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Modified Treatment of Initial Boundary Value Problems for One Dimensional Heat-Like and Wave-Like Equations Using Variational Iteration Method

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Abstract

In this paper, a new technique is applied to modified treatment of initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional) by mixed initial and boundary conditions together to obtain a new initial solution at every iteration using variational iteration method (VIM). The structure of a new successive initial solutions can give a more accurate solution in a first step.

Keywords: initial boundary value problems, one dimensional, heat-like and wave-like partial differential equations, variational iteration method

1. Introduction

Many researchers discussed the initial and boundary value problems. The Adomian decomposition method discussed for solving higher dimensional initial boundary value problems by Wazwaz A. M. [2000]. Analytic treatment for variable coefficient fourth-order parabolic partial differential equations discussed by Wazwaz A. M. [2001]. The solution of fractional heat-like and wave-like equations with variable coefficients using the decomposition method was found by Momani S. [2005] and so as by using variational iteration method was found by Yulita Molliq R. et.al [2009]. Solving higher dimensional initial boundary value problems by variational iteration decomposition method by Noor M.A. and Mohyud-Din S.T. [2008]. Exact and numerical solutions for non-linear Burger's equation by variational iteration method was applied by Biazar J. and Aminikhah H. [2009]. Weighted algorithm based on the homotopy analysis method is applied to inverse heat conduction problems and discussed by Shidfar A. and Molabahrami A. [2010]. The boundary value problems was applied by Niu Z. and Wang C. [2010] to calculate a one step optimal homotopy analysis method for linear and nonlinear differential equations with boundary conditions only, and homotopy perturbation technique for solving two-point boundary value problems–compared it with other methods was discussed by Chun C. and Sakthivel R. [2010]. Fractional differential equations with initial boundary conditions by modified Riemann–Liouville derivative was solved by Wu G. and Lee E. W. [2010]. It is worth mentioning that the origin of variational iteration method can be traced back by Inokuti et al. [1978].

It is interesting to point out that all these researchers obtained the solutions of initial and boundary value problems by using either initial or boundary conditions only. So we present a reliable framework by applying a new technique for treatment initial and boundary value problems by mixed initial conditions with boundary conditions together to obtain a new initial solution at every iteration using variational iteration method. Such as technique was applied by Ali E. J. [2011] for treatment of initial boundary value problems. In this paper, a new technique is applied to modified treatment of initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional) to construct a new successive initial solutions which can give a more accurate solution, some examples are given in this paper to illustrate the effectiveness and convenience of this technique.

We give some basic definitions and properties of the fractional calculus theory which are used further in this paper.

Definition 2.1. Jumarie is defined the fractional derivative [Jumarie G., 2009] as the following limit form

$$f^{(\alpha)} = \lim_{h\to 0} \frac{\Delta^{\alpha}[f(x) - f(0)]}{h^{\alpha}}. \quad (1.1)$$

This definition is close to the standard definition of derivatives, and as a direct result, the αth derivative of a constant, $0 < \alpha < 1$ is zero.

Definition 2.2. Fractional integral operator of order $\alpha \geq 0$ is defined as

$$I_x^{\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_0^x (x-\tau)^{\alpha-1} f(\tau) d\tau, \quad \alpha > 0. \quad (1.2)$$

Where Γ is a gamma function.

Definition 2.3. Fractional derivative of $f(x)$ in the Caputo sense [Caputo M.,1967] is defined as

$$D_x^{\alpha} f(x) = \frac{1}{\Gamma(m-\alpha)} \int_0^x (x-\tau)^{m-\alpha-1} \frac{d^m f(\tau)}{d\tau^m} d\tau, m-1 < \alpha \leq m, m \in \mathbb{N}, x > 0. \quad (1.3)$$

Definition 2.4. Fractional derivative of compounded functions [Jumarie G., 2009] is defined as

$$d^{\alpha} f \cong \Gamma(1+\alpha) df, \ \ 0 < \alpha < 1. \quad (1.4)$$

Definition 2.5. The integral with respect to $(dx)^{\alpha}$ [Jumarie G., 2009] is defined as the solution of the fractional differential equation

$$dy \cong f(x)(dx)^{\alpha}, \quad x \geq 0, \quad y(0) = 0, \ \ 0 < \alpha < 1. \quad (1.5)$$

Lemma 2.1. Let $f(x)$ denote a continuous function [Jumarie G., 2009] then the solution of the Eq. (1.5) is defined as

$$y = \int_0^x f(\tau)(d\tau)^\alpha = \alpha \int_0^x (x-\tau)^{\alpha-1} f(\tau)d\tau, \;\; 0 < \alpha < 1. \tag{1.6}$$

For example $f(x) = x^\alpha$ in Eq. (2.6) one obtains

$$\int_0^x \tau^\gamma (d\tau)^\alpha = \frac{\Gamma(\alpha+1)\Gamma(\gamma+1)}{\Gamma(\alpha+\gamma+1)} x^{\alpha+\gamma}, \;\; 0 < \alpha \le 1. \tag{1.7}$$

3. Variational iteration method

In this section, we introduce the basic idea underlying the variational iteration method (VIM) for solving nonlinear equations. Consider the general nonlinear differential equation

$$Lu(x,t) + Nu(x,t) = g(x,t), \tag{2.1}$$

where L is a linear differential operator, N is a nonlinear operator, and g is a given analytical function. The essence of the method is to construct a correction functional of the form [Gomez C. A. and Salas A. H. (2010)]

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(t,\tau)\big(Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,\tau)\big)d\tau. \tag{2.2}$$

It is obvious that the successive approximations $u_n, n \ge 0$ can be established by determining λ, a general Lagrange's multiplier, which can be identified optimally via the variational theory. The function $\tilde{u}_n$ is a restricted variation which means $\delta\tilde{u}_n = 0$. Therefore, we first determine the Lagrange multiplier λ that will be identified optimally via integration by parts. The successive approximations $u_{n+1}(x,t), n \ge 0$ of the solution $u(x,t)$ will be readily obtained upon using the obtained Lagrange multiplier and by using any selective function u_0. The initial values are usually used for selecting the zeroth approximation u_0, but in this paper well accredit a new technique to calculate the zeroth approximation which explain in the next section. With λ determined, then several approximations $u_n(x,t), n \geq 0$, follows immediately. Consequently, the exact solution may be obtained by using

$$u = \lim_{n\to\infty} u_n. \tag{2.3}$$

To describe the solution procedure of the fractional variational iteration method, we consider the following fractional differential equation

$$\frac{\partial^\alpha}{\partial t^\alpha} u(x,t) = \mu(x)\, u(x,t) + p(x,t), \quad x \in \mathbb{R}, \quad t > 0, \quad \alpha > 0, \tag{2.4}$$

where $\mu(x)$ is the differential operator in x, $p(x,t)$ is continuous function. According to the variational iteration method [Faraz N. et.al (2010)], we can construct a correct functional for Eq. (2.4) as follows

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha)} \int_0^t \lambda(t,\tau)(t-\tau)^{\alpha-1} \left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \mu(x)\, u_n(x,\tau) - p(x,\tau) \right) d\tau, \tag{2.5}$$

by Eq. (1.6), we have a new correction functional

$$u_{n+1}(x,t) = u_n(x,t) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(t,\tau) \left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \mu(x)\, \tilde{u}_n(x,\tau) - p(x,\tau) \right) (d\tau)^\alpha. \tag{2.6}$$

In general, we have a correction functional of the form

$$u_{n+1}(x,\tau) = \begin{cases} u_n(x,\tau) + \int_0^t \lambda(t,\tau) \left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \mu(x)\, \tilde{u}_n(x,\tau) - p(x,\tau) \right) d\tau, & \text{for } \alpha = m \in \mathbb{N}. \\ u_n(x,\tau) + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(t,\tau) \left(\frac{\partial^\alpha u_n(x,\tau)}{\partial \tau^\alpha} - \mu(x)\, \tilde{u}_n(x,\tau) - p(x,\tau) \right) (d\tau)^\alpha, & \text{for } m-1 < \alpha < m. \end{cases} \tag{2.7}$$

Where the Caputo time-fractional derivative operator of order $\alpha > 0$ is defined as

$$D_t^\alpha u(x,\tau) = \frac{\partial^\alpha}{\partial \tau^\alpha} u(x,\tau) = \begin{cases} \frac{\partial^m u(x,\tau)}{\partial \tau^m}, & \text{for } \alpha = m \in \mathbb{N}. \\ \frac{1}{\Gamma(m-\alpha)} \int_0^t (t-\tau)^{m-\alpha-1} \frac{\partial^m u(x,\tau)}{\partial \tau^m} d\tau, & \text{for } m-1 < \alpha < m. \end{cases} \tag{2.8}$$

3. New technique for solving one dimensional heat-like and wave-like equations (ordinary or fractional) using VIM

To convey the basic idea for modified treatment of initial boundary value problems by variational iteration method to solve one dimensional heat-like and wave-like equation of the form

$$\frac{\partial^\alpha}{\partial t^\alpha}u(x,t) = \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2}, \quad t > 0, \;\; 0 < x < 1, \;\; \alpha > 0, \tag{3.1}$$

the initial conditions associated with Eq. (3.1) are of the form

$$u(x,0) = f_0(x), \qquad \frac{\partial u(x,0)}{\partial t} = f_1(x), \qquad , \quad 0 < x < 1, \tag{3.2}$$

and the boundary conditions are given by

$$u(0,t) = g_0(t), \qquad u(1,t) = g_1(t), \quad t > 0, \tag{3.3}$$

where $f_0(x), f_1(x), g_0(t)$ and $g_1(t)$ are given functions. The initial solution can be written as $u_0(x,t) = f_0(x) + tf_1(x)$.

The initial values are usually used for selecting the zeroth approximation u_0 but in this paper we accredit a new technique to calculate the zeroth approximation u_0^* by construct a new initial solutions u_n^* by mixed initial conditions in Eq. (3.2) with boundary conditions in Eq. (3.3) at every iteration as follows [Ali E. J. (2011)]

$$u_n^*(x,t) = u_n(x,t) + (1-x)[g_0(t) - u_n(0,t)] + x[g_1(t) - u_n(1,t)], \;\; n \geq 0. \tag{3.4}$$

It is obvious that the new successive initial solutions u_n^* in Eq. (3.4) satisfying the initial and boundary conditions together as follows

$$if \;\; x = 0 \;\; then \;\; u_n^*(0,t) = g_0(t),$$

$$if \;\; x = 1 \;\; then \;\; u_n^*(1,t) = g_1(t),$$

$$if \;\; t = 0 \;\; then \;\; u_n^*(x,0) = u_n(x,0). \tag{3.5}$$

The second and third terms in right side of Eq. (3.4) well be vanish when we applying the second derivative by x which was appearing in a right side of Eq. (3.1), so to establish these terms we can be modified Eq. (3.4) and rewritten in a new formulation as

$u_n^*(x,t) = u_n(x,t) + (1-x^2)[g_0(t) - u_n(0,t)] + x^2[g_1(t) - u_n(1,t)], n \geq 0.$ (3.6)
Eq. (2.7) associated with Eqs. (3.1) and (3.6) can be rewritten in a new formulation to obtain the correct functional as

$$u_{n+1} = \begin{cases} u_n^* + \int_0^t \lambda(t,\tau)\left(\frac{\partial^\alpha u_n^*(x,\tau)}{\partial \tau^\alpha} - \frac{1}{2}x^2 \frac{\partial^2 \tilde{u}_n^*(x,\tau)}{\partial x^2}\right) d\tau, & \text{for } \alpha = m \in \mathbb{N}. \\ u_n^* + \frac{1}{\Gamma(\alpha+1)} \int_0^t \lambda(t,\tau)\left(\frac{\partial^\alpha u_n^*(x,\tau)}{\partial \tau^\alpha} - \frac{1}{2}x^2 \frac{\partial^2 \tilde{u}_n^*(x,\tau)}{\partial x^2}\right) (d\tau)^\alpha, & \text{for } m-1 < \alpha < m. \end{cases} \quad (3.7)$$

Such as treatment is a very effective as shown in this paper.

4. Applications and results

Example 1: Consider the following one-dimensional heat-like problem

$$\frac{\partial u}{\partial t} - \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, t > 0, \quad (4.1)$$

subject to the initial conditions

$u(x,0) = x^2, \qquad 0 < x < 1,$

and the boundary conditions

$u(0,t) = 0, \quad u(1,t) = \mathrm{e}^{t}, \quad t > 0,$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[\mathrm{e}^{t} - u_n(1,t)], \quad (4.2)$$

where $n = 0,1,2,\ldots$. The initial approximation is $u_0(x,t) = x^2$.
To begin with a new initial approximation u_0^* we applying Eq. (4.2) at $n = 0$ such as
$u_0^*(x,t) = x^2\, \mathrm{e}^{t}.$ (4.3)
the Lagrange multiplier in the first term of Eq. (3.7) can be determined as $\lambda = -1$, and by substituting Eq. (4.1) we have a correction functional as follows

$$u_{n+1}(x,t) = u_n^*(x,t) - \int_0^t \left(\frac{\partial u_n^*(x,\tau)}{\partial \tau} - \frac{1}{2}x^2 \frac{\partial^2 u_n^*(x,\tau)}{\partial x^2}\right) d\tau. \quad (4.4)$$

So as soon we have

$u_1(x,t) = x^2 e^t$. (4.5)
Which is the exact solution.

***Example 2*:** We next consider the one-dimensional wave-like equation

$$\frac{\partial^2 u}{\partial t^2} - \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2} = 0, \qquad 0 < x < 1, t > 0, \tag{4.6}$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad 0 < x < 1,$$

and the boundary conditions

$$u(0,t) = 0 \quad , \quad u(1,t) = 1 + \sinh t, \quad t > 0,$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1 - x^2)[0 - u_n(0,t)] + x^2[1 + \sinh t - u_n(1,t)], \tag{4.7}$$

where $n = 0,1,2, \ldots$. The initial approximation is $u_0(x,t) = x + x^2 t$.
To begin with a new initial approximation u_0^* we applying Eq. (4.7) at $n = 0$ such as

$$u_0^*(x,t) = x + x^2 \sinh t. \tag{4.8}$$

the Lagrange multiplier in the first term of Eq. (3.7) can be determined as $\lambda = \tau - t$, and by substituting Eq. (4.6) we have a correction functional as follows

$$u_{n+1}(x,t) = u_n^*(x,t) + \int_0^t (\tau - t)\left(\frac{\partial^2 u_n^*(x,\tau)}{\partial \tau^2} - \frac{1}{2}x^2 \frac{\partial^2 u_n^*(x,\tau)}{\partial x^2}\right) d\tau. \tag{4.9}$$

So as soon we have

$$u_1(x,t) = x + x^2 \sinh t. \tag{4.10}$$

Which is the exact solution.

***Example 3*:** Consider the following one-dimensional fractional heat-like problem

$$\frac{\partial^\alpha u}{\partial t^\alpha} - \frac{1}{2}x^2 \frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, 0 < \alpha < 1, t > 0, \tag{4.11}$$

subject to the initial conditions

$$u(x,0) = x^2, \qquad 0 < x < 1,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = \mathrm{E}_\alpha(\mathrm{t}^\alpha), \quad t > 0,$$

where

$$\mathrm{E}_{\alpha}(\mathrm{t}^{\alpha}) = \lim_{\mathrm{m}\to\infty} \sum_{\mathrm{k}=0}^{\mathrm{m}} \frac{\mathrm{t}^{\mathrm{k}\alpha}}{\Gamma(1+\mathrm{k}\alpha)}\,. \tag{4.12}$$

By applying a new approximations u_n^* in Eq. (3.6) we have

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2[\mathrm{E}_{\alpha}(\mathrm{t}^{\alpha}) - u_n(1,t)], \tag{4.13}$$

where $n = 0,1,2,\dots$. The initial approximation is $u_0(x,t) = x^2$.
To begin with a new initial approximation u_0^* we applying Eq. (4.13) at $n = 0$ such as

$$\begin{aligned} u_0^*(x,t) &= x^2 \mathrm{E}_{\alpha}(\mathrm{t}^{\alpha}) \\ &= x^2\left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots\right). \end{aligned} \tag{4.14}$$

Substituting Eq. (4.11) in the second term of Eq. (3.7) we have a correction functional as follows

$$\begin{aligned} u_{n+1}(x,t) &= u_n^*(x,t) \\ &\quad + \frac{1}{\Gamma(1+\alpha)} \int_0^t \lambda(t,\tau)\left(\frac{\partial^{\alpha} \tilde{u}_n^*(x,\tau)}{\partial \tau^{\alpha}} - \frac{1}{2}x^2 \frac{\partial^2 \tilde{u}_n^*(x,\tau)}{\partial x^2}\right)(d\tau)^{\alpha}. \end{aligned} \tag{4.15}$$

the Lagrange multiplier can be determined as $\lambda(t,\tau) = -1$, so we have

$$u_{n+1}(x,t) = u_n^*(x,t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^{\alpha} u_n^*(x,\tau)}{\partial \tau^{\alpha}} - \frac{1}{2}x^2 \frac{\partial^2 u_n^*(x,\tau)}{\partial x^2}\right)(d\tau)^{\alpha}. \tag{4.16}$$

We can derive

$$u_1(x,t) = u_0^*(x,t) - \frac{1}{\Gamma(1+\alpha)} \int_0^t \left(\frac{\partial^{\alpha} u_0^*(x,\tau)}{\partial \tau^{\alpha}} - \frac{1}{2}x^2 \frac{\partial^2 u_0^*(x,\tau)}{\partial x^2}\right)(d\tau)^{\alpha}. \tag{4.17}$$

the term $\frac{\partial^{\alpha} u_0^*(x,\tau)}{\partial \tau^{\alpha}}$ is calculating by Eq. (2.8) as following

$$\begin{aligned} \frac{\partial^{\alpha} u_0^*(x,t)}{\partial t^{\alpha}} &= \frac{x^2}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \frac{\partial}{\partial \tau}\left(1 + \frac{\tau^{\alpha}}{\Gamma(1+\alpha)} + \frac{\tau^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots\right) d\tau \\ &= \frac{x^2}{\Gamma(1-\alpha)} \int_0^t (t-\tau)^{-\alpha} \left(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)} + \frac{\tau^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots\right) d\tau \end{aligned}$$

$$= \frac{x^2}{\Gamma(1-\alpha)}\left\{\frac{\beta(1-\alpha,\alpha)}{\Gamma(\alpha)} + \frac{\beta(1-\alpha,2\alpha)t^{\alpha}}{\Gamma(2\alpha)} + \cdots\right\}$$

$$= x^2\left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \cdots\right). \quad (4.18)$$

Where β is Beta function. Eq.(4.17) becomes

$$u_1(x,t) = x^2\left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots\right)$$

$$-\frac{1}{\Gamma(1+\alpha)}\int_0^t \left[x^2\left(1 + \frac{\tau^{\alpha}}{\Gamma(1+\alpha)} + \cdots\right)\right.$$

$$\left. - x^2\left(1 + \frac{\tau^{\alpha}}{\Gamma(1+\alpha)} + \cdots\right)\right](d\tau)^{\alpha}$$

$$= x^2\left(1 + \frac{t^{\alpha}}{\Gamma(1+\alpha)} + \frac{t^{2\alpha}}{\Gamma(1+2\alpha)} + \cdots\right)$$

$$= \lim_{m\to\infty}\sum_{k=0}^{m}\frac{x^2 t^{k\alpha}}{\Gamma(1+k\alpha)} = x^2 E_{\alpha}(t^{\alpha}). \quad (4.19)$$

Which is the exact solution.

***Example 4*:** Consider the one-dimensional fractional wave-like equation

$$\frac{\partial^{\alpha}u}{\partial t^{\alpha}} - \frac{1}{2}x^2\frac{\partial^2 u}{\partial x^2} = 0, \quad 0 < x < 1, 1 < \alpha < 2, t > 0, \quad (4.20)$$

subject to the initial conditions

$$u(x,0) = x, \quad \frac{\partial u(x,0)}{\partial t} = x^2, \quad 0 < x < 1,$$

and the boundary conditions

$$u(0,t) = 0, \quad u(1,t) = 1 + t\, E_{\alpha,2}(t^{\alpha}), \quad t > 0,$$

where

$$E_{\alpha,2}(t^{\alpha}) = \lim_{m\to\infty}\sum_{k=0}^{m}\frac{t^{k\alpha}}{\Gamma(2+k\alpha)}. \quad (4.21)$$

By applying a new approximations u_n^* in Eq. (3.6) we obtain

$$u_n^*(x,t) = u_n(x,t) + (1-x^2)[0 - u_n(0,t)] + x^2\big[1 + t\,\mathrm{E}_{\alpha,2}(\mathrm{t}^{\alpha}) - u_n(1,t)\big], \qquad (4.22)$$

where $n = 0,1,2,\ldots$. The initial approximation is $u_0(x,t) = x^2$.
To begin with a new initial approximation u_0^* we applying Eq. (4.22) at $n = 0$ such as

$$u_0^*(x,t) = x + x^2 t\,\mathrm{E}_{\alpha,2}(\mathrm{t}^{\alpha})$$
$$= x + x^2 t\left(1 + \frac{t^{\alpha}}{\Gamma(2+\alpha)} + \frac{t^{2\alpha}}{\Gamma(2+2\alpha)} + \cdots\right). \qquad (4.23)$$

Substituting Eq. (4.20) in the second term of Eq. (3.7) we have a correction functional as follows

$$u_{n+1}(x,t) = u_n^*(x,t)$$
$$+ \frac{1}{\Gamma(1+\alpha)}\int_0^t \lambda(t,\tau)\left(\frac{\partial^{\alpha}\tilde{u}_n^*(x,\tau)}{\partial\tau^{\alpha}} - \frac{1}{2}x^2\frac{\partial^2\tilde{u}_n^*(x,\tau)}{\partial x^2}\right)(d\tau)^{\alpha}. \qquad (4.24)$$

the Lagrange multiplier can be determined as $\lambda(t,\tau) = (\tau - t)$, so we have

$$u_{n+1}(x,t) = u_n^*(x,t)$$
$$- \frac{1}{\Gamma(1+\alpha)}\int_0^t (\tau - t)\left(\frac{\partial^{\alpha}u_n^*(x,\tau)}{\partial\tau^{\alpha}} - \frac{1}{2}x^2\frac{\partial^2 u_n^*(x,\tau)}{\partial x^2}\right)(d\tau)^{\alpha}. \qquad (4.25)$$

We can derive

$$u_1(x,t) = u_0^*(x,t)$$
$$- \frac{1}{\Gamma(1+\alpha)}\int_0^t (\tau - t)\left(\frac{\partial^{\alpha}u_0^*(x,\tau)}{\partial\tau^{\alpha}} - \frac{1}{2}x^2\frac{\partial^2 u_0^*(x,\tau)}{\partial x^2}\right)(d\tau)^{\alpha}. \qquad (4.26)$$

the term $\frac{\partial^{\alpha}u_0^*(x,\tau)}{\partial\tau^{\alpha}}$ is calculating by Eq. (2.8) as following

$$\frac{\partial^{\alpha}u_0^*(x,t)}{\partial t^{\alpha}} = \frac{1}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\frac{\partial^2}{\partial\tau^2}\left(x + x^2\left(\tau + \frac{\tau^{\alpha+1}}{\Gamma(2+\alpha)} + \frac{\tau^{2\alpha+1}}{\Gamma(2+2\alpha)} + \cdots\right)\right)d\tau$$
$$= \frac{x^2}{\Gamma(2-\alpha)}\int_0^t (t-\tau)^{1-\alpha}\left(\frac{\tau^{\alpha-1}}{\Gamma(\alpha)} + \frac{\tau^{2\alpha-1}}{\Gamma(2\alpha)} + \cdots\right)d\tau$$
$$= \frac{x^2}{\Gamma(2-\alpha)}\left(\frac{\beta(2-\alpha,\alpha)t}{\Gamma(\alpha)} + \frac{\beta(2-\alpha,2\alpha)t^{1+\alpha}}{\Gamma(2\alpha)} + \cdots\right)$$
$$= x^2\left(t + \frac{t^{1+\alpha}}{\Gamma(2+\alpha)} + \cdots\right). \qquad (4.27)$$

Where β is Beta function. Eq.(4.26) becomes

$$u_1(x,t) = x + x^2 t\left(1 + \frac{t^{\alpha}}{\Gamma(2+\alpha)} + \frac{t^{2\alpha}}{\Gamma(2+2\alpha)} + \cdots\right)$$
$$- \frac{x^2}{\Gamma(1+\alpha)} \int_0^t (\tau - t)\left[\left(\tau + \frac{\tau^{1+\alpha}}{\Gamma(2+\alpha)} + \cdots\right) - \left(\tau + \frac{\tau^{\alpha+1}}{\Gamma(2+\alpha)} + \cdots\right)\right](d\tau)^{\alpha}$$
$$= x + x^2 t\left(1 + \frac{t^{\alpha}}{\Gamma(2+\alpha)} + \frac{t^{2\alpha}}{\Gamma(2+2\alpha)} + \cdots\right)$$
$$= x + x^2 t\, \mathrm{E}_{\alpha,2}(\mathrm{t}^{\alpha}). \qquad (4.28)$$

Which is the exact solution.

5. Conclusions

Very effective to construct a new initial successive solutions u_n^* by mixed initial and boundary conditions together which explained in formula (3.6) and used it to find successive approximations u_n of the solution u in a new correct functional which explained in Eq.(3.7) by applying variational iteration method to solve initial boundary value problems for one dimensional heat-like and wave-like partial differential equations (ordinary or fractional). This technique to construct of a new successive initial solutions can give a more accurate solution. Some examples are given in this paper to illustrate the effectiveness and convenience of a new technique. It is important and obvious to show that the exact solutions have found directly from a first iteration of these examples by applying a new technique which is determined in this paper, but if used initial conditions only [Momani S.(2005), Faraz N. et.al (2010)] or applied formula of Eq. (3.4) [Ali E. J. (2011)] we will have exact solution by calculating infinite successive solutions u_n which closed form by Eq. (2.3). In the other hand we note that the first a new initial approximation u_0^* which calculating in last examples are appearing the same exact solution.

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