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Numerical Approach to Solve Second Kind Nonlinear Integral Equations Using Lagrange Functions

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Abstract

In this paper, an efficient method based on Lagrange interpolation is used for solving nonlinear Fredholm integral equations of the power function type. For this purpose simple quadrature rules such as Trapezoidal and Simpson rules are used. However, numerical results with good accuracy is obtained. The method is applied to some numerical examples.

Keywords: Second kind Fredholm integral equations; Lagrange functions; Simpson rule; Trapezoidal rule

1 Introduction

In this paper we present a numerical approach for solving nonlinear Fredholm integral equations. Several numerical methods for approximating the solution of linear and nonlinear integral equations and specially Hammerstein integral equations are known [1-11]. The aim of this work is to present a numerical method for approximating the solution of nonlinear Fredholm integeral equation of the second kind:

$$f(x) = g(x) + \int_0^1 k(x,t)[f(t)]^m dt, \quad m > 1, \tag{1}$$

where, $g \in L^2[0,1)$, $k \in L^2[0,1)^2$ and f is the unknown function to be determined and m is a positive integer.

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2 Function Approximation

Lagrange interpolation of the function $f(x)$ is given by

$$f(x) \simeq \sum_{i=0}^{n} f_i L_i(x), \tag{2}$$

with $f_i = f(x_i)$ and

$$L_i(x) = \prod_{j=0, j\neq i}^{n} (\frac{x - x_j}{x_i - x_j}),$$

also, $L_i(x_j) = \delta_{ij}$ where δ_{ij} is the Kronecker delta.
We can write (2) in the matrix form

$$f(x) \simeq \mathbf{f}^t \mathbf{L}(x), \tag{3}$$

where, $\mathbf{f} = [f_0, f_1, \ldots, f_n]^t$ and $\mathbf{L}(x) = [L_0(x), L_1(x), \ldots, L_n(x)]^t$. Similarly, $k(x,t) \in L^2[0,1]^2$ may be approximated in the matrix form as:

$$k(x,t) \simeq \mathbf{L}^t(x)\mathbf{K}\mathbf{L}(t), \tag{4}$$

where $\mathbf{K} = [k_{ij}]_{0\le i,j\le n}$ and $k_{ij} = k(x_i, t_j)$. For a positive integer m, $[f(t)]^m$ may be approximated as:

$$[f(t)]^m \simeq \sum_{i=0}^{n} \tilde{f}_i L_i(t) = \tilde{\mathbf{f}}^t \mathbf{L}(t)$$

where $\tilde{\mathbf{f}}$ is a column vector whose elements are nonlinear combinations of the elements of the vector $\mathbf{f}$. Now, we consider evaluation of $\tilde{\mathbf{f}}$ in terms of $\mathbf{f}$. Using the subject already discussed in this section,

$$f(t) = \mathbf{f}^t \mathbf{L}(t) \text{ and } [f(t)]^m = \tilde{\mathbf{f}}^t \mathbf{L}(t),$$

so,

$$\tilde{\mathbf{f}}^t \mathbf{L}(t) = [\mathbf{f}^t \mathbf{L}(t)]^m, \tag{5}$$

if we choose $t = x_j, j = 0, 1, ..., n$ and using the fact that $\mathbf{L}(x_j) = \mathbf{e}_j$, where $\mathbf{e}_j$ is the j-th column of the identity matrix, equation (5) gives

$$\tilde{\mathbf{f}}_j = \mathbf{f}_j^m,$$

so, $\tilde{\mathbf{f}} = [\tilde{f}_0, \tilde{f}_1, ..., \tilde{f}_n] = [f_0^m, f_1^m, ..., f_n^m]$.

3 Nonlinear Fredholm integral equations of power function type

Now consider the nonlinear Fredholm integral equation of the second kind with nonlinear regular part:

$$f(x) = g(x) + \int_0^1 k(x,t)[f(t)]^m dt, \quad m > 1, \tag{6}$$

by approximating functions $f(t)$, $k(x,t)$ and $[f(t)]^m$, as before, in the matrix form we have:

$$f(t) \simeq \mathbf{f}^t \mathbf{L}(t) \tag{7}$$

$$k(x,t) \simeq \mathbf{L}^t(x)\mathbf{K}\mathbf{L}(t) \tag{8}$$

$$[f(t)]^m \simeq \tilde{\mathbf{f}}^t \mathbf{L}(t) \tag{9}$$

by substituting the approximations (7)-(9) into (6) we obtain:

$$\begin{aligned}\mathbf{L}^t(x)\mathbf{f} &= g(x) + \int_0^1 \mathbf{L}^t(x)\mathbf{K}\mathbf{L}(t)\mathbf{L}^t(t)\tilde{\mathbf{f}}dt \\ &= g(x) + \mathbf{L}^t(x)\mathbf{K}\int_0^1 \mathbf{L}(t)\mathbf{L}^t(t)dt\tilde{\mathbf{f}} \\ &= g(x) + \mathbf{L}^t(x)\mathbf{K}\mathbf{D}\tilde{\mathbf{f}} \end{aligned} \tag{10}$$

where $\mathbf{D} = \int_0^1 \mathbf{L}(t)\mathbf{L}^t(t)dt$, so, $\mathbf{D}$ is a $(n+1)\times(n+1)$ matrix with elements $\mathbf{D}_{ij} = \int_0^1 L_i(t)L_j(t)dt, \quad i,j = 0,1,...,n$. Also, we approximate the integral of f on $[a,b]$ as:

$$\int_a^b f(x)dx \approx \sum_{r=0}^{k} w_r f(x_r). \tag{11}$$

In this work we consider the Trapezoidal and Simpson quadrature rules with $k = n$. By approximating the integrals $\mathbf{D}_{ij}$ with quadrature rule (11) we get

$$\begin{aligned}\mathbf{D}_{ij} &\approx \sum_{r=0}^{n} w_r L_i(x_r)L_j(x_r) \\ &= \sum_{r=0}^{n} w_r \delta_{ir}\delta_{jr} \\ &= \begin{cases} 0, & i \neq r \text{ } or \text{ } i \neq r \\ w_p, & i = j = r = p, \ p = 0,1,\ldots,n \end{cases}\end{aligned}$$

therefore, **D** is a diagonal matrix:

$$\mathbf{D} = Diag(w_0, w_1, ..., w_n). \tag{12}$$

By collocating (10) at the points $x = x_j, j = 0, 1, ..., n$ simply we get

$$f_j = g(x_j) + \mathbf{e}_j^t\mathbf{KD}\tilde{\mathbf{f}}, \tag{13}$$

system (13) gives $n + 1$ nonlinear equations for $f_j, j = 0, 1, ..., n$ which can be easily solved by Newton iterative method. Therefore, desired approximation for $f(t)$ can be obtained by (3) for every $t \in [0, 1]$.

4 Numerical Examples

Now for implementing the presented method for solving nonlinear Fredholm integral equation, we choose some numerical examples, for which the exact solution is known for comparison with the approximate solution.

Example 1:

$$f(x) + \int_0^1 e^{x-2t}[f(t)]^3 dt = e^{x+1}, \quad 0 \leq x < 1,$$

with exact solution $f(x) = e^x$.

Example 2:

$$f(x) - \int_0^1 xt[f(t)]^3 dt = e^x - \frac{(1 + 2e^3)x}{9}, \quad 0 \leq x < 1,$$

with exact solution $f(x) = e^x$.
Table 1 shows the computed error $|e| = |u_{exact}(x) - u_n(x)|$ for examples 1-2 with Trapezoidal quadrature rule and n=6.

Table 1

t	Example 1	Example 2
0.0	6×10^{-4}	0.0
0.1	7×10^{-4}	4×10^{-3}
0.2	7×10^{-4}	9×10^{-3}
0.3	8×10^{-4}	1×10^{-2}
0.4	9×10^{-4}	1×10^{-2}
0.5	1×10^{-3}	2×10^{-2}
0.6	1×10^{-3}	2×10^{-2}
0.7	1×10^{-3}	3×10^{-2}
0.8	1×10^{-3}	3×10^{-2}
0.9	1×10^{-3}	4×10^{-2}
1.0	1×10^{-3}	4×10^{-2}

Table 2 shows the computed error $|e| = |u_{exact}(x) - u_n(x)|$ for examples 1-2 with Simpson quadrature rule and n=6.

Table 2

t	Example 1	Example 2
0.0	1×10^{-6}	0.0
0.1	1×10^{-6}	1×10^{-4}
0.2	1×10^{-6}	2×10^{-4}
0.3	1×10^{-6}	3×10^{-4}
0.4	1×10^{-6}	4×10^{-4}
0.5	1×10^{-6}	5×10^{-4}
0.6	2×10^{-6}	6×10^{-4}
0.7	2×10^{-6}	8×10^{-4}
0.8	2×10^{-6}	9×10^{-4}
0.9	2×10^{-6}	1×10^{-3}
1.0	3×10^{-6}	1×10^{-3}

Table 3 shows the computed error $|e| = |u_{exact}(x) - u_n(x)|$ for examples 1-2 with Trapezoidal quadrature rule and n=10.

Table 3

t	Example 1	Example 2
0.0	2×10^{-4}	0.0
0.1	2×10^{-4}	1×10^{-3}
0.2	2×10^{-4}	3×10^{-3}
0.3	3×10^{-4}	5×10^{-3}
0.4	3×10^{-4}	6×10^{-3}
0.5	3×10^{-4}	8×10^{-3}
0.6	4×10^{-4}	1×10^{-2}
0.7	4×10^{-4}	1×10^{-2}
0.8	5×10^{-4}	1×10^{-2}
0.9	5×10^{-4}	1×10^{-2}
1.0	6×10^{-4}	1×10^{-2}

Table 4 shows the computed error $|e| = |u_{exact}(x) - u_n(x)|$ for examples 1-2 with Simpson quadrature rule and n=10.

Table 4

t	Example 1	Example 2
0.0	1×10^{-7}	0.0
0.1	1×10^{-7}	1×10^{-5}
0.2	1×10^{-7}	3×10^{-5}
0.3	2×10^{-7}	4×10^{-5}
0.4	2×10^{-7}	6×10^{-5}
0.5	2×10^{-7}	7×10^{-5}
0.6	2×10^{-7}	9×10^{-5}
0.7	3×10^{-7}	1×10^{-4}
0.8	3×10^{-7}	1×10^{-4}
0.9	3×10^{-7}	1×10^{-4}
1.0	4×10^{-7}	1×10^{-4}

5 Conclusion

In this paper, Lagrange functions and simple quadrature rules were used to solve nonlinear integral equations of the power function type. The presented approach leads to solve nonlinear system of equations which may easily be solved by Newton iterative method. The advantages of presented method make it very simple and cheap as computational point of view. Also, the accuracy of the method may be increased by increasing n (the number of Lagrange functions or quadrature nodes), furthermore we can increase accuracy of the method by using more precise quadrature rules.

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