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A Note on Infinite Divisibility of Zeta Distributions

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Abstract

The Riemann zeta distribution, defined as the one whose characteristic function is the normalised Riemann zeta function, is an interesting example of an infinitely divisible distribution. The infinite divisibility of the distribution has been proved with recourse to the Euler product of the Riemann zeta function. In this paper, we look at multiple zeta-star function, which is a multi-dimensional generalisation of the Riemann zeta function and is believed to have no Euler product, and show that the corresponding distribution is not infinitely divisible.

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1 Introduction

Khinchin proved that the normalised Riemann zeta function is the characteristic function of an infinitely divisible distribution on the real line. This result now has several proofs, all of which appeal to the Euler product of the Riemann zeta function. Hu, Iksanov, Lin, and Zakusylo [1] studied the Hurwitz zeta distribution and showed that it is not infinitely divisible, except cases where the Euler product proves infinite divisibility.

In this article, we look at the multiple zeta-star distribution and prove that it is not infinitely divisible, bearing in mind that the multiple zeta-star function is believed to have no Euler product.

2 Infinite Divisibility of the Riemann Zeta Distribution

We first review the Riemann zeta distribution and its infinite divisibility. Let $\mathbb{N}$ denote the set of all positive integers.

Definition 2.1. *Let $s > 1$. The* Riemann zeta distribution *with parameter s is the discrete probability distribution μ_s with support $\{-\log m \mid m \in \mathbb{N}\}$ given by*

$$\mu_s(\{-\log m\}) = \frac{1}{\zeta(s)m^s}.$$

Remark 2.2. *The characteristic function of the Riemann zeta distribution μ_s is*

$$\varphi_{\mu_s}(t) = \sum_{m=1}^{\infty} e^{it(-\log m)} \frac{1}{\zeta(s)m^s} = \frac{\zeta(s+it)}{\zeta(s)},$$

hence the name of the distribution.

Definition 2.3. *A probability distribution μ on $\mathbb{R}^d$ is said to be* infinitely divisible *if for every $n \in \mathbb{N}$ there exist independent and identically distributed $\mathbb{R}^d$-valued random variables $X_1, \ldots, X_n$ whose sum $X_1 + \cdots + X_n$ has the distribution μ.*

Remark 2.4. *Infinitely divisible distributions are in one-to-one correspondence with Lévy processes; see [2] for details.*

We prove the infinite divisibility of the Riemann zeta distribution by explicitly giving the distribution of the factors:

Theorem 2.5. *Let $s > 1$ and $n \in \mathbb{N}$. Suppose that $X_1, \ldots, X_n$ are independent and identically distributed discrete random variables with support $\{-\log m \mid m \in \mathbb{N}\}$ such that*

$$P(X_1 = -\log m) = \zeta(s)^{-1/n} m^{-s} \prod_{l \in \mathbb{P}} \binom{1/n + \mathrm{ord}_l(m) - 1}{\mathrm{ord}_l(m)}$$

for all $m \in \mathbb{N}$, where $\mathbb{P}$ is the set of all prime numbers and $\mathrm{ord}_l(m)$ is the largest nonnegative integer k for which l^k divides m. Then $X_1 + \cdots + X_n$ has the Riemann zeta distribution μ_s. In particular, the Riemann zeta distribution is infinitely divisible.

Remark 2.6. *For each $m \in \mathbb{N}$, since $\mathrm{ord}_l(m) = 0$ for all but finitely many $l \in \mathbb{P}$ the infinite product is essentially a finite product.*

First proof of Theorem 2.5. Setting $p_m = P(X_1 = -\log m)$ for $m \in \mathbb{N}$, we need to show that

$$\sum_{\substack{-\log m_1 - \cdots - \log m_n = -\log m \\ m_1, \ldots, m_n \in \mathbb{N}}} p_{m_1} \cdots p_{m_n} = \frac{1}{\zeta(s) m^s} \tag{1}$$

for all $m \in \mathbb{N}$; note that this also assures us that $\sum_{m=1}^{\infty} p_m = 1$. If we put $c_m = \zeta(s)^{1/n} m^s$, then (1) is equivalent to

$$\sum_{\substack{m_1 \cdots m_n = m \\ m_1, \ldots, m_n \in \mathbb{N}}} c_{m_1} \cdots c_{m_n} = 1.$$

Define a formal power series by

$$f = \sum_{m=1}^{\infty} c_m \prod_{l \in \mathbb{P}} x_l^{\mathrm{ord}_l(m)} \in \mathbb{R}[[x_l \mid l \in \mathbb{P}]].$$

Since

$$\begin{aligned} f^n &= \sum_{m_1, \ldots, m_n = 1}^{\infty} c_{m_1} \cdots c_{m_n} \prod_{l \in \mathbb{P}} x_l^{\mathrm{ord}_l(m_1) + \cdots + \mathrm{ord}_l(m_n)} \\ &= \sum_{m=1}^{\infty} \left(\sum_{\substack{m_1 \cdots m_n = m \\ m_1, \ldots, m_n \in \mathbb{N}}} c_{m_1} \cdots c_{m_n} \right) \prod_{l \in \mathbb{P}} x_l^{\mathrm{ord}_l(m)}, \end{aligned}$$

we need to show that $f^n = \sum_{m=1}^{\infty} \prod_{l \in \mathbb{P}} x_l^{\mathrm{ord}_l(m)}$. Since

$$c_m = \prod_{l \in \mathbb{P}} \binom{1/n + \mathrm{ord}_l(m) - 1}{\mathrm{ord}_l(m)} = \prod_{l \in \mathbb{P}} \binom{-1/n}{\mathrm{ord}_l(m)} (-1)^{\mathrm{ord}_l(m)},$$

we have

$$f = \sum_{m=1}^{\infty} \prod_{l \in \mathbb{P}} \binom{-1/n}{\mathrm{ord}_l(m)} (-x_l)^{\mathrm{ord}_l(m)} = \prod_{l \in \mathbb{P}} \sum_{k=0}^{\infty} \binom{-1/n}{k} (-x_l)^k = \prod_{l \in \mathbb{P}} (1 - x_l)^{-1/n}$$

and so

$$f^n = \prod_{l \in \mathbb{P}} (1 - x_l)^{-1} = \prod_{l \in \mathbb{P}} \sum_{k=0}^{\infty} x_l^k = \sum_{m=1}^{\infty} \prod_{l \in \mathbb{P}} x_l^{\mathrm{ord}_l(m)}.$$

□

Second proof of Theorem 2.5. Suppose that $\{Y_{l,j} \mid l \in \mathbb{P}, j = 1, \ldots, n\}$ is a family of independent random variables such that each $Y_{l,j}$ has the negative binomial distribution with parameters $1/n$ and l^{-s}:

$$P(Y_{l,j} = k) = \binom{1/n + k - 1}{k} (1 - l^{-s})^{1/n} (l^{-s})^k$$

for nonnegative integers k. Since

$$\sum_{l \in \mathbb{P}} \sum_{j=1}^{n} P(Y_{l,j} > 0) = \sum_{l \in \mathbb{P}} \sum_{j=1}^{n} \left(1 - (1 - l^{-s})^{1/n}\right) \le \sum_{l \in \mathbb{P}} \sum_{j=1}^{n} l^{-s} \le n \sum_{k=1}^{\infty} k^{-s} = n\zeta(s) < \infty,$$

the Borel-Cantelli theorem shows that $Y_{l,j} = 0$ for all but finitely many (l, j) almost surely.

Set $X'_j = \sum_{l \in \mathbb{P}} (-\log l) Y_{l,j}$ for $j = 1, \ldots, n$ and $X' = \sum_{l=1}^{n} X'_j$. We prove the theorem by showing that X_j and X'_j have the same distribution for $j = 1, \ldots, n$ and that X' has the Riemann zeta distribution.

The first claim follows from the observation that for every $m \in \mathbb{N}$, we have

$$\begin{aligned} P(X'_j = -\log m) &= \prod_{l \in \mathbb{P}} P\left(Y_{l,j} = \mathrm{ord}_l(m)\right) \\ &= \prod_{l \in \mathbb{P}} \binom{1/n + \mathrm{ord}_l(m) - 1}{\mathrm{ord}_l(m)} (1 - l^{-s})^{1/n} (l^{-s})^{\mathrm{ord}_l(m)} \\ &= \zeta(s)^{-1/n} m^{-s} \prod_{l \in \mathbb{P}} \binom{1/n + \mathrm{ord}_l(m) - 1}{\mathrm{ord}_l(m)}. \end{aligned}$$

For the second claim, we compute the characteristic function of X':

$$\begin{aligned} \varphi_{X'}(t) &= \prod_{l \in \mathbb{P}} \prod_{j=1}^{n} \varphi_{(-\log l) Y_{l,j}}(t) = \prod_{l \in \mathbb{P}} \prod_{j=1}^{n} \varphi_{Y_{l,j}}(-t \log l) \\ &= \prod_{l \in \mathbb{P}} \prod_{j=1}^{n} \left(\frac{1 - l^{-s}}{1 - l^{-s} e^{-it \log l}} \right)^{1/n} = \prod_{l \in \mathbb{P}} \frac{1 - l^{-s}}{1 - l^{-s-it}} = \frac{\zeta(s + it)}{\zeta(s)} = \varphi_{\mu_s}(t). \end{aligned}$$

□

Remark 2.7. *We obviously used the Euler product of the Riemann zeta function in the second proof. In the first proof, the equalities*

$$\sum_{m=1}^{\infty} \prod_{l \in \mathbb{P}} \binom{-1/n}{\mathrm{ord}_l(m)} (-x_l)^{\mathrm{ord}_l(m)} = \prod_{l \in \mathbb{P}} \sum_{k=0}^{\infty} \binom{-1/n}{k} (-x_l)^k,$$

$$\prod_{l \in \mathbb{P}} \sum_{k=0}^{\infty} x_l^k = \sum_{m=1}^{\infty} \prod_{l \in \mathbb{P}} x_l^{\mathrm{ord}_l(m)}$$

are essentially the Euler product.

3 Multiple Zeta Distributions

In this section, we study a multi-dimensional generalisation of the Riemann zeta distribution. Throughout this section, we let d be an integer with $d \geq 2$.

Definition 3.1. *Set* $I_d = \{(m_1, \dots, m_d) \in \mathbb{N}^d \mid m_1 \geq \cdots \geq m_d\}$. *The* multiple zeta-star function *is defined by*

$$\zeta^\star(s_1, \dots, s_d) = \sum_{(m_1,\dots,m_d)\in I_d} \frac{1}{m_1^{d_1} \cdots m_d^{s_d}}$$

for complex numbers $s_1, \dots, s_d$ *with* $\operatorname{Re} s_1 > 1$ *and* $\operatorname{Re} s_2, \dots, \operatorname{Re} s_d \geq 1$.

Definition 3.2. *Write* $-\log \mathbf{m} = (-\log m_1, \dots, -\log m_d) \in \mathbb{R}^d$ *for each* $\mathbf{m} = (m_1, \dots, m_d) \in I_d$, *and set* $S_d = \{-\log \mathbf{m} \mid \mathbf{m} \in I_d\}$. *Let* $s_1 > 1$ *and* $s_2, \dots, s_d \geq 1$ *be real numbers. The* multiple zeta-star distribution *with parameters* $s_1, \dots, s_d$ *is defined as the discrete probability distribution* $\mu_{s_1,\dots,s_d}$ *with support* S_d *given by*

$$\mu_{s_1,\dots,s_d}(\{-\log \mathbf{m}\}) = \frac{1}{\zeta^\star(s_1, \dots, s_d) m_1^{s_1} \cdots m_d^{s_d}}$$

for $\mathbf{m} \in I_d$.

Remark 3.3. *The characteristic function of* $\mu_{s_1,\dots,s_d}$ *is*

$$\begin{aligned}
\varphi_{\mu_{s_1,\dots,s_d}}(t_1, \dots, t_d) &= \sum_{\mathbf{m}=(m_1,\dots,m_d)\in I_d} e^{i(t_1,\dots,t_d)\cdot(-\log \mathbf{m})} \mu_{s_1,\dots,s_d}(\{-\log \mathbf{m}\}) \\
&= \sum_{\mathbf{m}=(m_1,\dots,m_d)\in I_d} \frac{1}{\zeta^\star(s_1, \dots, s_d) m_1^{s_1+it_1} \cdots m_d^{s_d+it_d}} \\
&= \frac{\zeta^\star(s_1 + it_1, \dots, s_d + it_d)}{\zeta^\star(s_1, \dots, s_d)}.
\end{aligned}$$

Remark 3.4. *The multiple zeta-star function is less commonly investigated than the multiple zeta function defined by*

$$\zeta(s_1, \dots, s_d) = \sum_{\substack{m_1 > \cdots > m_d \\ m_1,\dots,m_d\in\mathbb{N}}} \frac{1}{m_1^{s_1} \cdots m_d^{s_d}},$$

in which $m_1, \dots, m_d$ *must all be distinct. We favour the multiple zeta-star function because it puts* 0 *in the support of the corresponding distribution; the proof of the next theorem will explain why* 0 *is important.*

Theorem 3.5. *Let $s_1 > 1$ and $s_2, \dots, s_d \geq 1$ be real numbers. Then the multiple zeta-star distribution $\mu_{s_1,\dots,s_d}$ is not infinitely divisible. More precisely, whenever $n \geq 2$, there do not exist independent and identically distributed $\mathbb{R}^d$-valued random variables $X_1, \dots, X_n$ whose sum $X_1 + \cdots + X_n$ has the distribution $\mu_{s_1,\dots,s_d}$.*

Proof. Fix $s_1, \dots, s_d$ and n, and suppose that such $X_1, \dots, X_n$ exist.

We first show that $\mathrm{supp}(X_1) \subset S_d$. Write $S'_d = \mathrm{supp}(X_1)$ for simplicity. By [2, Lemma 24.1], we have

$$S_d = \mathrm{supp}(X_1 + \cdots + X_n) = \overline{\{x_1 + \cdots + x_n \mid x_1, \dots, x_n \in S'_d\}}. \tag{2}$$

Since (2) implies that $S'_d \subset \{x/n \mid x \in S_d\}$ and since every element of $\{x/n \mid x \in S\} \backslash \{0\}$ has a negative first coordinate, we have

$$P(X_1 = 0)^n = P(X_1 + \cdots + X_n = 0) = \mu_{s_1,\dots,s_d}(\{0\}) > 0,$$

from which it follows that $0 \in S'_d$. Therefore (2) and $n \geq 2$ give

$$S_d \supset \{x + 0 + \cdots + 0 \mid x \in S'_d\} = S'_d.$$

Now we put $p_{\mathbf{m}} = P(X_1 = -\log \mathbf{m})$ for $\mathbf{m} \in I_d$, so that $\sum_{\mathbf{m} \in I_d} p_{\mathbf{m}} = 1$. Then the required condition is that

$$\sum_{\substack{\mathbf{m}_1 \cdots \mathbf{m}_n = \mathbf{m} \\ \mathbf{m}_1, \dots, \mathbf{m}_n \in I_d}} p_{\mathbf{m}_1} \cdots p_{\mathbf{m}_n} = \frac{1}{\zeta^\star(s_1, \dots, s_d) m_1^{s_1} \cdots m_d^{s_d}} \tag{3}$$

for all $\mathbf{m} = (m_1, \dots, m_d) \in I_d$, where the product of elements of I_d is defined coordinatewise. If we set $c_{\mathbf{m}} = \zeta^\star(s_1, \dots, s_d)^{1/n} m_1^{s_1} \cdots m_d^{s_d} p_{\mathbf{m}}$ for $\mathbf{m} = (m_1, \dots, m_d)$, then (3) is equivalent to

$$\sum_{\substack{\mathbf{m}_1 \cdots \mathbf{m}_n = \mathbf{m} \\ \mathbf{m}_1, \dots, \mathbf{m}_n \in I_d}} c_{\mathbf{m}_1} \cdots c_{\mathbf{m}_n} = 1. \tag{4}$$

We use (4) to find $c_{\mathbf{m}}$ recursively:

- Use (4) for $\mathbf{m} = (1, \dots, 1)$ to get $c_{(1,\dots,1)}^n = 1$. Therefore $c_{(1,\dots,1)} = 1$.
- Use (4) for $\mathbf{m} = (2, 1, \dots, 1), (3, 1, \dots, 1), (3, 3, 1, \dots, 1), (4, 3, 1, \dots, 1)$ to get $n c_{\mathbf{m}} c_{(1,\dots,1)}^{n-1} = 1$. Therefore $c_{\mathbf{m}} = 1/n$ for these $\mathbf{m}$.
- Use (4) for $\mathbf{m} = (4, 1, \dots, 1)$ to get

  $$n c_{(4,1,\dots,1)} c_{(1,\dots,1)}^{n-1} + \frac{n(n-1)}{2} c_{(2,1,\dots,1)}^2 c_{(1,\dots,1)}^{n-2} = 1.$$

  Therefore $c_{(4,1,\dots,1)} = (n+1)/2n^2$.

- Use (4) for $\mathbf{m} = (6, 3, 1, \dots, 1)$ to get

$$nc_{(6,3,1,\dots,1)}c_{(1,\dots,1)}^{n-1} + n(n-1)c_{(3,3,1,\dots,1)}c_{(2,1,\dots,1)}c_{(1,\dots,1)}^{n-2} = 1.$$

  Therefore $c_{(6,3,1,\dots,1)} = 1/n^2$.

- Use (4) for $\mathbf{m} = (12, 3, 1, \dots, 1)$ to get

$$\begin{aligned} &nc_{(12,3,1,\dots,1)}c_{(1,\dots,1)}^{n-1} + n(n-1)c_{(6,3,1,\dots,1)}c_{(2,1,\dots,1)}c_{(1,\dots,1)}^{n-2} \\ &\quad + n(n-1)c_{(4,3,1,\dots,1)}c_{(3,1,\dots,1)}c_{(1,\dots,1)}^{n-2} + n(n-1)c_{(3,3,1,\dots,1)}c_{(4,1,\dots,1)}c_{(1,\dots,1)}^{n-2} \\ &\qquad + \frac{n(n-1)(n-2)}{2}c_{(3,3,1,\dots,1)}c_{(2,1,\dots,1)}^2 c_{(1,\dots,1)}^{n-3} = 1. \end{aligned}$$

  Therefore $c_{(12,3,1,\dots,1)} = (-2n^2 + 3n + 1)/2n^3$.

Since $(-2n^2+3n+1)/2n^3 < 0$ for $n \geq 2$, we have arrived at a contradiction. □

Remark 3.6. *There are many other* $\mathbf{m} \in I_d$ *for which* $c_{\mathbf{m}} < 0$.

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