Applied Mathematical Sciences, Vol. 6, 2012, no. 15, 731 - 738

A Note on the q-Genocchi Numbers and Polynomials with Weak Weight α

C. S. Ryoo

Department of Mathematics
Hannam University, Daejeon 306-791, Korea
ryoocs@hnu.kr

J. Y. Kang

Department of Mathematics
Hannam University, Daejeon 306-791, Korea

Abstract

In this paper we construct a new type of q-Genocchi numbers and polynomials with weak weight α: $G_{n,q}^{(\alpha)}$, $G_{n,q}^{(\alpha)}(x)$ respectively. Some interesting results and relationships are obtained.

Mathematics Subject Classification: 11B68, 11S40, 11S80

Keywords: Genocchi numbers and polynomials, q-Genocchi numbers and polynomials, q-Genocchi numbers and polynomials with weak weight α

1 Introduction

The Genocchi numbers and polynomials possess many interesting properties and arising in many areas of mathematics and physics. Recently, many mathematicians have studied in the area of the q-Genocchi numbers and polynomials(see [1-13]). In this paper, we construct a new type of q-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x)$ weak weight α.

Throughout this paper we use the following notations. By $\mathbb{Z}_p$ we denote the ring of p-adic rational integers, $\mathbb{Q}_p$ denotes the field of p-adic rational numbers, $\mathbb{C}_p$ denotes the completion of algebraic closure of $\mathbb{Q}_p$, $\mathbb{N}$ denotes the set of natural numbers, $\mathbb{Z}$ denotes the ring of rational integers, $\mathbb{Q}$ denotes the field of rational numbers, $\mathbb{C}$ denotes the set of complex numbers, and $\mathbb{Z}_+ = \mathbb{N} \cup \{0\}$. Let ν_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one talks of q-extension, q is considered in many ways such as an indeterminate, a complex number $q \in \mathbb{C}$, or p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$ one normally assume that $|q| < 1$. If $q \in \mathbb{C}_p$, we normally assume that $|q-1|_p < p^{-\frac{1}{p-1}}$ so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. Throughout this paper we use the notation:

$$[x]_q = \frac{1-q^x}{1-q}, \quad [x]_{-q} = \frac{1-(-q)^x}{1+q} \text{ (cf. [1-13]) }.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case. For

$$g \in UD(\mathbb{Z}_p) = \{g | g : \mathbb{Z}_p \to \mathbb{C}_p \text{ is uniformly differentiable function}\},$$

the fermionic p-adic q-integral on $\mathbb{Z}_p$ is defined by Kim as follows:

$$I_{-q}(g) = \int_{\mathbb{Z}_p} g(x) d\mu_{-q}(x) = \lim_{N\to\infty} \frac{1}{[p^N]_{-q}} \sum_{x=0}^{p^N-1} g(x)(-q)^x \text{ (cf. [3-6]) }. \tag{1.1}$$

If we take $g_1(x) = g(x+1)$ in (1.1), then we easily see that

$$qI_{-q}(g_1) + I_{-q}(g) = [2]_q g(0). \tag{1.2}$$

From (1.2), we obtain

$$q^n I_{-q}(g_n) + (-1)^{n-1} I_{-q}(g) = [2]_q \sum_{l=0}^{n-1} (-1)^{n-1-l} q^l g(l), \tag{1.3}$$

where $g_n(x) = g(x+n)$(cf. [3-6]).

As well known definition, the Genocchi polynomials are defined by

$$F(t) = \frac{2t}{e^t+1} = e^{Gt} = \sum_{n=0}^{\infty} G_n \frac{t^n}{n!},$$

$$F(t,x) = \frac{2t}{e^t+1} e^{xt} = e^{G(x)t} = \sum_{n=0}^{\infty} G_n(x) \frac{t^n}{n!},$$

with the usual convention of replacing $G^n(x)$ by $G_n(x)$. In the special case, $x = 0$, $G_n(0) = G_n$ are called the n-th Genocchi numbers(cf. [1-13]).

Our aim in this paper is to define q-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α. We investigate some properties which are related to q-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α. We also derive the existence of a specific interpolation function which interpolate q-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α at negative integers.

2 q-Genocchi numbers and polynomials with weak weight α

Our primary goal of this section is to define q-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α. We also find generating functions of q-Genocchi numbers $G_{n,q}^{(\alpha)}$ and polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α.

For $\alpha \in \mathbb{Z}$ and $q \in \mathbb{C}_p$ with $|1-q|_p \leq 1$, q-Genocchi numbers $G_{n,q}^{(\alpha)}$ are defined by

$$G_{n,q}^{(\alpha)} = n\int_{\mathbb{Z}_p}[x]_q^{n-1}d\mu_{-q^\alpha}(x). \tag{2.1}$$

By using p-adic q-integral on $\mathbb{Z}_p$, we obtain,

$$\begin{aligned} n\int_{\mathbb{Z}_p}[x]_q^{n-1}d\mu_{-q^\alpha}(x) &= n\lim_{N\to\infty}\frac{1}{[p^N]_{-q^\alpha}}\sum_{x=0}^{p^N-1}[x]_q^{n-1}(-q^\alpha)^x \\ &= n[2]_{q^\alpha}\left(\frac{1}{1-q}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{1}{1+q^{\alpha+l}} \qquad (2.2) \\ &= n[2]_{q^\alpha}\sum_{m=0}^{\infty}(-1)^m q^{\alpha m}[m]_q^{n-1}. \end{aligned}$$

By (2.1), we have

$$\begin{aligned} G_{n,q}^{(\alpha)} &= n[2]_{q^\alpha}\left(\frac{1}{1-q}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{1}{1+q^{\alpha+l}} \\ &= n[2]_{q^\alpha}\sum_{m=0}^{\infty}(-1)^m q^{\alpha m}[m]_q^{n-1}. \end{aligned}$$

We set

$$F_q^{(\alpha)}(t) = \sum_{n=0}^{\infty}G_{n,q}^{(\alpha)}\frac{t^n}{n!}.$$

By using above equation and (2.2), we have

$$\begin{aligned} F_q^{(\alpha)}(t) &= \sum_{n=0}^{\infty}G_{n,q}^{(\alpha)}\frac{t^n}{n!} \\ &= [2]_{q^\alpha}\sum_{n=0}^{\infty}\left(n\left(\frac{1}{1-q}\right)^{n-1}\sum_{l=0}^{n-1}\binom{n-1}{l}(-1)^l\frac{1}{1+q^{\alpha+l}}\right)\frac{t^n}{n!} \qquad (2.3) \\ &= t[2]_{q^\alpha}\sum_{m=0}^{\infty}(-1)^m q^{\alpha m}e^{[m]_q t}. \end{aligned}$$

Thus q-Genocchi numbers with weak weight α, $G_{n,q}^{(\alpha)}$ are defined by means of the generating function

$$F_q^{(\alpha)}(t) = t[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m]_q t}. \tag{2.4}$$

By using (2.1), we have

$$\begin{aligned} \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)} \frac{t^n}{n!} &= \sum_{n=0}^{\infty} n \int_{\mathbb{Z}_p} [x]_q^{n-1} d\mu_{-q^\alpha}(x) \frac{t^n}{n!} \\ &= t \int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q^\alpha}(x). \end{aligned} \tag{2.5}$$

By (2.3), (2.5), we have

$$t \int_{\mathbb{Z}_p} e^{[x]_q t} d\mu_{-q^\alpha}(x) = t[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m]_q t}.$$

Next, we introduce q-Genocchi polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α. The q-Genocchi polynomials $G_{n,q}^{(\alpha)}(x)$ with weak weight α are defined by

$$G_{n,q}^{(\alpha)}(x) = n \int_{\mathbb{Z}_p} [x+y]_q^{n-1} d\mu_{-q^\alpha}(y). \tag{2.6}$$

By using p-adic q-integral, we obtain

$$G_{n,q}^{(\alpha)}(x) = n[2]_{q^\alpha} \left(\frac{1}{1-q}\right)^{n-1} \sum_{l=0}^{n-1} \binom{n-1}{l} (-1)^l q^{xl} \frac{1}{1+q^{\alpha+l}}. \tag{2.7}$$

We set

$$F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x) \frac{t^n}{n!}. \tag{2.8}$$

By using (2.7) and (2.8), we obtain

$$F_q^{(\alpha)}(t,x) = \sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x) \frac{t^n}{n!} = t[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} e^{[m+x]_q t}. \tag{2.9}$$

Obverse that, if $q \to 1$, then $F_q^{(\alpha)}(t,x) \to F(t,x)$ and $F_q^{(\alpha)}(t) \to F(t)$.

Since $[x+y]_q = [x]_q + q^x[y]_q$, we easily obtain that

$$\begin{aligned} G^{(\alpha)}_{n+1,q}(x) &= n\int_{\mathbb{Z}_p}[x+y]_q^{n-1}d\mu_{-q^\alpha}(y) \\ &= q^{-x}\sum_{k=0}^{n+1}\binom{n+1}{k}[x]_q^{n+1-k}q^{xk}G^{(\alpha)}_{l,q} \\ &= q^{-x}\left([x]_q + q^x G^{(\alpha)}_q\right)^{n+1} \\ &= (n+1)[2]_{q^\alpha}\sum_{m=0}^{\infty}(-1)^m q^{\alpha m}[x+m]_q^n. \end{aligned} \tag{2.10}$$

By (2.7), we have the following complement relation:

Theorem 2.1 *Property of complement*

$$G^{(\alpha)}_{n,q^{-1}}(1-x) = (-1)^{n-1}q^{n-1}G^{(\alpha)}_{n,q}(x)$$

By (2.7), we have the following distribution relation:

Theorem 2.2 *For any positive integer m(=odd), we have*

$$G^{(\alpha)}_{n,q}(x) = \frac{[2]_{q^\alpha}}{[2]_{q^{\alpha m}}}[m]_q^{n-1}\sum_{i=0}^{m-1}(-1)^i q^{\alpha i}G^{(\alpha)}_{n,q^m}\left(\frac{i+x}{m}\right), \quad n \in \mathbb{Z}_+.$$

By (1.3), (2.1) , and (2.6), we easily see that

$$m[2]_{q^\alpha}\sum_{l=0}^{n-1}(-1)^{n-1-l}q^{\alpha l}[l]_q^{m-1} = q^{\alpha n}G^{(\alpha)}_{m,q}(n) + (-1)^{n-1}G^{(\alpha)}_{m,q}.$$

Hence, we have the following theorem.

Theorem 2.3 *Let $m \in \mathbb{Z}_+$. If $n \equiv 0 \pmod 2$, then*

$$q^{\alpha n}G^{(\alpha)}_{m,q}(n) - G^{(\alpha)}_{m,q} = m[2]_{q^\alpha}\sum_{l=0}^{n-1}(-1)^{l+1}q^{\alpha l}[l]_q^{m-1}.$$

If $n \equiv 1 \pmod 2$, then

$$q^{\alpha n}G^{(\alpha)}_{m,q}(n) + G^{(\alpha)}_{m,q} = m[2]_{q^\alpha}\sum_{l=0}^{n-1}(-1)^{l}q^{\alpha l}[l]_q^{m-1}.$$

From (1.2), we note that

$$
\begin{aligned}
[2]_{q^\alpha} &= q^\alpha \int_{\mathbb{Z}_p} te^{[x+1]_q t} d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} te^{[x]_q t} d\mu_{-q^\alpha}(x) \\
&= \sum_{n=0}^{\infty} \left(q^\alpha \int_{\mathbb{Z}_p} n[x+1]_q^{n-1} d\mu_{-q^\alpha}(x) + \int_{\mathbb{Z}_p} n[x]_q^{n-1} d\mu_{-q^\alpha}(x) \right) \frac{t^n}{n!} \\
&= \sum_{n=0}^{\infty} \left(q^\alpha G_{n,q}^{(\alpha)}(1) + G_{n,q}^{(\alpha)} \right) \frac{t^n}{n!}.
\end{aligned}
$$

Therefore, we obtain the following theorem.

Theorem 2.4 *For $n \in \mathbb{Z}_+$, we have*

$$
q^\alpha G_{n,q}^{(\alpha)}(1) + G_{n,q}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1. \end{cases}
$$

By Theorem 2.4 and (2.10), we have the following corollary.

Corollary 2.5 *For $n \in \mathbb{Z}_+$, we have*

$$
q^\alpha (q G_q^{(\alpha)} + 1)^n + G_{n,q}^{(\alpha)} = \begin{cases} [2]_{q^\alpha}, & \text{if } n = 1, \\ 0, & \text{if } n \neq 1, \end{cases}
$$

with the usual convention of replacing $(G_q^{(\alpha)})^n$ by $G_{n,q}^{(\alpha)}$.

3 The analogue of the Genocchi zeta function

By using q-Genocchi numbers and polynomials with weak weight α, q-Genocchi zeta function with weak weight α and Hurwitz q-Genocchi zeta functions with weak weight α are defined. These functions interpolate the q-Genocchi numbers and q-Genocchi polynomials with weak weight α, respectively. In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. From (2.4), we note that

$$
\begin{aligned}
\left. \frac{d^{k+1}}{dt^{k+1}} F_q^{(\alpha)}(t) \right|_{t=0} &= (k+1)[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^n q^{\alpha m} [m]_q^k \\
&= G_{k+1,q}^{(\alpha)}, (k \in \mathbb{N}).
\end{aligned}
$$

By using the above equation, we are now ready to define q-Genocchi zeta functions.

Definition 3.1 *Let $s \in \mathbb{C}$.*

$$\zeta_q^{(\alpha)}(s) = [2]_{q^\alpha} \sum_{n=1}^{\infty} \frac{(-1)^n q^{\alpha n}}{[n]_q^s}. \tag{3.1}$$

Note that $\zeta_q^{(\alpha)}(s)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_q^{(\alpha)}(s)$ and $G_{k,q}^{(\alpha)}$ is given by the following theorem.

Theorem 3.2 *For $k \in \mathbb{N}$, we have*

$$\zeta_q^{(\alpha)}(-k) = \frac{G_{k+1,q}^{(\alpha)}}{k+1}.$$

Observe that $\zeta_q^{(\alpha)}(s)$ function interpolates $G_{k,q}^{(\alpha)}$ numbers at non-negative integers. By using (2.9), we note that

$$\begin{aligned} \left.\frac{d^{k+1}}{dt^{k+1}} F_q^{(\alpha)}(t,x)\right|_{t=0} &= (k+1)[2]_{q^\alpha} \sum_{m=0}^{\infty} (-1)^m q^{\alpha m} [x+m]_q^k \\ &= G_{k+1,q}^{(\alpha)}(x), (k \in \mathbb{N}) \end{aligned} \tag{3.2}$$

and

$$\left.\left(\frac{d}{dt}\right)^{k+1} \left(\sum_{n=0}^{\infty} G_{n,q}^{(\alpha)}(x) \frac{t^n}{n!}\right)\right|_{t=0} = G_{k+1,q}^{(\alpha)}(x), \text{ for } k \in \mathbb{N}. \tag{3.3}$$

By (3.2) and (3.3), we are now ready to define the Hurwitz q-Genocchi zeta functions.

Definition 3.3 *Let $s \in \mathbb{C}$.*

$$\zeta_q^{(\alpha)}(s,x) = [2]_{q^\alpha} \sum_{n=0}^{\infty} \frac{(-1)^n q^{\alpha n}}{[n+x]_q^s}. \tag{3.4}$$

Note that $\zeta_q^{(\alpha)}(s,x)$ is a meromorphic function on $\mathbb{C}$. Relation between $\zeta_q^{(\alpha)}(s,x)$ and $G_{k,q}^{(\alpha)}(x)$ is given by the following theorem.

Theorem 3.4 *For $k \in \mathbb{N}$, we have*

$$\zeta_q^{(\alpha)}(-k,x) = \frac{G_{k+1,q}^{(\alpha)}(x)}{k+1}.$$

Observe that $\zeta_q^{(\alpha)}(-k,x)$ function interpolates $G_{k,q}^{(\alpha)}(x)$ numbers at non-negative integers.

References

[1] I. N. Cangul, V. Kurt, H. Ozden, Y. Simsek, On the higher-order w-q-Genocchi numbers, *Adv. Stud. Contemp. Math.*, **19**(2009), 39-57.

[2] L.-C. Jang, A study on the distribution of q-Genocchi polynomials, *Adv. Stud. Contemp. Math.* **18**(2009), 181-189.

[3] T. Kim, An analogue of Bernoulli numbers and their applications, *Rep. Fac. Sci. Engrg. Saga Univ. Math.* **22** (1994) , 21-26.

[4] T. Kim , On the q-extension of Euler and Genocchi numbers , *Journal of Mathematical Analysis and Applications*, **326**(2007), 1458-1465.

[5] T. Kim, q-Volkenborn integration, *Russ. J. Math. Phys.* **9**(2002) , 288-299.

[6] T. Kim, On the multiple q-Genocchi and Euler numbers,*Russ. J. Math. Phys.*, **15**(2008), 481-486.

[7] T. Kim, J. Choi, Y-.H. Kim, C. S. Ryoo, A Note on the weighted p-adic q-Euler measure on $\mathbb{Z}_p$, *Advan. Stud. Contemp. Math.* **21** (2011), 35-40.

[8] C. S. Ryoo, T. Kim, L.-C. Jang, Some relationships between the analogs of Euler numbers and polynomials, *J. Inequal. Appl.* **2007**(2007), Art ID 86052, 22pp.

[9] C.S. Ryoo, A numerical investigation on the zeros of the Genocchi polynomials, *Journal of Applied Mathematics and Computing*, **22**(2006), 125-132.

[10] C.S. Ryoo, A numerical computation on the structure of the roots of q-extension of Genocchi polynomials, *Applied Mathematics Letters*, **21**(2008), 348-354.

[11] C.S. Ryoo, Calculating zeros of the twisted Genocchi polynomials, *Adv. Stud. Contemp. Math.*, **17**(2008), 147-159.

[12] S.-H. Rim, S. J. Lee, E. J. Moon, J. H. Jin, On the q-Genocchi numbers and polynomials associated with q-zeta function, *Proceedings of the Jangjeon Mathematical Society*, **12**(2009), 261-267.

[13] S. H. Rim, J. H. Jin, E. J. Moon, S. J. Lee, Some identities on the q-Genocchi polynomials of higher-order and q-Stirling numbers by the fermionic p-adic integral on $\mathbb{Z}_p$, *International Journal of Mathematics and Mathematical Sciences*, **2010**(2010), Art. ID 860280.

Received: August, 2011