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Positive Solutions of a Nonlinear m-Point Integral Boundary Value Problem

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Abstract

We study sufficient conditions for the existence of positive solutions to the m-point integral boundary value problem

$$u'' + a(t)f(u) = 0, \qquad t \in (0,1),$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds,$$

where $0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \eta_{m-1} = 1$, $0 < \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) < 2$, $\alpha_i \geq 0$ for $i \in \{1,\ldots,m-3\} \cup \{m-1\}$ and $\alpha_{m-2} > 0$. $a \in C([0,1],[0,\infty))$ and $f \in C([0,\infty),[0,\infty))$. We show the existence of at least one positive solution if f is either superlinear or sublinear by applying the fixed point theorem in cones.

Mathematics Subject Classification: 34B10; 34B15

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1 Introduction

The study of multi-point boundary-value problems for linear second order ordinary differential equations was initiated by Il'in and Moiseev [1-2]. Gupta [3] studied three-point boundary-value problems for nonlinear second order ordinary differential equations. Since then, the existence of solutions for nonlinear three-point, m-point, integral and nonlocal boundary value problems has been studied by several authors by using the Leray-Schauder continuation theorem, nonlinear alternatives of Leray-Schauder, coincidence degree theory and the fixed-point theorem in cones. We refer the reader to [4-21, 23-29] and the references therein.

However, most of those papers have studied the existence and multiplicity of solutions (or positive solutions) to boundary value problems with boundary conditions specified as solution values or the slope of solution values at given points or as more general nonlocal integral boundary conditions. There are also a few papers that consider nonlinear second order ordinary differential equations with boundary conditions specified as an area under the curve of solutions; see for example, [22].

In this paper, we consider the existence of at least one positive solution to the equation

$$u'' + a(t)f(u) = 0, \qquad t \in (0,1), \tag{1}$$

with the following boundary conditions that include an m-point integral condition

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds, \tag{2}$$

where $0 = \eta_0 < \eta_1 < \eta_2 < \cdots < \eta_{m-2} < \eta_{m-1} = 1$, $0 < \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) < 2$, $\alpha_i \geq 0$ for $i \in \{1, \ldots, m-3\} \cup \{m-1\}$ and $\alpha_{m-2} > 0$. We note that the m-point boundary condition (2) are related to $m-1$ intervals of the area under the curve of solution $u(t)$ from $t = \eta_{i-1}$ to $t = \eta_i$ for $i = 1, \ldots, m-1$.

By the positive solution of (1)-(2) we mean that a function $u(t)$ is positive on $0 < t < 1$ and satisfies the problem (1)-(2). Throughout this paper, we also assume the following.

$(H1)$ $f \in C([0,\infty), [0,\infty))$ and the limits;

$$f_0 := \lim_{u \to 0^+} \frac{f(u)}{u}, \qquad f_\infty := \lim_{u \to \infty} \frac{f(u)}{u},$$

exist. We denote that $f_0 = 0$ and $f_\infty = \infty$ correspond to the superlinear case, and $f_0 = \infty$ and $f_\infty = 0$ correspond to the sublinear case.

$(H2)$ $a \in C([0,1], [0,\infty))$, and there exists $t_0 \in [\eta_{m-2}, 1]$ such that $a(t_0) > 0$.

Moreover, we will work in the Banach space $C[0,1]$, and only the sup norm is used. The proof of the main theorem is based upon an application of the following well-known Guo-Krasnoselskii fixed point theorem [30].

Theorem 1.1 *Let E be a Banach space, and let $K \subset E$ be a cone. Assume Ω_1, Ω_2 are open bounded subsets of E with $0 \in \Omega_1$, $\overline{\Omega}_1 \subset \Omega_2$, and let*

$$A : K \cap (\overline{\Omega}_2 \setminus \Omega_1) \longrightarrow K$$

be a completely continuous operator such that
(i) $\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_1,$ *and* $\|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_2;$ *or*
(ii) $\|Au\| \ge \|u\|, \quad u \in K \cap \partial\Omega_1,$ *and* $\|Au\| \le \|u\|, \quad u \in K \cap \partial\Omega_2.$
Then A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$.

2 Preliminary Notes

In this section, we present some lemmas that will be used in the proof of our main results.

Lemma 2.1 *Let* $\alpha_i \ge 0$ *for* $i = 1, 2, \dots, m-1$, *and* $\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) \ne 2$. *If* $y \in C[0,1]$, *then the problem*

$$u''(t) + y(t) = 0, \qquad 0 < t < 1, \tag{3}$$

$$u(0) = 0, \qquad u(1) = \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds, \tag{4}$$

has a unique solution

$$\begin{aligned} u(t) = &-\int_0^t (t-s)y(s)ds + \frac{2t}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_0^1 (1-s)y(s)ds \\ &- \frac{t}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_i} (\eta_i - s)^2 y(s)ds \\ &+ \frac{t}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^2 y(s)ds. \end{aligned} \tag{5}$$

Proof. From (3), we have

$$u''(t) = -y(t).$$

For $t \in [0,1)$, integration from 0 to t, yields

$$u'(t) = -\int_0^t y(s)ds + u'(0).$$

For $t \in [0,1]$, integration from 0 to t, gives

$$u(t) = -\int_0^t \Big(\int_0^r y(s)ds\Big)dr + u'(0)t,$$

i.e.,

$$u(t) = -\int_0^t (t-s)y(s)ds + u'(0)t. \tag{6}$$

So

$$u(1) = -\int_0^1 (1-s)y(s)ds + u'(0).$$

Integrating (6) from η_{i-1} to η_i for $0 \leq \eta_{i-1} < \eta_i \leq 1$, $i = 1, \ldots, m-1$, and reversing the order of double integration, we get

$$\begin{aligned}\int_{\eta_{i-1}}^{\eta_i} u(s)ds &= -\int_{\eta_{i-1}}^{\eta_i} \Big(\int_0^r (r-s)y(s)ds\Big)dr + u'(0)\frac{\eta_i^2 - \eta_{i-1}^2}{2} \\ &= -\int_0^{\eta_i} \frac{(\eta_i - s)^2}{2} y(s)ds + \int_0^{\eta_{i-1}} \frac{(\eta_{i-1} - s)^2}{2} y(s)ds \\ &\quad + u'(0)\frac{\eta_i^2 - \eta_{i-1}^2}{2}, \qquad i = 1, \ldots, m-1.\end{aligned}$$

From condition (4), we have

$$\begin{aligned}u'(0) - \int_0^1 (1-s)y(s)ds = &-\frac{1}{2}\sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_i} (\eta_i - s)^2 y(s)ds \\ &+ \frac{1}{2}\sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^2 y(s)ds \\ &+ \frac{u'(0)}{2}\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2).\end{aligned}$$

Hence,

$$\begin{aligned}u'(0) = &\frac{2}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_0^1 (1-s)y(s)ds \\ &- \frac{1}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_i} (\eta_i - s)^2 y(s)ds \\ &+ \frac{1}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^2 y(s)ds.\end{aligned}$$

Therefore, (3)-(4) has a unique solution

$$\begin{aligned} u(t) = &- \int_0^t (t-s)y(s)ds + \frac{2t}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_0^1 (1-s)y(s)ds \\ &- \frac{t}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_i} (\eta_i - s)^2 y(s)ds \\ &+ \frac{t}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i \int_0^{\eta_{i-1}} (\eta_{i-1} - s)^2 y(s)ds. \end{aligned}$$

□

Lemma 2.2 *Let* $\alpha_i \geq 0$ *for* $i = 1, 2, \dots, m-1$, *and* $\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) < 2$. *If* $y \in C([0,1],[0,\infty))$, *then the unique solution* u *of* (3)-(4) *satisfies* $u(t) \geq 0$ *for* $t \in [0,1]$.

Proof. From the fact that $u''(t) = -y(t) \leq 0$, we know that the graph of $u(t)$ is concave down on $(0,1)$. From this fact, we have that

$$\frac{u(\eta_1)}{\eta_1} \geq \frac{u(\eta_2)}{\eta_2} \geq \cdots \geq \frac{u(\eta_{i-1})}{\eta_{i-1}} \geq \frac{u(\eta_i)}{\eta_i} \geq \cdots \geq \frac{u(1)}{1}, \tag{7}$$

and

$$\int_{\eta_{i-1}}^{\eta_i} u(s)ds \geq \frac{1}{2}(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1})), \tag{8}$$

where $\frac{1}{2}(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1}))$ is the area of the trapezoid under the curve $u(t)$ from $t = \eta_{i-1}$ to $t = \eta_i$ for $i = 1, 2, \dots, m-1$. Combining conditions (4), (7) and (8), we get

$$\begin{aligned} u(1) &= \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds \\ &\geq \frac{1}{2} \sum_{i=1}^{m-1} \alpha_i(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1})) \\ &\geq \frac{1}{2} \sum_{i=1}^{m-1} \alpha_i(\eta_i - \eta_{i-1})\Big(u(\eta_i) + \frac{\eta_{i-1}}{\eta_i} u(\eta_i)\Big) \\ &= \frac{1}{2} \sum_{i=1}^{m-1} \frac{\alpha_i}{\eta_i}(\eta_i^2 - \eta_{i-1}^2)u(\eta_i). \end{aligned} \tag{9}$$

If $u(1) \geq 0$, then the concavity of u together with the boundary condition $u(0) = 0$ implies that $u(t) \geq 0$ for $t \in [0,1]$.

If $u(1) < 0$, then conditions (4), (7) and (8) imply

$$\begin{aligned}
u(1) &= \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} u(s)ds \\
&\geq \frac{1}{2}\sum_{i=1}^{m-1} \alpha_i(\eta_i - \eta_{i-1})(u(\eta_i) + u(\eta_{i-1})) \\
&\geq \frac{1}{2}\sum_{i=1}^{m-1} \alpha_i(\eta_i - \eta_{i-1})(\eta_i u(1) + \eta_{i-1}u(1)) \\
&= \frac{1}{2}\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)u(1).
\end{aligned}$$

This contradicts the fact that $\sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1}^2) < 2$. □

Lemma 2.3 *Let $\alpha_i \geq 0$ for $i \in \{1, \dots, m-3\} \cup \{m-1\}, \alpha_{m-2} > 0$ and $\sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1}^2) > 2$. If $y \in C[0,1]$ and $y(t) \geq 0$ for $t \in (0,1)$, then* (3)-(4) *has no positive solution.*

Proof. Assume that (3)-(4) has a positive solution u, then $u(\eta_i) > 0$ for $i = 1, \dots, m-1$. By condition (9), we have

$$\begin{aligned}
u(1) &\geq \frac{1}{2}\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)\frac{u(\eta_i)}{\eta_i} \\
&\geq \frac{1}{2}\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)\frac{u(\overline{\eta})}{\overline{\eta}} \\
&> \frac{u(\overline{\eta})}{\overline{\eta}},
\end{aligned}$$

where $\overline{\eta} \in \{\eta_1, \dots, \eta_{m-1}\}$ satisfies $\frac{u(\overline{\eta})}{\overline{\eta}} = \min\{\frac{u(\eta_i)}{\eta_i} \,|\, i = 1, \dots, m-1\}$. This contradicts the concavity of u.

If $u(1) = 0$, then applying $\alpha_{m-2} > 0$, we know that $\int_{\eta_{m-3}}^{\eta_{m-2}} u(s)ds = 0$ which implies $u(\eta_{m-2})u(\eta_{m-3}) \leq 0$. This contradicts the concavity of u. The proof is complete. □

Lemma 2.4 *Let $\alpha_i \geq 0$ for $i \in \{1, 2, \dots, m-3\} \cup \{m-1\}$, $\alpha_{m-2} > 0$ and $\sum_{i=1}^{m-1}\alpha_i(\eta_i^2 - \eta_{i-1})^2 < 2$. If $y \in C[0,1]$ and $y(t) \geq 0$ for $t \in (0,1)$, then the unique solution u of problem* (3)-(4) *satisfies*

$$\inf_{t\in[\eta_{m-2},1]} u(t) \geq \gamma\|u\|,$$

where

$$\gamma = \min\left\{\eta_{m-2},\ \frac{1}{2}\sum_{i=1}^{m-2}\alpha_i(\eta_i^2-\eta_{i-1}^2),\ \frac{\alpha_{m-2}(\eta_{m-2}^2-\eta_{m-3}^2)(1-\eta_{m-2})}{\eta_{m-2}(2-\alpha_{m-2}(\eta_{m-2}^2-\eta_{m-3}^2))},\right.$$
$$\left.\frac{1}{2}\sum_{i=1}^{m-2}\frac{\alpha_i}{\eta_i}(\eta_i^2-\eta_{i-1}^2)(1-\eta_i)\right\}. \tag{10}$$

Proof. Since $u''(t) = -y(t) \le 0$, the graph of $u(t)$ is concave down on $(0,1)$. If $u(t)$ is maximum at $t = \bar{t}$, then $\|u\| = u(\bar{t})$. We divide the proof into four cases.

Case I. If $0 < \eta_1 < \cdots < \eta_{m-2} < \bar{t} \le 1$ and $\inf_{t\in[\eta_{m-2},1]} u(t) = u(\eta_{m-2})$, then the concavity of u implies that

$$\frac{u(\eta_{m-2})}{\eta_{m-2}} \ge \frac{u(\bar{t})}{\bar{t}} \ge u(\bar{t}).$$

Therefore,

$$\inf_{t\in[\eta_{m-2},1]} u(t) \ge \eta_{m-2}\|u\|.$$

Case II. If $0 < \eta_1 < \cdots < \eta_{m-2} \le \bar{t} < 1$ and $\inf_{t\in[\eta_{m-2},1]} u(t) = u(1)$. Then (4), (8) and the concavity of u imply

$$\begin{aligned}
u(1) &= \sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} u(s)ds\\
&\ge \frac{1}{2}\sum_{i=1}^{m-1}\alpha_i(\eta_i-\eta_{i-1})(u(\eta_i)+u(\eta_{i-1}))\\
&\ge \frac{1}{2}\sum_{i=1}^{m-2}\alpha_i(\eta_i-\eta_{i-1})(u(\eta_i)+u(\eta_{i-1}))\\
&= \frac{1}{2}\sum_{i=1}^{m-2}\alpha_i(\eta_i^2-\eta_{i-1}^2)\frac{\left(\frac{u(\eta_i)+u(\eta_{i-1})}{2}\right)}{\left(\frac{\eta_i+\eta_{i-1}}{2}\right)}\\
&\ge \frac{1}{2}\sum_{i=1}^{m-2}\alpha_i(\eta_i^2-\eta_{i-1}^2)\frac{u(\bar{t})}{\bar{t}}\\
&\ge \frac{1}{2}\sum_{i=1}^{m-2}\alpha_i(\eta_i^2-\eta_{i-1}^2)u(\bar{t}).
\end{aligned}$$

This implies

$$\inf_{t\in[\eta_{m-2},1]} u(t) \ge \frac{1}{2}\sum_{i=1}^{m-2}\alpha_i(\eta_i^2-\eta_{i-1}^2)\|u\|.$$

Case III. If $0 < \eta_1 < \cdots < \eta_i \leq \overline{t} < \eta_{i+1} < \cdots < \eta_{m-2} < 1$ for some $i \in \{1, \ldots, m-3\}$, and $\inf_{t\in[\eta_{m-2},1]} u(t) = u(1)$. From (9), we get that

$$u(1) \geq \frac{1}{2}\sum_{i=1}^{m-1} \frac{\alpha_i(\eta_i^2 - \eta_{i-1}^2)u(\eta_i)}{\eta_i} \geq \frac{\alpha_{m-2}}{2\eta_{m-2}}\Big(\eta_{m-2}^2 - \eta_{m-3}^2\Big)u(\eta_{m-2}). \tag{11}$$

From the concavity of u and (11), we get

$$\begin{aligned} u(\overline{t}) &\leq u(1) + \frac{u(1) - u(\eta_{m-2})}{1-\eta_{m-2}}(0-1) \\ &= u(1) - \frac{u(1)}{1-\eta_{m-2}} + \frac{u(\eta_{m-2})}{1-\eta_{m-2}} \\ &\leq u(1)\Big[1 - \frac{1}{1-\eta_{m-2}} + \frac{2\eta_{m-2}}{\alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)(1-\eta_{m-2})}\Big] \\ &= \frac{\eta_{m-2}(2 - \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2))}{\alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)(1-\eta_{m-2})}u(1). \end{aligned}$$

Thus,

$$\inf_{t\in[\eta_{m-2},1]} u(t) \geq \frac{\alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2)(1-\eta_{m-2})}{\eta_{m-2}(2 - \alpha_{m-2}(\eta_{m-2}^2 - \eta_{m-3}^2))}\|u\|.$$

Case IV. If $\overline{t} \leq \eta_1 < \eta_2 < \cdots < \eta_{m-2} < 1$ and $\inf_{t\in[\eta_{m-2},1]} u(t) = u(1)$. It is easy to see from the concavity of u that

$$\frac{u(\eta_{m-2})}{1-\eta_{m-2}} \geq \frac{u(\eta_{m-3})}{1-\eta_{m-3}} \geq \cdots \geq \frac{u(\eta_1)}{1-\eta_1} \geq \frac{u(\overline{t})}{1-\overline{t}} \geq u(\overline{t}).$$

This, together with (9), implies that

$$\begin{aligned} u(1) &\geq \frac{1}{2}\sum_{i=1}^{m-1} \alpha_i \frac{(\eta_i^2 - \eta_{i-1}^2)}{\eta_i}(1-\eta_i)u(\overline{t}) \\ &= \frac{1}{2}\sum_{i=1}^{m-2} \frac{\alpha_i}{\eta_i}(\eta_i^2 - \eta_{i-1}^2)(1-\eta_i)u(\overline{t}). \end{aligned}$$

This implies that

$$\inf_{t\in[\eta_{m-2},1]} u(t) \geq \frac{1}{2}\sum_{i=1}^{m-2} \frac{\alpha_i}{\eta_i}(\eta_i^2 - \eta_{i-1}^2)(1-\eta_i)\|u\|.$$

This completes the proof. □

3 Main Results

In this section we discuss the existence of at least one positive solution for the problem (1)-(2). We obtain the following existence results.

Theorem 3.1 *Assume* $(H1)$ *and* $(H2)$ *hold. Then the problem* (1)-(2) *has at least one positive solution in the case*
(i) $f_0 = 0$ *and* $f_\infty = \infty$ *(superlinear), or*
(ii) $f_0 = \infty$ *and* $f_\infty = 0$ *(sublinear).*

Proof. Superlinear case. Suppose then that $f_0 = 0$ and $f_\infty = \infty$. We wish to show the existence of a positive solution of (1)-(2). Now (1)-(2) has a solution $u = u(t)$ if and only if u solves the operator equation

$$\begin{aligned} u(t) = & -\int_0^t (t-s)a(s)f(u(s))ds \\ & + \frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1 (1-s)a(s)f(u(s))ds \\ & - \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds \\ & + \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds \\ := & Au(t). \end{aligned} \tag{12}$$

Denote

$$K = \Big\{u | u \in C[0,1], u \geq 0, \min_{\eta_{m-2}\leq t\leq 1} u(t) \geq \gamma\|u\|\Big\},$$

where γ is defined by (10). It is obvious that K is a cone in $C[0,1]$. Moreover, by Lemma 2.4, $AK \subset K$. It is also easy to check that $A : K \to K$ is completely continuous.

Since $f_0 = 0$, we may choose $H_1 > 0$ so that $f(u) \leq \epsilon u$, for $0 < u < H_1$, where $\epsilon > 0$ satisfies

$$\left(\frac{2\int_0^1(1-s)a(s)ds + \sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)\epsilon \leq 1. \tag{13}$$

Thus, if $u \in K$ and $\|u\| = H_1$, then from (12) and (13), we get

$$\begin{aligned}
Au(t) \leq & \frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)f(u(s))ds \\
& + \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds \\
\leq & \frac{2}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)\epsilon u(s)ds \\
& + \frac{1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)\epsilon u(s)ds \\
\leq & \frac{2\epsilon}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)ds\|u\| \\
& + \frac{\epsilon}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds\|u\| \\
\leq & \left(\frac{2\int_0^1(1-s)a(s)ds+\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)\epsilon H_1. \quad (14)
\end{aligned}$$

Let $\Omega_1 = \{u \in C[0,1] \mid \|u\| < H_1\}$, then (14) shows that $\|Au\| \leq \|u\|$, for $u \in K \cap \partial\Omega_1$.

Further, since $f_\infty = \infty$, there exists $\widehat{H}_2 > 0$ such that $f(u) \geq \rho u$, for $u \geq \widehat{H}_2$, where $\rho > 0$ is chosen so that

$$\rho\gamma\frac{\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_{\eta_{m-2}}^1(1-s)a(s)ds \geq 1. \quad (15)$$

Let $H_2 = \max\{2H_1, (\widehat{H}_2/\gamma)\}$ and $\Omega_2 = \{u \in C[0,1] \mid \|u\| < H_2\}$, then $u \in K$ and $\|u\| = H_2$ implies

$$\min_{t\in[\eta_{m-2},1]} u(t) \geq \gamma\|u\| \geq \widehat{H}_2.$$

So,

$$
\begin{aligned}
Au(s) = &-\int_0^s (s-r)a(r)f(u(r))dr \\
&+ \frac{2s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1 (1-r)a(r)f(u(r))dr \\
&- \frac{s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-r)^2 a(r)f(u(r))dr \\
&+ \frac{s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-r)^2 a(r)f(u(r))dr.
\end{aligned}
$$

This implies

$$
\begin{aligned}
u(1) = &\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} u(s)ds \\
= &\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\Bigg(-\int_0^s (s-r)a(r)f(u(r))dr \\
&+ \frac{2s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1 (1-r)a(r)f(u(r))dr \\
&- \frac{s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-r)^2 a(r)f(u(r))dr \\
&+ \frac{s}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-r)^2 a(r)f(u(r))dr\Bigg)ds \\
= &-\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}\int_0^s (s-r)a(r)f(u(r))drds \\
&+ \frac{2\int_0^1(1-r)a(r)f(u(r))dr}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i} sds\right)
\end{aligned}
$$

$$
\begin{aligned}
&-\frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-r)^2a(r)f(u(r))dr}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}sds\right)\\
&+\frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-r)^2a(r)f(u(r))dr}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i\int_{\eta_{i-1}}^{\eta_i}sds\right)\\
=&\ \frac{1}{2}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds\\
&-\frac{1}{2}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds\\
&+\frac{\int_0^1(1-s)a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\right)\\
&-\frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\frac{\alpha_i}{2}(\eta_i^2-\eta_{i-1}^2)\right)\\
&+\frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\left(\sum_{i=1}^{m-1}\frac{\alpha_i}{2}(\eta_i^2-\eta_{i-1}^2)\right)\\
=&\ \frac{\int_0^1(1-s)a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)\\
&-\frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\\
&+\frac{\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\\
=&\ \frac{1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\Bigg[\int_0^{\eta_{i-1}}(\eta_i-\eta_{i-1})\\
&\times(2-(\eta_i+\eta_{i-1}))sa(s)f(u(s))ds\\
&+\int_{\eta_{i-1}}^{\eta_i}\big[(\eta_i^2-\eta_{i-1}^2)(1-s)-(\eta_i-s)^2\big]a(s)f(u(s))ds\\
&+\int_{\eta_i}^{1}(\eta_i^2-\eta_{i-1}^2)(1-s)a(s)f(u(s))ds\Bigg]\\
\geq&\ \frac{1}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\Bigg[\int_{\eta_{i-1}}^{\eta_i}(\eta_i-\eta_{i-1})\\
&\times\big[\eta_{i-1}(1-s)+(1-\eta_i)s\big]a(s)f(u(s))ds
\end{aligned}
$$

$$
\begin{aligned}
&\quad + (\eta_i^2 - \eta_{i-1}^2) \int_{\eta_i}^{1} (1-s)a(s)f(u(s))ds \Bigg] \\
&\geq \frac{1}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) \int_{\eta_i}^{1} (1-s)a(s)f(u(s))ds \\
&\geq \frac{\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_{\eta_{m-2}}^{1} (1-s)a(s)f(u(s))ds. \qquad (16)
\end{aligned}
$$

Hence, for $u \in K \cap \partial\Omega_2$,

$$
\|Au\| \geq |u(1)| \geq \rho\gamma \frac{\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_{\eta_{m-2}}^{1} (1-s)a(s)ds\|u\| \geq \|u\|.
$$

By the first part of Theorem 1.1, it follows that A has a fixed point in $K \cap (\overline{\Omega}_2 \setminus \Omega_1)$, such that $H_1 \leq \|u\| \leq H_2$.

Sublinear case. Suppose that $f_0 = \infty$ and $f_\infty = 0$. We first choose $H_3 > 0$ such that $f(u) \geq Mu$ for $0 < u < H_3$, where

$$
\gamma M \frac{\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_{\eta_{m-2}}^{1} (1-s)a(s)ds \geq 1.
$$

For $u \in K$ and $\|u\| = H_3$, by a similar method used to obtain (16), we can get that

$$
\begin{aligned}
Au(1) &= \sum_{i=1}^{m-1} \alpha_i \int_{\eta_{i-1}}^{\eta_i} Au(s)ds \\
&\geq \frac{1}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2) \int_{\eta_i}^{1} (1-s)a(s)f(u(s))ds \\
&\geq \frac{\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_{\eta_{m-2}}^{1} (1-s)a(s)f(u(s))ds \\
&\geq \frac{\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_{\eta_{m-2}}^{1} (1-s)a(s)Mu(s)ds \\
&\geq \gamma M \frac{\sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)}{2 - \sum_{i=1}^{m-1} \alpha_i(\eta_i^2 - \eta_{i-1}^2)} \int_{\eta_{m-2}}^{1} (1-s)a(s)ds\|u\| \\
&\geq H_3. \qquad (17)
\end{aligned}
$$

Let $\Omega_3 = \{u \in C[0,1] \mid \|u\| < H_3\}$. Thus, (17) shows that $\|Au\| \geq \|u\|$, $u \in K \cap \partial\Omega_3$.

Next, in view of $f_\infty = 0$, there exists $\widehat{H}_4 > 0$ such that $f(u) \le \lambda u$ for $u \ge \widehat{H}_4$, where $\lambda > 0$ satisfies

$$\left(\frac{2\int_0^1(1-s)a(s)ds + \sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)\lambda \le 1.$$

We consider the following two cases.
Case (i). Suppose f is bounded, say $f(u) \le N$ for all $u \in [0,\infty)$. In this case, choose

$$H_4 = \max\left\{2H_3,\ \left(\frac{2\int_0^1(1-s)a(s)ds + \sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)N\right\},$$

such that, for $u \in K$ with $\|u\| = H_4$, we have

$$\begin{aligned}
Au(t) = &-\int_0^t (t-s)a(s)f(u(s))ds \\
&+\frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)f(u(s))ds \\
&-\frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2a(s)f(u(s))ds \\
&+\frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds \\
\le &\frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1(1-s)a(s)f(u(s))ds \\
&+\frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)f(u(s))ds \\
\le &\left(\frac{2\int_0^1(1-s)a(s)ds + \sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)N \\
\le &H_4.
\end{aligned}$$

Therefore, $\|Au\| \le \|u\|$.
Case (ii). If f is unbounded, which implies that there is H_4, $H_4 > \max\{2H_3, (\widehat{H}_4/\gamma)\}$, such that $f(u) \le f(H_4)$ for $0 < u \le H_4$. Then for $u \in K$ and $\|u\| = H_4$, we

have

$$\begin{aligned}
Au(t) = & -\int_0^t (t-s)a(s)f(u(s))ds \\
& + \frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1 (1-s)a(s)f(u(s))ds \\
& - \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_i}(\eta_i-s)^2 a(s)f(u(s))ds \\
& + \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2 a(s)f(u(s))ds \\
\le & \frac{2t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\int_0^1 (1-s)a(s)f(H_4)ds \\
& + \frac{t}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2 a(s)f(H_4)ds \\
\le & \left(\frac{2\int_0^1(1-s)a(s)ds+\sum_{i=1}^{m-1}\alpha_i\int_0^{\eta_{i-1}}(\eta_{i-1}-s)^2a(s)ds}{2-\sum_{i=1}^{m-1}\alpha_i(\eta_i^2-\eta_{i-1}^2)}\right)\lambda H_4 \\
\le & H_4.
\end{aligned}$$

Hence, in either case, we may always let $\Omega_4 = \{u \in C[0,1] \mid \|u\| < H_4\}$, and for $u \in K \cap \partial\Omega_4$. By the second part of Theorem 1.1, we can conclude that problem (1)-(2) has at least one positive solution. □

4 Some examples

In this section, in order to illustrate our result, we consider some examples.

Example 4.1 Consider the eleven-point boundary value problem

$$u''(t) + e^{2t}(e^u u^e \sin u + u^2 \ln(1+u)) = 0, \qquad 0 < t < 1, \tag{18}$$

$$\begin{aligned}
u(0) = 0, \qquad u(1) = & 10e\int_0^{\frac{1}{10}} u(s)ds + \frac{4e}{5}\int_{\frac{1}{5}}^{\frac{3}{10}} u(s)ds + \frac{8e}{9}\int_{\frac{2}{5}}^{\frac{5}{10}} u(s)ds \\
& + \frac{12e}{13}\int_{\frac{3}{5}}^{\frac{7}{10}} u(s)ds + \frac{16e}{17}\int_{\frac{4}{5}}^{\frac{9}{10}} u(s)ds.
\end{aligned} \tag{19}$$

Set $\eta_i = \frac{i}{10}$ for $i = 0, \ldots, 10$, $a(t) = e^{2t}$ and $f(u) = e^u u^e \sin u + u^2 \ln(1+u)$, $\alpha_{2j} = 0$ for $j = 1, \ldots, 5$, and $\alpha_1 = 10e$, $\alpha_3 = \frac{4e}{5}$, $\alpha_5 = \frac{8e}{9}$, $\alpha_7 = \frac{12e}{13}$, $\alpha_9 = \frac{16e}{17}$.

We can show that

$$\sum_{i=1}^{10} \alpha_i(\eta_i^2 - \eta_{i-1}^2) = \frac{e}{2} < 2.$$

Through a simple calculation we can get $f_0 = 0$ and $f_\infty = \infty$. Thus, by the first part of Theorem 3.1, we can get that the problem (18)-(19) has at least one positive solution.

Example 4.2 Consider the eleven-point boundary value problem

$$u''(t) + \pi^t t^\pi f(u) = 0, \qquad 0 < t < 1, \tag{20}$$

$$u(0) = 0, \qquad u(1) = 2\pi \int_{\frac{1}{10}}^{\frac{1}{5}} u(s)ds + \pi \int_{\frac{3}{10}}^{\frac{2}{5}} u(s)ds + \frac{10\pi}{11} \int_{\frac{5}{10}}^{\frac{3}{5}} u(s)ds + \frac{3\pi}{5} \int_{\frac{7}{10}}^{\frac{4}{5}} u(s)ds + \frac{8\pi}{17} \int_{\frac{4}{5}}^{\frac{9}{10}} u(s)ds + \frac{10\pi}{19} \int_{\frac{9}{10}}^{1} u(s)ds, \tag{21}$$

where

$$f(u) = \begin{cases} \frac{\pi \sin u + 2\cos u}{u^2}, & 0 < u \le \frac{\pi}{2} \\ \frac{\sqrt{8u} + 2\sqrt{\pi}\sin^2 u}{\pi^{\frac{3}{2}}}, & u > \frac{\pi}{2}. \end{cases}$$

Set $\eta_i = \frac{i}{10}$ for $i = 0, \ldots, 10$, $a(t) = \pi^t t^\pi$, $\alpha_{2j-1} = 0$ for $j = 1, \ldots, 4$, and $\alpha_2 = 2\pi$, $\alpha_4 = \pi$, $\alpha_6 = \frac{10\pi}{11}$, $\alpha_8 = \frac{3\pi}{5}$, $\alpha_9 = \frac{8\pi}{17}$, $\alpha_{10} = \frac{10\pi}{19}$. We can show that

$$\sum_{i=1}^{10} \alpha_i(\eta_i^2 - \eta_{i-1}^2) = \frac{\pi}{2} < 2.$$

Through a simple calculation we can get $f_0 = \infty$ and $f_\infty = 0$. Thus, by the second part of Theorem 3.1, we can get that the problem (20)-(21) has at least one positive solution.

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