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On the p-Adic Integral Representation of Bernstein Polynomials Associated with (h,q)-Genocchi Numbers and Polynomials

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Abstract

Recently, we introduced some interesting relations between Genocchi number and Bernstein polynomials(see [6]). In this paper, we give some interesting identities on the (h,q)-Genocchi polynomials and Bernstein polynomials.

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1 Introduction

Throughout this paper, let p be a fixed odd prime number. The symbol, $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ denote the ring of p-adic integers, the field of p-adic rational numbers and the completion of algebraic closure of $\mathbb{Q}_p$. Let $\mathbb{N}$ be the set of natural numbers and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. As well known definition, the p-adic absolute value is given by $|x|_p = p^{-r}$ where $x = p^r \frac{t}{s}$ with $(t,p) = (s,p) = (t,s) = 1$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. In this paper we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < 1$. We assume that $UD(\mathbb{Z}_p)$ is the space of the uniformly differentiable function on $\mathbb{Z}_p$. For $f \in UD(\mathbb{Z}_p)$, Kim defined the fermionic p-adic invariant integral on $\mathbb{Z}_p$

$$I_{-1}(f) = \int_{\mathbb{Z}_p} f(x) d\mu_{-1}(x) = \lim_{N\to\infty} \sum_{x=0}^{p^N-1} f(x)(-1)^x, \text{ see [1, 2, 3, 4]}. \tag{1.1}$$

The Genocchi polynomials are defined by the generating function as follows:

$$\frac{2t}{e^t+1}e^{xt}=\sum_{n=0}^{\infty}G_n(x)\frac{t^n}{n!}, \text{ see [6].} \tag{1.2}$$

In the special case, $x=0$, $G_n(0)=G_n$ are called the n-th Euler numbers.

From (1.1), we have

$$t\int_{\mathbb{Z}_p}q^y e^{(x+y)t}d\mu_{-1}(y)=\frac{2t}{qe^t+1}e^{xt}=\sum_{n=0}^{\infty}G_{n,q}(x)\frac{t^n}{n!}. \tag{1.3}$$

By (1.3), we obtain

$$G_{0,q}(x)=0, \quad \int_{\mathbb{Z}_p}q^{hy}(x+y)^n d\mu_{-1}(y)=\frac{G_{n+1,q}(x)}{n+1}, \text{ for } n\in\mathbb{N}. \tag{1.4}$$

In [1], Kim introduced p-adic extension of Bernstein polynomials as follows:

$$B_{k,n}(x)=\binom{n}{k}x^k(1-x)^{n-k}, \text{ where } x\in\mathbb{Z}_p \text{ and } n,k\in\mathbb{Z}_+. \tag{1.5}$$

In this paper, we investigate some properties for the (h,q)-Genocchi numbers and polynomials. By using these properties, we give some interesting identities on the (h,q)-Genocchi polynomials and Bernstein polynomials.

2 Some identities on the Bernstein and (h,q)-Genocchi polynomials

From (1.3), we can derive the following recurrence formula for the (h,q)-Genocchi numbers:

$$G_{0,q}^{(h)}=0, \text{ and } q^h(G_q^{(h)}+1)^n+G_{n,q}^{(h)}=\begin{cases}2, & \text{if } n=1,\\ 0, & \text{if } n>1,\end{cases} \tag{2.1}$$

with usual convention about replacing $(G_q^{(h)})^n$ by $G_{n,q}^{(h)}$.

By (1.3), we easily get

$$\sum_{n=0}^{\infty}G_{n,q^{-1}}^{(h)}(1-x)(-1)^n\frac{t^n}{n!}=(-1)\frac{2tq^h}{q^he^t+1}e^{xt}=(-1)q^h\sum_{n=0}^{\infty}G_{n,q}^{(h)}(x)\frac{t^n}{n!}. \tag{2.2}$$

By (2.2), we obtain the following theorem.

Theorem 2.1 *For $n \in \mathbb{Z}_+$, we have*

$$G_{n,q}^{(h)}(x) = (-1)^{n-1} q^{-h} G_{n,q^{-1}}^{(h)}(1-x).$$

From (1.4), we note that

$$G_{0,q}^{(h)} = 0, \qquad \int_{\mathbb{Z}_p} q^{hx} x^n d\mu_{-1}(x) = \frac{G_{n+1,q}^{(h)}}{n+1}, \text{ for } n \in \mathbb{N}. \tag{2.3}$$

By (2.1), for $n \in \mathbb{N}$ with $n > 1$, we have

$$\begin{aligned} G_{n,q}^{(h)}(2) &= (G_q^{(h)} + 1 + 1)^n = \sum_{l=0}^{n} \binom{n}{l} G_{l,q}(1) \\ &= \frac{1}{q^h}(nq^h G_{1,q}^{(h)}(1)) + \frac{1}{q^h} \sum_{l=2}^{n} \binom{n}{l} G_{l,q}^{(h)}(1) \\ &= \frac{2n}{q^h} + \frac{1}{(q^h)^2} G_{n,q}^{(h)}. \end{aligned} \tag{2.4}$$

Therefore, by (2.4), we obtain the following theorem.

Theorem 2.2 *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$q^h G_{n,q}^{(h)}(2) = 2n + \frac{1}{q^h} G_{n,q}^{(h)}.$$

By (2.3) and Theorem 2.2, we obtain the following corollary.

Corollary 2.3 *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\frac{1}{q^h} \int_{\mathbb{Z}_p} q^{-hx} (x+2)^n d\mu_{-1}(x) = 2 + q^h \frac{G_{n+1,q^{-1}}^{(h)}}{n+1}.$$

By (1.4), (2.3) and Corollary 2.3, we know that

$$\begin{aligned} \int_{\mathbb{Z}_p} q^{hx} (1-x)^n d\mu_{-1}(x) &= (-1)^n \int_{\mathbb{Z}_p} q^{hx} (x-1)^n d\mu_{-1}(x) \\ &= 2 + q^h \int_{\mathbb{Z}_p} q^{-hx} x^n d\mu_{-1}(x). \end{aligned}$$

Therefore, we have the following theorem.

Theorem 2.4 *For $n \in \mathbb{N}$ with $n > 1$, we have*

$$\int_{\mathbb{Z}_p} q^{hx} (1-x)^n d\mu_{-1}(x) = 2 + q^h \int_{\mathbb{Z}_p} q^{-hx} x^n d\mu_{-1}(x).$$

In (1.5), we take the fermionic p-adic invariant integral on $\mathbb{Z}_p$ for one Bernstein polynomials as follows:

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{hx} B_{k,n}(x) d\mu_{-1}(x) &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{n-k-l} \frac{G_{n-l+1,q}^{(h)}}{n-l+1} \\ &= \binom{n}{k} \sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} \frac{G_{k+l+1,q}^{(h)}}{k+l+1}, \text{ where } n,k \in \mathbb{Z}_+.\end{aligned} \tag{2.5}$$

From the reflection symmetric properties of Bernstein polynomials, we note that

$$B_{k,n}(x) = B_{n-k,n}(1-x), \text{ where } n,k \in \mathbb{Z}_+ \text{ and } x \in \mathbb{Z}_p. \tag{2.6}$$

For $n,k \in \mathbb{Z}_+$ with $n > k+1$, we have

$$\begin{aligned}\int_{\mathbb{Z}_p} q^{hx} B_{k,n}(x) d\mu_{-1}(x) &= \int_{\mathbb{Z}_p} q^{hx} B_{n-k,n}(1-x) d\mu_{-1}(x) \\ &= \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + q^h \int_{\mathbb{Z}_p} q^{-hx} x^{n-l} d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.5 *For $n,k \in \mathbb{Z}_+$ with $n > k+1$, we have*

$$\int_{\mathbb{Z}_p} q^{hx} B_{k,n}(x) d\mu_{-1}(x) = \binom{n}{k} \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + q^h \frac{G_{n-l+1,q^{-1}}^{(h)}}{n-l+1}\right).$$

By (2.5) and Theorem 2.5, we have the following theorem.

Theorem 2.6 *Let $n,k \in \mathbb{Z}_+$ with $n > k+1$. Then we have*

$$\sum_{l=0}^{n-k} \binom{n-k}{l} (-1)^{l} \frac{G_{k+l+1,q}^{(h)}}{k+l+1} = \sum_{l=0}^{k} \binom{k}{l} (-1)^{k-l} \left(2 + q^h \frac{G_{n-l+1,q^{-1}}^{(h)}}{n-l+1}\right).$$

Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k+1$. Then we get

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x) B_{k,n_2}(x) q^{hx} d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \int_{\mathbb{Z}_p} q^{hx} (1-x)^{n_1+n_2-l} d\mu_{-1}(x) \\ &= \binom{n_1}{k}\binom{n_2}{k} \sum_{l=0}^{2k} \binom{2k}{l} (-1)^{l+2k} \left(2 + q^h \int_{\mathbb{Z}_p} q^{-hx} x^{n_1+n_2-l} d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we obtain the following theorem.

Theorem 2.7 *For $n_1, n_2\, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)q^{hx}d\mu_{-1}(x) = \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2 + q^h\frac{G^{(h)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1}\right).$$

By simple calculation, we easily see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)q^{hx}d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\int_{\mathbb{Z}_p} q^{hx}x^{l+2k}d\mu_{-1}(x) \qquad (2.7)\\ &= \binom{n_1}{k}\binom{n_2}{k}\sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{G^{(h)}_{l+2k+1,q}}{l+2k+1}.\end{aligned}$$

Therefore, by (2.7) and Theorem 2.7, we obtain the following theorem.

Theorem 2.8 *Let $n_1, n_2, k \in \mathbb{Z}_+$ with $n_1 + n_2 > 2k + 1$. Then we have*

$$\sum_{l=0}^{2k}\binom{2k}{l}(-1)^{l+2k}\left(2 + q^h\frac{G^{(h)}_{n_1+n_2-l+1,q^{-1}}}{n_1+n_2-l+1}\right) = \sum_{l=0}^{n_1+n_2-2k}(-1)^l\binom{n_1+n_2-2k}{l}\frac{G^{(h)}_{l+2k+1,q}}{l+2k+1}.$$

For $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1 + n_2 + n_3 > 3k + 1$, by the symmetry of Bernstein polynomials, we see that

$$\begin{aligned}&\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\int_{\mathbb{Z}_p} q^{hx}(1-x)^{n_1+n_2+n_3-l}d\mu_{-1}(x)\\ &= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2 + q^h\int_{\mathbb{Z}_p} q^{-hx}x^{n_1+n_2+n_3-l}d\mu_{-1}(x)\right).\end{aligned}$$

Therefore, we have the following theorem.

Theorem 2.9 *For $n_1, n_2, n_2, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$, we have*

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2+q^h\frac{G^{(h)}_{n_1+n_2+n_3-l,q^{-1}}}{n_1+n_2+n_3-l+1}\right).$$

In the same manner, multiplication of three Bernstein polynomials can be given by the following relation:

$$\int_{\mathbb{Z}_p} B_{k,n_1}(x)B_{k,n_2}(x)B_{k,n_3}(x)q^{hx}d\mu_{-1}(x)$$
$$= \binom{n_1}{k}\binom{n_2}{k}\binom{n_3}{k}\sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}\frac{G^{(h)}_{l+3k+1,q}}{l+3k+1},$$

where $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$.

Therefore, by Theorem 2.9, we have the following theorem.

Theorem 2.10 *Let $n_1, n_2, n_3, k \in \mathbb{Z}_+$ with $n_1+n_2+n_3 > 3k+1$. Then we have*

$$\sum_{l=0}^{3k}\binom{3k}{l}(-1)^{l+3k}\left(2+q^h\frac{G^{(h)}_{n_1+n_2+n_3-l+1,q^{-1}}}{n_1+n_2+n_3-l+1}\right)$$
$$= \sum_{l=0}^{n_1+n_2+n_3-3k}(-1)^l\binom{n_1+n_2+n_3-3k}{l}\frac{G^{(h)}_{l+3k+1,q}}{l+3k+1}.$$

Using the above theorem and mathematical induction, we have the following theorem.

Theorem 2.11 *Let $m \in \mathbb{N}$. For $n_1, n_2, \ldots, n_m, k \in \mathbb{Z}_+$ with $n_1+\cdots+n_m > mk+1$, the multiplication of the sequence of Bernstein polynomials $B_{k,n_1}(x), \ldots, B_{k,n_m}(x)$ with different degrees under fermionic p-adic invariant integral on $\mathbb{Z}_p$ can be given as*

$$\int_{\mathbb{Z}_p}\left(\prod_{i=1}^m B_{k,n_i}(x)\right)q^{hx}d\mu_{-1}(x)$$
$$= \left(\prod_{i=1}^m\binom{n_i}{k}\right)\sum_{l=0}^{mk}\binom{mk}{l}(-1)^{l+mk}\left(2+q^h\frac{G^{(h)}_{n_1+\cdots+n_m-l+1,q^{-1}}}{n_1+\cdots+n_m-l+1}\right).$$

We also easily see that

$$\int_{\mathbb{Z}_p}\left(\prod_{i=1}^{m}B_{k,n_i}(x)\right)q^{hx}d\mu_{-1}(x) = \left(\prod_{i=1}^{m}\binom{n_i}{k}\right)\sum_{l=0}^{n_1+\cdots+n_m-mk}\binom{n_1+\cdots+n_m-mk}{l}(-1)^l\frac{G_{l+mk+1,q}^{(h)}}{l+mk+1}. \tag{2.8}$$

By Theorem 2.11 and (2.8), we have the following corollary.

Corollary 2.12 *Let* $m\in\mathbb{N}$. *For* $n_1,n_2,\ldots,n_m,k\in\mathbb{Z}_+$ *with* $n_1+\cdots+n_m>mk+1$, *we have*

$$\sum_{l=0}^{mk}\binom{mk}{l}(-1)^{l+mk}\left(2+q^h\frac{G_{n_1+\cdots+n_m-l+1,q^{-1}}^{(h)}}{n_1+\cdots+n_m-l+1}\right) = \sum_{l=0}^{n_1+\cdots+n_m-mk}\binom{n_1+\cdots+n_m-mk}{l}(-1)^l\frac{G_{l+mk+1,q}^{(h)}}{l+mk+1}.$$

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