Applied Mathematical Sciences, Vol. 6, 2012, no. 36, 1771 - 1777

The Variational Iteration Method for Solving Nonlinear Oscillators

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Abstract

In this paper, the variational iteration method (VIM) is applied to solve nonlinear oscillators. Only one iteration leads to high accuracy of the solutions. The accuracy of the obtained solutions is demonstrated by several examples.

Keywords: Variational iteration method; General Lagrange's multiplier; Nonlinear oscillator

1 Introduction

The study of nonlinear equations is of crucial importance in many areas of physics and engineering. Many phenomena such as solid state physics, plasma physics, fluid dynamics, mathematical biology and chemical kinetics, are modelled by nonlinear differential equations. A broad class of analytical and numerical methods is utilized to handle these problems [1, 2, 3].However, these nonlinear models are still difficult to solve.

Our attention focuses on the general nonlinear oscillator in the form

$$u'' - f(u) = 0, \qquad (1)$$

where $f(u)$ is a nonlinear term, a function of u only, and it requires $f(u)/u > 0$, subject to initial conditions

$$u(0) = A, \qquad u'(0) = 0. \tag{2}$$

Considerable research has been conducted in the study of strongly nonlinear oscillators. Several methods have been proposed to find approximate solutions to these problems. Such methods include the harmonic balance method (HB) [4], the elliptic Lindstedt-Poincare method (LP) [4, 5], the Krylov-Bogolioubov-Mitropolsky method (KBM) [6], the multiple scales method (MSM) [5], He's parameter-expanding methods [7], Adomian decomposition method [8] and homotopy perturbation method (HPM)[9].

Another powerful analytical method, called the variational iteration method (VIM), was first proposed by He [10] (see also [11, 12, 13, 14]). VIM has successfully been applied to many situations. For example, He [11] solved the classical Blasius' equation using VIM. He [12] used VIM to give approximate solutions for some well-known non-linear problems. He [14] solved strongly nonlinear equations using VIM. Momani *et al.* [15] applied VIM to the Helmholtz equation. VIM has been applied for solving nonlinear differential equations of fractional order by Odibat *et al.* [16]. Bildik *et al.* [17] used VIM to solve different types of nonlinear partial differential equations. Abbasbandy [18] solved the quadratic Riccati differential equation by VIM incorporating Adomian's polynomials. Junfeng [19] applied VIM to solve singular two-point boundary value problems. Abdou and Soliman used VIM to obtain the solution of Burger's equation and coupled Burger's equations. Batiha *et al.* [20] applied VIM to solve the generalized Huxley equation and in [21] they employed VIM to the generalized Burgers-Huxley equation. Many authors [22, 23, 24] used VIM to solve the nonlinear oscillator.

The purpose of this paper is to apply VIM to find the approximate analytical solution for nonlinear oscillators (1).

2 Variational iteration method

VIM is based on the general Lagrange's multiplier method [25]. The main feature of the method is that the solution of a mathematical problem with linearization assumption is used as initial approximation or trial function. Then a more highly precise approximation at some special point can be obtained. This approximation converges rapidly to an accurate solution [14].

To illustrate the basic concepts of VIM, we consider the following nonlinear differential equation:

$$Lu + Nu = g(x), \tag{3}$$

where L is a linear operator, N is a nonlinear operator, and $g(x)$ is an inhomogeneous term. According to VIM [12, 13, 14], we can construct a correction functional as follows:

$$u_{n+1}(x) = u_n(x) + \int_0^x \lambda(\tau)[Lu_n(\tau) + N\tilde{u}_n(\tau) - g(\tau)]\mathrm{d}\tau, \quad n \geq 0, \tag{4}$$

where λ is a general Lagrangian multiplier [25] which can be identified optimally via the variational theory, the subscript n denotes the nth-order approximation, $\tilde{u}_n$ is considered as a restricted variation [12, 13], i.e. $\delta\tilde{u}_n = 0$.

3 Analysis of general nonlinear oscillator

In this section, we solve Eq.(1) subject to initial condition (2) using VIM.

The iteration formulation can be constructed as follows [22]:

$$u_{n+1}(t) \;=\; u_0(t) + \int_0^t (s-t) f(u_n)\,\mathrm{d}s, \quad n \geq 0. \tag{5}$$

We can take $u_0(t) = u(0) + u'(0)t = A$, so the iteration becomes

$$u_{n+1}(t) \;=\; A + \int_0^t (s-t) f(u_n)\,\mathrm{d}s, \quad n \geq 0. \tag{6}$$

Its period can be obtained as follows:

$$T = 4\sqrt{2A/f(A)}. \tag{7}$$

And approximate solution reads

$$u(t) = A\cos\frac{2\pi}{T}t. \tag{8}$$

We give some examples to illustrate the effectiveness and convenience of these formulations.

4 Examples

In order to assess advantages and the accuracy of VIM for solving nonlinear oscillators, we have applied it to a variety of initial-value problems arising in nonlinear dynamics.

4.1 Example 1

First, we consider the following Helmholtz equation:

$$u''(t) + 2u(t) + u^2(t) = 0, \tag{9}$$

with the following initial conditions:

$$u(0) = 0.1, \quad u'(0) = 0. \tag{10}$$

Here, $f(u) = 2u + u^2$.

Using iteration (5) we get:

$$u_{n+1}(t) \;=\; 0.1 + \int_0^t (s-t)[2u_n + u_n^2]\,\mathrm{d}s, \tag{11}$$

we obtain its first-order approximate solution

$$\begin{aligned} u_1(t) &= 0.1 - 0.1050t^2, & (12)\\ u_2(t) &= 0.1 - 0.1050t^2 + 0.01925t^4 - 0.0003675t^6, & (13)\\ &\vdots \end{aligned}$$

Other components can be easily obtained.

The above results are in agreement with those obtained by the homotopy perturbation method reported in [26].

Then its period can be approximately obtained as:

$$\begin{aligned} T &= 4\sqrt{2A/f(A)} = 4\sqrt{0.2/0.19} = 4.1039136, & (14)\\ u(t) &= A\cos\frac{2\pi}{T}t = 0.0039763041t. & (15) \end{aligned}$$

4.2 Example 2

We now consider the nonlinear equation

$$u''(t) + u(t) + \varepsilon u^2(t)u'(t) = 0, \tag{16}$$

subject to the initial conditions:

$$u(0) = 1, \quad u'(0) = 0. \tag{17}$$

This equation can be called the "unplugged" van der Pol equation, and all of its solutions are expected to oscillate with decreasing amplitude to zero. For comparison with the solution obtained in [26], we set the parameter ε to 0.1.

Here, $f(u) = u + \varepsilon u^2 u'$.

Using iteration (5) we get:

$$u_{n+1}(t) = 1 + \int_0^t (s-t)[u_n + 0.1u_n^2 u_n']\,\mathrm{d}s, \tag{18}$$

we obtain its first-order approximate solution

$$u_1(t) = 1 - 0.5t^2, \tag{19}$$

$$u_2(t) = 1 - 0.5t^2 + 0.016667t^3 + 0.041667t^4 - 0.005t^5 + 0.000595t^7, \tag{20}$$

$$\vdots$$

Other components can be easily obtained.

The above results are in agreement with those obtained by the homotopy perturbation method reported in [26].

Then its period can be approximately obtained as:

$$T = 4\sqrt{2A/f(A)} = 5.65684, \tag{21}$$

$$u(t) = A\cos\frac{2\pi}{T}t = 0.444016t. \tag{22}$$

5 Conclusions

The variation iteration method (VIM) is an efficient tool for calculating periodic solutions to nonlinear oscillatory systems. The results of the present method when applied to several examples are in excellent agreement with those obtained by the homotopy perturbation method (HPM). The basic idea described in this paper is expected to be further employed to solve other similar strongly nonlinear oscillators.

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Received: October, 2011