{h}^4, \mathrm{k}^2, \mathrm{h}^2\mathrm{k}^2)\end{aligned} \tag{33}$$

By using the differential (27) all terms inside the brackets are equal zero, then we have

$$\mathrm{T}_m^n = \mathrm{O}(\mathrm{h}^4, \mathrm{k}^2, \mathrm{h}^2\mathrm{k}^2) \tag{34}$$

Thus T_m^n tends to zero as *h, k* tend to zero, we deduce that the proposed scheme is a second order in time and a fourth order in space, it is consistent since the local truncation error T_m^n tends to zero.

5 Stability

To study the accuracy of the proposed scheme, the von Neumann stability analysis will be used. This method can only be applied for linear scheme. By freezing all terms which make the scheme nonlinear [3] then eq. (27) has the form

$$\frac{\partial \mathbf{u}}{\partial t} + \frac{1}{2} C_1 \frac{\partial^2 \mathbf{u}}{\partial x^2} + \propto C_2 \mathbf{u} = 0 \tag{35}$$

where

$$C_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & -2 & 0 \end{pmatrix}, C_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1/\alpha \\ -1 & -1 & M^2/\alpha & 0 \end{pmatrix}$$

where, $\propto = \max\{\propto_1, \propto_2, \propto_3\}$

The finite difference scheme of (35) is

$$\left(1 + \frac{1}{12}\delta_x^2\right)(\mathbf{U}_m^{n+1} - \mathbf{U}_m^n) + \frac{1}{2} r C_1 \frac{\partial^2}{\partial x^2}(\mathbf{U}_m^{n+1} + \mathbf{U}_m^n) + \frac{k\propto}{2} C_2 \left(1 + \frac{1}{12}\delta_x^2\right)(\mathbf{U}_m^{n+1} + \mathbf{U}_m^n) = 0 \tag{36}$$

where, $r = \frac{k}{2h^2}$

Assume that

$$\mathrm{U}_m^n = \mathrm{E}^n H e^{i\beta m h}$$

be the test function $\beta \in \mathbb{R}, H \in \mathbb{R}$ and $E \in \mathbb{R}^{4\times4}$ be the amplification matrix. The necessary condition for stability of the scheme is

$$max_j |\lambda_j| \leq 1, \ \ j = 1, 2, 3, 4$$

Substituting in (35) we have

$$E = \left(\gamma I - (\omega_1 C_1 - \omega_2 C_2)\right)^{-1}\left(\gamma I + (\omega_1 C_1 - \omega_2 C_2)\right)$$

where, $\omega_1 = \frac{1}{2} r\mu, \quad \omega_2 = \frac{k}{2}\alpha\gamma, \quad \gamma = 1 - \frac{1}{3}sin^2\frac{\beta h}{2}, \ \ \mu = -4sin^2\frac{\beta h}{2}$

The eigenvalues of the matrix E are

$$\lambda_1 = \frac{\gamma - i\omega_2}{\gamma + i\omega_2}, \ \ \lambda_2 = \frac{\gamma - \omega_1 + i\omega_2}{\gamma + \omega_1 - i\omega_2}, \ \ \lambda_3 = \frac{\gamma - i\frac{M\omega_2}{\alpha}}{\gamma + i\frac{M\omega_2}{\alpha}}, \ \ \lambda_4 = \frac{\gamma + i\frac{M\omega_2}{\alpha}}{\gamma - i\frac{M\omega_2}{\alpha}}$$

It is clear that all the modulus of the eigenvalues equal 1, then the scheme is unconditionally stable, the scheme is also consistent, and then according to Lax theorem the scheme is convergent .

6 Numerical results

We study the accuracy of the proposed scheme by calculating L_∞ and L_2 the norm of $\emptyset$ which are given by

$$\|Er\|_\infty = max\{|(\|\phi(x_m,t_n)\| - \|U_{1,m}^n + iU_{2,m}^n\|)|\}$$

$$\|Er\|_2 = [\sum_{m=1}^n(\|\phi(x_m,t_n)\| - \|U_{1,m}^n + iU_{2,m}^n\|)]^{\frac{1}{2}}$$

Two cases are considered single soliton and two soliton interactions. We take the parameters $x_L = -50,\ x_R = 50,\ h = 0.2,\ k = 0.01$ and $\theta = 0$.

6.1 Single soliton solution

In this test we take the initial conditions

$$\phi(x,0) = \frac{3\sqrt{2}M^2}{4\sqrt{(1-q^2)}} sech^2\left(\frac{M}{2\sqrt{(1-q^2}}x\right)\exp(iqx+\theta),$$

$$\psi(x,0) = \frac{3M^2}{4(1-q^2)} sech^2\left(\frac{M}{2\sqrt{(1-q^2}}x\right)$$

In Table 1 and 2, we calculate the errors and the conserved quantities for two values M=0.5, M=0.75 respectively. It is very clear that the method is very quite and conserved quantities almost constants. Figure 1, 2 show the evolution of single ϕ and ψ at t=1, 2,…, 60 for M=0.5 and M=0.75 respectively moving to the right.

Time	Q(t)	E(t)	$L_\infty(Er)$	$L_2(Er)$
10	0.658037	0.132241	0.152269E-06	0.835208 E-06
20	0.658037	0.132241	0.205442 E-06	0.124585 E-06
30	0.658037	0.132241	0.280562 E-06	0.176373 E-06
40	0.658037	0.132241	0.361386 E-06	0.229091 E-06
50	0.658037	0.132241	0.434425 E-06	0.284363 E-06
60	0.658037	0.132241	0.551632 E-06	0.347312 E-06

Table 1: Accuracy and conserved quantity of the scheme with *q=0.5, M=0.5*

Time	Q(t)	E(t)	$L_\infty(Er)$	$L_2(Er)$
10	1.331777	0.127366	0.281881E-05	0.126431 E-04
20	1.331777	0.127366	0.511850 E-05	0.284741 E-04
30	1.331777	0.127366	0.737983 E-05	0.368405 E-04
40	1.331777	0.127366	0.980864 E-05	0.490385 E-04
50	1.331777	0.127366	0.121989 E-04	0.615725 E-04
60	1.331777	0.127366	0.147490 E-04	0.470227 E-04

Table 2: Accuracy and conserved quantity of the scheme with q=0.5, M=0.75

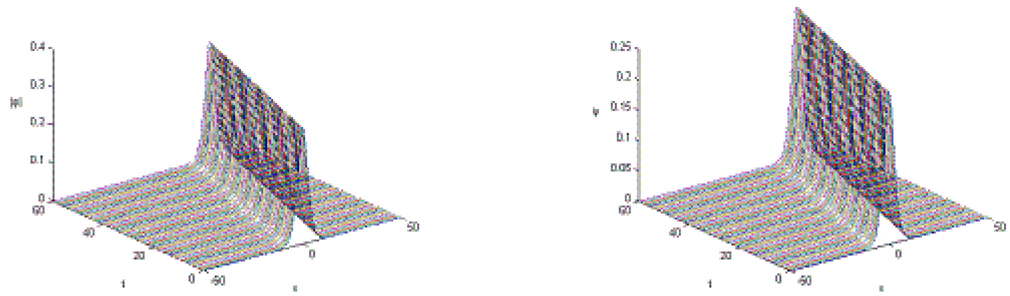

Figure 1: Single soliton solution with *q=0.5, M=0.5* left $|\phi|$ right ψ

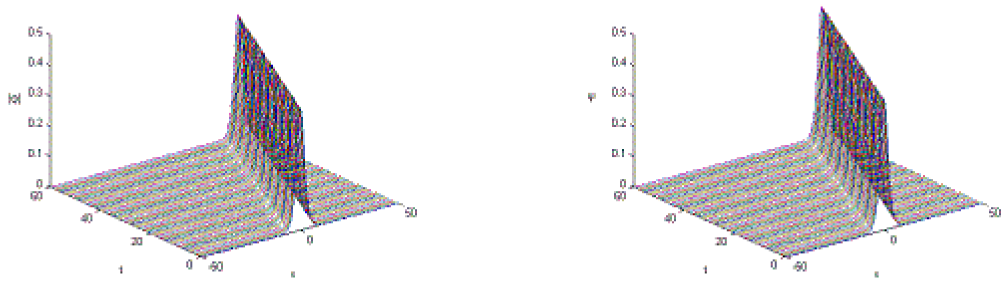

Figure 2: Single soliton solution with *q=0.5, M=0.75* left $|\phi|$ right ψ

6.2 Two solitons interaction

In this test we take the interaction of two solitons, where the initial conditions are assumed of the form

$$\phi(x,0) = \sum_{j=1}^{2} \frac{3\sqrt{2}\mathrm{M}^2}{4\sqrt{(1-\mathrm{q}_\mathrm{j}^2)}} sech^2\left(\frac{\mathrm{M}}{2\sqrt{(1-\mathrm{q}_\mathrm{j}^2}}(\mathrm{x}-\mathrm{x_j})\right)\exp\left(\mathrm{iq_j}(\mathrm{x}-\mathrm{x_j})\right),$$

$$\psi(x,0) = \sum_{j=1}^{2} \frac{3\mathrm{M}^2}{4(1-\mathrm{q}_\mathrm{j}^2)} sech^2\left(\frac{\mathrm{M}}{2\sqrt{(1-\mathrm{q}_\mathrm{j}^2}}(\mathrm{x}-\mathrm{x_j})\right)$$

which represents the sum of two single solitons located x_1 at and x_2 respectively. Figure 3 display the interactions of two solitons ϕ and ψ at t=0, 1, 2,…, 60 with the parameters $q_1 = 0.8,\ q_2 = -0.8, x_1 = -30, x_2 = 30$. We noticed that the two waves approaches, each other inteact and leave the interaction unchanged in shape and velocity.

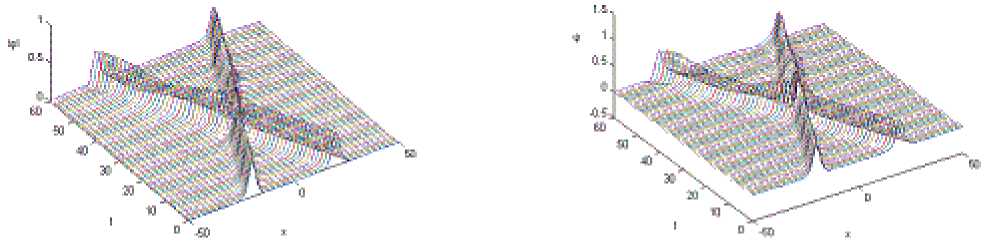

Figure 3: Interaction of two soliton solutions left $|\phi|$ right ψ

7 Conclusions

In this paper we used the ansatz method to obtain the exact solution of coupled nonlinear Klein-Gordon-Schrödinger-equations. We derive the numerical solution by the finite difference scheme which is a fourth order in space and second order in time. The accuracy of the scheme is also discussed. We show that the scheme is unconditionally stable. Numerical test are given the show the validity of the method.

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