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On the Well Posedness and Further Regularity of a Diffusive Three Species Aquatic Model

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Abstract

We consider Upadhay's three species aquatic food chain model, with the inclusion of spatial spread. This is a well established food chain model for the interaction of three given aquatic species. It exhibits rich dynamical behavior, including chaos. We prove the existence of a global weak solution to the diffusive system, followed by existence of local mild and strong solution.

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1 Introduction

Predator-prey interactions have a long standing history in the biological sciences. The various cycles that occur in these populations have fascinated naturalists and mathematicians alike. Their study has been regarded as one of the most intense and sought after in ecology, [BSM97]. A special class of predator-prey systems are three species coupled models. These are of interest to ecologists due to the interdependence of the species, be it predator or prey, and the rich vein of biological consequences therein. This includes competing predators, competing prey, cooperation and refuge [BMT96], [A01]. They are also of interest to mathematicians and the dynamical systems community because the coupling mathematically gives rise to rich dynamics [M87], [M01]. There is yet another reason why both ecologists and dynamical systems persons are attracted to investigate such models. This is due to much heated debate as to why the chaotic behavior of such systems although established theoretically, by dynamicists, is not observed in the wild, by ecologists. The larger issue here of course, is why chaos is not observed often enough in real ecological systems, although these systems possess all the needed ingredients to this end. The works [UKR09],[UR97], [C01] and [C03] have been devoted to such pursuit.

In [U08], Upadhyay proposed a generalized model for a aquatic ecological food chain system consisting of TPP, Zooplankton and Molluscs. He observed that an increase in the strength of toxic substance released by toxin producing phytoplankton population reduces the propensity of chaotic dynamics and changes the state of chaos to limit cycle and finally settles down to stable focus. Upadhyay introduced mutual interference in all the three populations, by adding an extra mortality term in the Zooplankton population and also taking into account the toxin liberation process of the TPP population. This model generalizes several other well known models in the literature like the Upadhyay and Rai model [UR97] and the Hastings and Powell model [HP91]. The results reported in [U08] are for Holling type II functional response.

Upadhyay's model, as it stands, is a ODE model. The purpose of the current manuscript is to include diffusion of the various species. Results for general cross diffusion systems were reported as early as 1998, see [LNW98]. More recent work presenting general theory has been done by Shim [S02], [S02]. However for specific applications various further estimates have to be made from time to time. This has resulted in a fair amount of work on systems with time delays [L06], [AF90], specific cross diffusion systems [LN06], systems with stage structure [LCL02],[ZCN00] , food chain models [YF08] and coupled models for invasive aquatic species [PG11]. Recently Zhang et al also investigated a cross diffusion PDE model with Holling type III functional response [ZGF09]. Ko and Ryu have investigated predator prey models with Holling type II responses, where there is scope of prey refuge [KR06].

Our goal in the current manuscript is to show that under the inclusion of spatial spread, a well defined dynamics is still preserved for the model. In essence we will prove its well posedness. Thus we first prove the existence of a weak solution to the system. This followed by further estimates on higher Sobolev norms will allow us to show existence of mild and then strong solution also.

2 The Mathematical Model

Upadhyay's three species aquatic model for a predator-prey system of three aquatic species is a ODE system, [U08]. The novel feature of the current work is to add diffusion terms into the model, thus incorporating spatial spread. Spatial spread captures the realistic movements of TPP when predated upon by Zooplankton, and Zooplankton when predated upon by Molluscs. We assume a bounded domain for the spatial spread, making it consistent with real scenarios. Thus our diffusive three species aquatic model takes the following form,

$$\frac{\partial x_1}{\partial t} = \Delta x_1 + a_1 x_1 - b_1 x_1^2 - w_0 \left(\frac{x_1}{x_1 + D_0}\right)^{m_1} (x_2)^{m_2} \tag{1}$$

$$\frac{\partial x_2}{\partial t} = \Delta x_2 - a_2 x_2 + w_1 \left(\frac{x_1}{x_1 + D_1}\right)^{m_1} (x_2)^{m_2} - w_2 \left(\frac{x_2}{x_2 + D_2}\right)^{m_2} (x_3)^{m_2} - \theta f_1(x_1) x_2 \tag{2}$$

$$\frac{\partial x_3}{\partial t} = \Delta x_3 + c x_3^{m_3} - w_3 f_2(x_2) x_3^{m_3} \tag{3}$$

The problem is posed on $\Omega \subset \mathbb{R}^3$. Ω is bounded, and $\partial\Omega$ is assumed to be lipschitz. We consider Dirchlet boundary conditions

$$x_1 = 0 \text{ on } \partial\Omega,\ x_2 = 0 \text{ on } \partial\Omega,\ x_3 = 0 \text{ on } \partial\Omega, \tag{4}$$

We also impose suitable positive initial conditions

$$x_1(x,0) = x_{10},\ x_2(x,0) = x_{20},\ x_3(x,0) = x_{30} \tag{5}$$

Remark 2.1 *Note that we can work with the Neumann type or mixed type boundary conditions just as easily. It will not effect the integration by parts carried out at many places in the subsequent analysis.*

As is customary in many biological systems we assume $(x_{10}, x_{20}, x_{30}) \in L^\infty(\Omega)$. It follows via application of standard parabolic theory, [P92], that x_1, x_2 are bounded by their carrying capacities. Thus

$$||x_1||_\infty \leq K \tag{6}$$

$$||x_2||_\infty \leq K \tag{7}$$

We also assume,

$$c \leq \frac{w_3}{D_3 + K} \leq w_3 f_2(x_2) \tag{8}$$

This prevents finite time blow up of (3), and thus

$$||x_3||_\infty \leq K \tag{9}$$

Where

$$K = \max(K_1, K_2, K_3) \tag{10}$$

Here K_1, K_2, K_3 are the carrying capacities of x_1, x_2, x_3 respectively.

Consider a situation where a prey population x_1 is predated by a population x_2. The population x_2, in turn serves as favourite food for a population x_3. This interaction is represented by the above system. Here $m_i > 0$ for $i = 1, 2, 3$, also $a_1, a_2, b_1, w_0, w_1, w_2, w_3, c$ and D_0, D_1, D_2, D_3, D_4 are the positive constants. The parameters m_i for $i = 1, 2, 3$ are mutual interference parameters that model the intraspecies competition among predators when hunting for prey. The model hopes to capture the dynamics between TPP population (prey) denoted x_1, which serves as the only food source for the specialist predator Zooplankton denoted x_2, which in turn, serves as the favorite food for the generalist predator Molluscs denoted x_3. For a detailed understanding of the parameters in the system, the reader is referred to [U08].

3 Weak Solutions and Galerkin Truncation

Our Goal in this section is show that the diffusive 3 species aquatic model possesses a weak solution in the requisite functional spaces. Also in all the estimates made henceforth C, C_1, C_2, C_3 are generic constants that can change in their value from line to line.

3.1 Uniform L^2 Estimates

The following equation holds for the Galerkin truncation of x_1, for $\forall\ 1 \leq j \leq n$,

$$\frac{\partial x_{1n}}{\partial t} = \Delta x_{1n} + a_1 x_{1n} - b_1 x_{1n}^2 - P_n \left[w_0 \left(\frac{x_{1n}}{x_{1n} + D_0} \right)^{m_1} (x_{2n})^{m_2} \right], \tag{11}$$

$$x_{1n}(0) = P_n(x_{1n0}). \tag{12}$$

Here P_n is the projection onto the space spanned by first n eigenvectors. Note in general

$$\langle x_{in}, P_n(F(x_{in}))\rangle = \langle P_n(x_{in}), F(x_{in})\rangle = \langle x_{in}, F(x_{in})\rangle \tag{13}$$

We multiply (11) by x_{1n} and integrate by parts over Ω. This yields

$$\frac{d}{dt}|x_{1n}|_2^2 = -|\nabla x_{1n}|_2^2 + a_1|x_{1n}|_2^2 - b_1|x_{1n}|_3^3 - w_0 \int_\Omega \frac{x_{1n}^{m_1+1}}{(x_{1n}+D_0)^{m_1}} x_{2n}^{m_2} d\mathbf{x} \tag{14}$$

we next rewrite

$$a_1|x_{1n}|_2^2 = a_1 \int_\Omega (\left(\left(\frac{b_1}{a_1}\right)^{\frac{2}{3}} x_{1n}\right) \left(\frac{a_1}{b_1}\right)^{\frac{2}{3}})^2 d\mathbf{x} \tag{15}$$

We then use Holder's inequality, followed by Young's inequality to yield

$$\frac{d}{dt}|x_{1n}|_2^2 = -|\nabla x_{1n}|_2^2 + b_1|x_{1n}|_3^3 + C_2 - b_1|x_{1n}|_3^3 - w_0 \int_\Omega \frac{x_{1n}^{m_1+1}}{(x_{1n}+D_0)^{m_1}} x_{2n}^{m_2} d\mathbf{x} \tag{16}$$

We now use Poincaire's inequality and the positivity of x_{1n} and x_{2n}, thus the fact that

$$0 < \int_\Omega \frac{x_{1n}^{m_1+1}}{(x_{1n}+D_0)^{m_1}} x_{2n}^{m_2} d\mathbf{x} \tag{17}$$

To yield

$$\frac{d}{dt}|x_{1n}|_2^2 + C_1|x_{1n}|_2^2 \leq C_2 \tag{18}$$

Thus application of Gronwall's Lemma gives us the following estimate

$$|x_{1n}|_2^2 \leq e^{-C_1 t}|x_{1n}(0)|_2^2 + \frac{C_2}{C_1} \tag{19}$$

On the other hand we can integrate (14) in the time interval $[0, T]$ to yield

$$\int_0^T |\nabla x_{1n}|_2^2 dt \leq |x_{1n}(0)|_2^2 + C_2 T \leq |x_{10}|_2^2 + C \tag{20}$$

Thus from (19) and (20) we have the following estimates

$$|x_{1n}|_{L^\infty(0,T;L^2(\Omega))} \leq C \tag{21}$$

$$|x_{1n}|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{22}$$

We can now use the uniform bounds in (21) and (22) to extract a subsequence x_{1n_j} such that

$$x_{1n_j} \overset{*}{\rightharpoonup} x_1 \ \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{23}$$

$$x_{1n_j} \rightharpoonup x_1 \ \text{ in } L^2(0,T;H_0^1(\Omega)) \tag{24}$$

$$x_{1n_j} \to x_1 \ \text{ in } L^2(0,T;L^2(\Omega)). \tag{25}$$

This last inequality follows via the compact embedding of $H_0^1(\Omega) \hookrightarrow L^2(\Omega)$. The estimates for x_2, x_3 are made similarly, and we obtain

$$|x_{2n}|_{L^\infty(0,T;L^2(\Omega))} \leq C \tag{26}$$

$$|x_{2n}|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{27}$$

We can now via the uniform bounds in (26) and (27) extract a subsequence x_{2n_j} such that

$$x_{2n_j} \overset{*}{\rightharpoonup} x_2 \ \text{ in } L^\infty(0,T;L^2(\Omega)) \tag{28}$$

$$x_{2n_j} \rightharpoonup x_2 \ \text{ in } L^2(0,T;H_0^1(\Omega)) \tag{29}$$

$$x_{2n_j} \to x_2 \ \text{ in } L^2(0,T;L^2(\Omega)). \tag{30}$$

This convergence in (30) follows via the compact Sobolev embedding of

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega). \tag{31}$$

and

$$|x_{3n}|_{L^\infty(0,T;L^2(\Omega))} \leq C \tag{32}$$

$$|x_{3n}|_{L^2(0,T;H_0^1(\Omega))} \leq C \tag{33}$$

We can now via the uniform bounds in (32) and (33) extract a subsequence x_{3n_j} such that

$$x_{3n_j} \stackrel{*}{\rightharpoonup} x_3 \text{ weak star in } L^\infty(0,T;L^2(\Omega)) \tag{34}$$

$$x_{3n_j} \rightharpoonup x_3 \text{ in } L^2(0,T;H_0^1(\Omega)) \tag{35}$$

$$x_{3n_j} \rightarrow x_3 \text{ in } L^2(0,T;L^2(\Omega)). \tag{36}$$

The convergence in (36) follows via the compact Sobolev embedding of

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega). \tag{37}$$

This completes the L^2 estimates.

3.2 Uniform H^1 estimates

We multiply the equation for the truncation of x_1 by $-\Delta x_{1n}$ and integrate by parts over Ω to obtain

$$\begin{aligned} &\frac{d}{dt}|\nabla x_{1n}|_2^2 = -|\Delta x_{1n}|_2^2 - a_1\int_\Omega x_{1n}\Delta x_1 d\mathbf{x} + b_1\int_\Omega (x_{1n})^2\Delta x_{1n} d\mathbf{x} \\ &+w_0\int_\Omega \frac{x_{1n}^{m_1}}{(x_{1n}+D_0)^{m_1}} x_{2n}^{m_2}\Delta x_{1n} d\mathbf{x} \end{aligned} \tag{38}$$

Poincaire's inequality, Integration by parts on the third term on the right hand side, followed by the application of Cauchy Schwartz and Youngs inequalities on the last term on the right hand side yield

$$\begin{aligned}
&\frac{d}{dt}|\nabla x_{1n}|_2^2 + C|\Delta x_{1n}|_2^2 \\
\leq\ & -b_1\int_\Omega (2(\nabla x_{1n})^2 x_{1n} d\mathbf{x} + a_1|x_{1n}|_2^2 + \frac{1}{2}|\Delta x_{1n}|_2^2 + w_0|\Omega|K^{2m_2} \\
\leq\ & \frac{1}{4}|\Delta x_{1n}|_2^2 \qquad (39)
\end{aligned}$$

The positivity of b_1 and x_1 implies

$$b_1\int_\Omega (2(\nabla x_{1n})^2 x_{1n} d\mathbf{x} > 0 \qquad (40)$$

Thus we obtain

$$\frac{d}{dt}|\nabla x_{1n}|_2^2 + \frac{C}{4}|\nabla x_{1n}|_2^2 \leq (a_1)^2|x_{1n}|_2^2 + C_1|\Omega|(K)^{2m_2} \qquad (41)$$

At this juncture an intermediate step is required. Recall from (14) we have

$$\frac{d}{dt}|x_{1n}|_2^2 + C_1|\nabla x_{1n}|_2^2 \leq C_2 \qquad (42)$$

Integrating the above in the time interval $[t_1, t_1+1]$ yields

$$\int_{t_1}^{t_1+1} |\nabla x_{1n}|_2^2 \leq C_2 + |x_{1n}(t_1)|_2^2 \leq C_3 \qquad (43)$$

Thus via intermediate value theorem for integrals there exists a $t^* \in [t_1, t_1+1]$ such that

$$|\nabla x_{1n}(t^*)|_2^2 \leq C_3 \qquad (44)$$

Thus we obtain

$$\begin{aligned}
&\frac{d}{dt}|\nabla x_{1n}|_2^2 + \frac{C}{4}|\nabla x_{1n}|_2^2 \\
\leq\ & C_1|\Omega|(K)^{2m_2} + (a_1)^2|x_{1n}|_2^2 \\
\leq\ & C_1|\Omega|(K)^{2m_2} + C \qquad (45)
\end{aligned}$$

Thus application of Gronwall's Lemma gives us the following estimate

$$|\nabla x_{1n}|_2^2 \leq e^{-\frac{C}{4}(t-t^*)}|\nabla x_{1n}(t^*)|_2^2 + \frac{2C_1|\Omega|(K)^{2m_2}}{Ca_1} \leq e^{-\frac{C}{4}(t-t^*)}|x_{1n}(0)|_2^2 \quad (46)$$

On the other hand we can use the Holder inequality along with Youngs inequality as done before, without resorting to Poincaire's inequality to yield

$$\frac{d}{dt}|\nabla x_{1n}|_2^2 + \frac{a_1 C}{2}|\Delta x_{1n}|_2^2 \leq C_2 \quad (47)$$

We can integrate (47) in the time interval $[t^*, T]$ to yield

$$\int_{t^*}^{T} |\Delta x_{1n}|_2^2 \leq |\nabla x_{1n}(t^*)|_2^2 + C_2 T \leq |\nabla x_1(t_1^*)|_2^2 + C \quad (48)$$

Thus from these estimates we obtain

$$|x_{1n}|_{L^\infty(t^*,T;H_0^1(\Omega))} \leq C \quad (49)$$

$$|x_{1n}|_{L^2(t^*,T;H^2(\Omega))} \leq C \quad (50)$$

Via the above uniform bounds we can now extract a subsequence x_{1n_j} such that

$$x_{1n_j} \stackrel{*}{\rightharpoonup} x_1 \ \text{ in } L^\infty(t^*, T; H_0^1(\Omega)) \quad (51)$$

$$x_{1n_j} \rightharpoonup x_1 \ \text{ in } L^2(t^*, T; H^2(\Omega)) \quad (52)$$

$$x_{1n_j} \to x_1 \ \text{ in } L^2(t^*, T; H_0^1(\Omega)). \quad (53)$$

The convergence in (53) follows via the compact embedding of $H^2(\Omega) \hookrightarrow H_0^1(\Omega)$.

The estimates on x_2, x_3 are made similarly. Essentially we obtain,

$$|x_{2n}|_{L^\infty(t_1^*,T;H_0^1(\Omega))} \leq C \quad (54)$$

$$|x_{2n}|_{L^2(t_1^*,T;H^2(\Omega))} \leq C \quad (55)$$

Via the above uniform estimates we can now extract a subsequence x_{2n_j} such that

$$x_{2n_j} \overset{*}{\rightharpoonup} x_2 \text{ in } L^\infty(t_1^*, T; H_0^1(\Omega)) \tag{56}$$

$$x_{2n_j} \rightharpoonup x_2 \text{ in } L^2(t_1^*, T; H^2(\Omega)) \tag{57}$$

$$x_{2n_j} \rightarrow x_2 \text{ in } L^2(t_1^*, T; H_0^1(\Omega)). \tag{58}$$

The compact embedding of $H^2(\Omega) \hookrightarrow H_0^1(\Omega)$ facilitates the above.

also we obtain

$$|x_{3n}|_{L^\infty(t_3^*, T; H_0^1(\Omega))} \leq C \tag{59}$$

$$|x_{3n}|_{L^2(t_3^*, T; H^2(\Omega))} \leq C \tag{60}$$

We can now extract a subsequence x_{3n_j} via the above uniform bounds such that

$$x_{3n_j} \overset{*}{\rightharpoonup} x_3 \text{ in } L^\infty(t_3^*, T; H_0^1(\Omega)) \tag{61}$$

$$x_{3n_j} \rightharpoonup x_3 \text{ in } L^2(t_3^*, T; H^2(\Omega)) \tag{62}$$

$$x_{3n_j} \rightarrow x_3 \text{ in } L^2(t_3^*, T; H_0^1(\Omega)). \tag{63}$$

This last inequality follows via the compact embedding of $H^2(\Omega) \hookrightarrow H_0^1(\Omega)$.

3.3 Uniform Estimate of the time derivatives

In order to complete the requirement for the well posedness we need estimates on the time derivatives of solutions in the Sobolev Space $H^{-1}(\Omega)$. To this end it is convenient to recast the truncation of (1) as

$$\frac{\partial x_{1n}}{\partial t} + b_1 x_{1n}^2 + P_n\left[w_0\left(\frac{x_{1n}}{x_{1n}+D_0}\right)^{m_1}(x_{2n})^{m_2}\right] = \Delta x_{1n} + a_1 x_{1n} \tag{64}$$

Note via the positivity of x_{1n},x_{2n},b_1 and D_0 we have

$$b_1 x_{1n}^2 + w_0 \left(\frac{x_{1n}}{x_{1n} + D_0} \right)^{m_1} (x_{2n})^{m_2} > 0 \tag{65}$$

Thus it trivially follows that

$$\begin{aligned} & \left|\left| \frac{\partial x_{1n}}{\partial t} \right|\right|_{H^{-1}(\Omega)} \\ \leq \; & \left|\left| \frac{\partial x_{1n}}{\partial t} + b_1 x_1^{2n} + w_0 \left(\frac{x_{1n}}{x_{1n} + D_0} \right)^{m_1} (x_{2n})^{m_2} \right|\right|_{H^{-1}(\Omega)} \\ \leq \; & ||\Delta x_{1n} + a_1 x_{1n}||_{H^{-1}(\Omega)} \\ \leq \; & ||\Delta x_{1n}||_{H^{-1}(\Omega)} + a_1 \, ||x_{1n}||_{H^{-1}(\Omega)} \\ \leq \; & C(|x_{1n}|_{H_0^1(\Omega)} + |x_{1n}|_2) \\ \leq \; & C \end{aligned} \tag{66}$$

This follows from the uniform $L^2(\Omega)$ and $H_0^1(\Omega)$ estimates already derived, and the Sobolev embedding of

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \hookrightarrow H^{-1}(\Omega) \tag{67}$$

Thus we obtain

$$\left|\left| \frac{\partial x_{1n}}{\partial t} \right|\right|_{H^{-1}(\Omega)} \leq C|x_{1n}|_{H_0^1(\Omega)} + C|x_{1n}|_2 \tag{68}$$

Integrating the above in the time interval $[0, T]$ yields

$$\begin{aligned} & \int_0^T \left|\left| \frac{\partial x_{1n}}{\partial t} \right|\right|_{H^{-1}(\Omega)} \\ \leq \; & C \int_0^T |x_{1n}|_{H_0^1(\Omega)} + C \int_0^T |x_{1n}|_2 \\ \leq \; & C|x_{1n}|_{L^2(0,T;H_0^1(\Omega))} + C|x_{1n}|_{L^\infty(0,T;L^2(\Omega))} \\ \leq \; & C \end{aligned} \tag{69}$$

This yields the following estimate

$$\frac{\partial x_{1n}}{\partial t} \in L^2(0, T; H^{-1}(\Omega)). \tag{70}$$

The time derivative of x_2 can be estimated similarly,

$$\frac{\partial x_{2n}}{\partial t} \in L^2(0,T;H^{-1}(\Omega)). \tag{71}$$

The estimate on the time derivative of x_3 is easily obtained and we have,

$$\frac{\partial x_{3n}}{\partial t} \in L^2(0,T;H^{-1}(\Omega)). \tag{72}$$

3.4 Estimate of the nonlinear terms

We are almost in a position to prove existence of weak solution to the three species aquatic model. The main technical difficulty is to show convergence of the nonlinear terms in the equations. To this end we state a Lemma that gives an estimate for the nonlinear term in (1). The estimates on the other nonlinear terms follow similarly.

Lemma 3.1 *Consider two distinct nonlinear terms as they appear in* (1), *from the diffusive three species aquatic model*

$$F(x_{1n},x_{2n}) = -w_0\left(\frac{x_{1n}}{x_{1n}+D_0}\right)^{m_1}(x_{2n})^{m_2}, \tag{73}$$

$$F(y_{1n},y_{2n}) = -w_0\left(\frac{y_{1n}}{y_{1n}+D_0}\right)^{m_1}(y_{2n})^{m_2} \tag{74}$$

Then for $m_1 \leq 3$ *and* $m_2 \leq 4$, *the following estimate holds*

$$|F(x_{1n},x_{2n}) - F(y_{1n},y_{2n})|_2 \leq C_1|x_{1n}-y_{1n}|_2 + C_2|x_{2n}-y_{2n}|_2 \tag{75}$$

This follows via standard algebraic manipulations for a binomial series, the lipscitz property of the nonlinearity, the bounds on x_{2n},y_{2n} and the compact Sobolev embedding

$$H_0^1 \hookrightarrow L^{2m_2}(\Omega) \hookrightarrow L^2(\Omega), \text{ for } m_2 \leq 3 \tag{76}$$

3.5 Passage to Weak Limit

Now that we have made the a priori estimates on the truncated equations, we will attempt to pass to the weak limit, as is customary. We will focus on (1) for demonstration. Recall via the earlier Galerkin theory we are seeking an approximate solution of the form

$$x_{1n}(t) = \sum_{j=1}^{n} x_{1nj}(t)w_j. \tag{77}$$

Such that for each $1 \leq j \leq n$, and $\forall \phi(t) \in C_0^\infty(0,T)$, the following holds.

$$\begin{aligned} & \left(\frac{d\ x_{1n}}{dt}, \phi(t)w_j\right) + \left(\nabla x_{1n_j}, \nabla w_j \phi(t)\right) - a_1 \left\langle x_{1n_j}, \phi(t)w_j\right\rangle + b_1 \left\langle x_{1n_j}^2, \phi(t)w_j\right\rangle \\ = & \left\langle F(x_{1n_j}, x_{2n_j}), \phi(t)w_j\right\rangle, \end{aligned} \tag{78}$$

$$x_{1n}(0) = P_n(x_{10}). \tag{79}$$

Upon passing to the weak limit of 78 we will obtain

$$\left(\frac{d\ x_1}{dt}, w_j\right) + (\nabla x_1, \nabla w_j) - a_1 \langle x_1, w_j\rangle + b_1 \left\langle x_1^2, w_j\right\rangle = \langle F(x_1, x_2), w_j\rangle. \tag{80}$$

This will imply the existence of a weak solution x_1 to 1. The existence of a weak solution x_2 to 2 and x_3 to 3 is achieved similarly. We proceed as follows. Consider a $\phi \in C_0^\infty(0,T)$. We multiply equation 78 by $\phi(t)$ and integrate by parts in time to yield

$$\begin{aligned} & -\int_0^T \left(x_{n_j}, \phi'(t)w_j\right) dt \\ = & -\int_0^T \left(\nabla x_{1n_j}, \nabla w_j\phi(t)\right) dt + \int_0^T \left\langle F(x_{1n_j}, x_{2n_j}), \phi(t)w_j\right\rangle dt \\ + & \ a_1 \int_0^T \left\langle x_{1n_j}, \phi(t)w_j\right\rangle dt - b_1 \int_0^T \left\langle x_{1n_j}^2, \phi(t)w_j\right\rangle dt.^1 \end{aligned} \tag{81}$$

We will use the earlier derived Lemma to show convergence of the nonlinear term. This is stated via the following lemma:

Lemma 3.2 *Consider the nonlinear term*

$$F(x_1, x_2) = \frac{x_1^{m_1+1}}{(x_1 + D_0)^{m_1}} x_2^{m_2} \tag{82}$$

The following convergence result holds

$$\lim_{j\to\infty}\int_0^T\int_\Omega F(x_{1n_j},x_{2n_j})\phi(t)w_j d\boldsymbol{x}\,dt=\int_0^T\int_\Omega F(x_1,x_2)\phi(t)w_j d\boldsymbol{x}\,dt$$

for all $\phi\in C_0^\infty(0,T)$

Proof 3.3 *Consider*

$$\begin{aligned}
& \left|\lim_{j\to\infty}\int_0^T\int_\Omega F(x_{1n_j},x_{2n_j})\phi(t)w_j d\boldsymbol{x}dt-\int_0^T\int_\Omega F(x_1,x_2)\phi(t)w_j d\boldsymbol{x}dt\right| \\
\leq\ & C\int_0^T\int_\Omega |(F(x_{1n_j},x_{2n_j}),\phi w_j)-(F(x_1,x_2),\phi w_j)|^2 d\boldsymbol{x}dt \\
\leq\ & C|\phi|_\infty|w_j|_\infty\int_0^T\int_\Omega |F(x_1,x_2)-F(x_{1n_j},x_{2n_j})|^2 d\boldsymbol{x}dt \\
=\ & C|\phi|_\infty|w_j|_\infty\int_0^T\int_\Omega |\frac{x_1^{m_1+1}}{(x_1+D_0)^{m_1}}x_2^{m_2}-\frac{x_{1n}^{m_1+1}}{(x_{1n}+D_0)^{m_1}}x_{2n}^{m_2}|^2 d\boldsymbol{x}dt \\
\leq\ & C\int_0^T\int_\Omega\left|\left(\frac{x_1}{x_1+D_0}\right)^{m_1}(x_2^{m_2}-x_{2n}^{m_2})\right|^2 dt \\
+\ & C\int_0^T\int_\Omega\left|\left(\left(\frac{x_1}{x_1+D_0}\right)^{m_1}-\left(\frac{x_{1n}}{x_{1n}+D_0}\right)^{m_1}\right)(x_{2n}^{m_2})\right|^2 dt \\
\leq\ & C\int_0^T |x_1-x_{1n_j}|_2^2+|x_2-x_{2n_j}|_2^2 dt \\
\leq\ & C|x_1-x_{1n_j}|_{L^2(0,T;L^2(\Omega))}+C|x_2-x_{2n_j}|_{L^2(0,T;L^2(\Omega))} \\
\leq\ & C(0+0) \\
=\ & 0
\end{aligned}\tag{83}$$

This follows via the lipscitz property of the nonlinearity and because we have demonstrated

$$x_{1n_j}\to x_1 \text{ strongly in } L^2(0,T;L^2(\Omega)),\tag{84}$$

$$x_{2n_j}\to x_2 \text{ strongly in } L^2(0,T;L^2(\Omega)),\tag{85}$$

Thus the convergence of the nonlinear term has been established. The convergence of the quadratic term can handled similarly. Essentially

$$\begin{aligned}
&\left|\lim_{j\to\infty}\int_0^T\int_\Omega x_{1n_j}^2\phi(t)w_j d\mathbf{x}dt - \int_0^T\int_\Omega x_1^2\phi(t)w_j d\mathbf{x}dt\right| \\
\leq\ & C\int_0^T\int_\Omega |x_{1n_j}^2\phi(t)w_j - x_1^2\phi(t)w_j|^2 d\mathbf{x}dt \\
\leq\ & C|\phi|_\infty|w_j|_\infty\int_0^T\int_\Omega |x_{1n_j}^2 - x_1^2|^2 d\mathbf{x}dt \\
\leq\ & C|x_2 + x_{2n_j}|_\infty^2\int_0^T |x_1 - x_{1n_j}|_2^2 dt \\
\leq\ & C|x_1 - x_{1n_j}|^2_{L^2(0,T;L^2(\Omega))} \\
\leq\ & 0
\end{aligned} \tag{86}$$

This follows from the compact Sobolev embedding of

$$H_0^1(\Omega) \hookrightarrow L^2(\Omega) \tag{87}$$

and from

$$x_{1n_j} \to x_1 \text{ strongly in } L^2(0,T;L^2(\Omega)), \tag{88}$$

Now taking the limit as $j \to \infty$ in equation (81) we obtain

$$\begin{aligned}
& \lim_{j\to\infty}\int_0^T (x_{1n_j}, \phi'(t)w_j)dt + \int_0^T (\nabla x_{1n_j}, \nabla w_j\phi)dt + a_1\int_0^T (x_{1n_j}, \phi w_j)dt \\
- \ & b_1\int_0^T ({x_{1n_j}}^2, \phi w_j)dt - \int_0^T (F(x_{1n_j}, x_{2n_j}), \phi w_j)dt \\
= \ & \int_0^T (x_1, \phi'(t)w_j)dt + \int_0^T (\nabla x_1, \nabla w_j\phi)dt - a_1\int_0^T (x_1, \phi w_j)dt + b_1\int_0^T (x_1^2, \phi w_j)dt \\
- \ & \int_0^T (F(x_1, x_2), \phi w_j)dt \\
= \ & 0
\end{aligned} \tag{89}$$

This implies that we have continuity with respect to w_j. Thus we obtain that for any $v \in H_0^1(\Omega)$ the following holds

$$\begin{aligned}
& -\int_0^T \left(x_1, \phi'(t)v\right) dt + \int_0^T (\nabla x_1, \nabla v\phi(t))\, dt - a_1\int_0^T (x_1, \phi(t)v)\, dt \\
& + b_1\int_0^T \left(x_1^2, \phi(t)v\right) dt \\
& = \int_0^T (F(x_1, x_2), \phi(t)v)\, dt.
\end{aligned} \tag{90}$$

This yields the existence of x_1 such that the following is true in a distributional sense

$$\frac{d}{dt}(x_1, v) + (\nabla x_1, \nabla v) - a_1(x_1, v) + b_1(x_1^2, v) = (F(x_1, x_2), v) \tag{91}$$

In other words there exists a weak solution x_1 to (1). Since

$$x_1 \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)), \tag{92}$$

and

$$\frac{\partial x_1}{\partial t} \in L^2(0, T; H^{-1}(\Omega)), \tag{93}$$

it follows via standard theory such as Lions-Aubin compactness Lemma [T97] that

$$x_1 \in C([0, T]; L^2(\Omega)). \tag{94}$$

This establishes that the solution belongs to the requisite functional spaces. The convergence for x_2 and x_3 is handled similarly. We can now state the following Theorem.

Theorem 3.4 *Consider the diffusive three species aquatic model as defined via* (1)- (3)*. For initial data in* $L^2(\Omega) \cap L^\infty(\Omega)$*, and any time* T*, there exists a unique weak solution* (x_1, x_2, x_3) *to the system such that*

$$(x_1, x_2, x_3) \in L^\infty(0, T; L^2(\Omega)) \cap L^2(0, T; H_0^1(\Omega)) \cap C([0, T]; L^2(\Omega)) \tag{95}$$

and

$$\frac{\partial x_1}{\partial t} \in L^2(0, T; H^{-1}(\Omega)), \tag{96}$$

Furthermore (x_1, x_2, x_3) *are continuous with respect to initial data.*

The continuity w.r.t initial data and uniqueness are shown by standard methods.

4 Mild and Strong Solutions

In this section we show that the diffusive three species aquatic model in addition to posessing a unique weak solution, posesses a unique mild solution which is also a strong solution. This is shown via additional requirements on the nonlinearity. We follow the approach of You and Sell in [SY02].

Lemma 4.1 (Sell, You) *Consider the nonlinear evolutionary equation*

$$\frac{\partial u}{\partial t} + Au = F(u,t),\ u(t_0) = u_0 \in W \tag{97}$$

Let the standing Hypothesis A be satisfied and let

$$F \in C_{Lip} = C_{Lip}(V^{2\beta} \times \mathbb{R}^+; W) \tag{98}$$

Where $0 \leq \beta < 1$ and $0 < \theta \leq 1$. Then for every $u_0 \in V^{2\beta}$, there is a $\tau > 0$ such that the above evolutionary equation has a unique mild solution $u = u(t)$ in $V^{2\beta}$ with

$$u \in C[t_0, t_0+\tau; V^{2\beta}) \cap C^{0,\theta_1}_{loc}(t_0, t_0+\tau; V^{2\alpha}) \cap C^{0,\theta}_{loc}(t_0, t_0+\tau; V^{2r}) \tag{99}$$

for all α and r with $0 \leq \alpha \leq \beta$ and $0 \leq r < 1$, where $\theta_1 > 0$ and $\theta > 0$.

Remark 4.2 *$C^{k,\lambda}$ is taken to be the standard function space of Holder continuous functions with Holder exponent λ*

We now attempt to apply some of the above theory to our system of interest. Recall the nonlinearity for (1),

$$F(x_1,t) = -b_1 x_1^2 - w_0 \left(\frac{x_1}{x_1+D_0}\right)^{m_1} x_2^{m_2} \tag{100}$$

We can make the following estimate for $m_1, m_2 \leq 3$

$$\begin{aligned} & |F(x_1,t)|_2 \\ = \ & \left|-b_1 x_1^2 - w_0 \left(\frac{x_1}{x_1+D_0}\right)^{m_1} x_2^{m_2}\right|_2 \\ \leq \ & b_1|x_1|_4^2 + w_0 \left|\left(\frac{x_1}{x_1+D_0}\right)^{m_1} x_2^{m_2}\right|_2 \\ \leq \ & b_1|x_1|_4^2 + w_0\,|x_2|_{2m_2}^{m_2} \\ \leq \ & C_1|x_1|_{H_0^1} + C_2|x_2|_{H_0^1}^{\frac{3}{2}} \\ \leq \ & K_0 \end{aligned} \tag{101}$$

This follows from the uniform H_0^1 estimates of x_1 and x_2 and the compact Sobolev embedding of

$$H_0^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega) \tag{102}$$

We now make the following estimate again for $m_1, m_2 \leq 3$

$$\begin{aligned}
& |F(x_1,t) - F(x_1^*,t)|_2 \\
\leq\ & |b_1(x_1^2 - (x_1^*)^2|_2 + w_0 \left| \left(\frac{x_1}{x_1 + D_0}\right)^{m_1} x_2^{m_2} - \left(\frac{x_1^*}{x_1^* + D_0}\right)^{m_1} (x_2^*)^{m_2} \right|_2 \\
\leq\ & C|x_1 - x_1^*|_2 + \left| \left(\frac{x_1}{x_1 + D_0}\right)^{m_1} (x_2^{m_2} - (x_2^*)^{m_2}) \right|_2 \\
& + \left| \left(\left(\frac{x_1}{x_1 + D_0}\right)^{m_1} - \left(\frac{x_1^*}{x_1^* + D_0}\right)^{m_1} \right) (x_2^*)^{m_2} \right|_2 \\
\leq\ & C_1|x_1 - x_1^*|_{H_0^1(\Omega)} + C_2|(x_2 - x_2^*)(x_2^{m_2-1} - (x_2^*)^{m_2-1})|_2 + K^{m_2}|x_1 - x_1^*|_2 \\
\leq\ & C_1|x_1 - x_1^*|_{H_0^1(\Omega)} + C_2|x_2 - x_2^*|_2 + C_3|x_1 - x_1^*|_2 \\
\leq\ & C|x_1 - x_1^*|_{H_0^1(\Omega)}
\end{aligned} \tag{103}$$

This follows from the uniform $H_0^1(\Omega)$ estimates of x_1 and x_2, the compact Sobolev embedding of

$$H_0^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^4(\Omega), \tag{104}$$

and because of the lipscitz property of $\left(\frac{x_1}{x_1+D_0}\right)^{m_1}$.

Theorem 4.3 *Consider the three species aquatic model. For every $(x_{10}, x_{20}, x_{30}) \in H_0^1$, there is a $\tau > 0$ such that the system equation has a unique mild solution (x_1, x_2, x_3) in H_0^1. Furthermore the solution has the following regularity*

$$(x_1, x_2, x_3) \in C[t_0, t_0 + \tau; H_0^1(\Omega)) \cap C_{loc}^{0,\theta_1}(t_0, t_0 + \tau; H_0^1(\Omega)) \cap C_{loc}^{0,\theta}(t_0, t_0 + \tau; H^{2-\delta}(\Omega)) \tag{105}$$

for all α and r with $0 \leq \alpha \leq \beta$ and $0 \leq r < 1$, where $\theta_1 > 0$, $\theta > 0$, $0 < \delta << 1$ and $t_0 > t^*$ [2].

Proof 4.4 *In our case the operator $A = -\Delta$. Thus the standing hypothesis A is satisfied. Furthermore with $W = L^2(\Omega)$, $\beta = \frac{1}{2}$, thus $V^{2\beta} = H_0^1(\Omega)$, we can now use the above estimates and in conjunction with Lemma 4.1 to prove the Theorem.*

[2] Here t^* is the same as appears in (44)

Lemma 4.5 (Sell, You) *Consider the nonlinear evolutionary equation*

$$\frac{\partial u}{\partial t} + Au = F(u,t),\ u(t_0) = u_0 \in W \tag{106}$$

Let the standing Hypothesis A be satisfied and let

$$F \in C_{Lip;\theta} = C_{Lip;\theta}(V^{2\beta} \times \mathbb{R}^+; W) \tag{107}$$

Where $0 \leq \beta < 1$ *and* $0 < \theta \leq 1$*. Then for every* $u_0 \in V^{2\beta}$*, there is a* $T > 0$ *such that the mild solution of the above evolutionary equation in* $V^{2\beta}$*, is a unique strong solution in* $V^{2\beta}$ *with*

$$u \in C[0,T;V^{2\alpha}) \cap C^{0,1-r}_{loc}(0,T;V^{2r}) \cap C(0,T;D(A)) \tag{108}$$

for all α *and* r *with* $0 \leq \alpha \leq \beta$ *and* $0 \leq r < 1$*.*

We now state the following Lemma

Lemma 4.6 *Consider* (1) *in the diffusive three species aquatic model. The following uniform estimate for the nonlinear term holds, for* $m_1, m_2 \leq 3$

$$|F(x_1,t_1) - F(x_1^*,t_2)|_2 \leq C(|x_1 - x_2|_2 + |t_1 - t_2|),\ \textit{for}\ t_1, t_2 \geq \max(t^*, t_1^*) \tag{109}$$

Proof 4.7

$$\begin{aligned}
& |F(x_1,t_1) - F(x_1^*,t_2)|_2 \\
\leq\ & |b_1(x_1^2(t_1) - (x_1^* t_2))^2|_2 \\
& +w_0 \left| \left(\frac{x_1(t_1)}{x_1(t_1)+D_0}\right)^{m_1} x_2(t_1)^{m_2} - \left(\frac{x_1^*(t_2)}{x_1^*(t_2)+D_0}\right)^{m_1} x_2^*(t_2)^{m_2} \right|_2 \\
\leq\ & C|x_1 - x_1^*|_2 + \left| \left(\frac{x_1(t_1)}{x_1(t_1)+D_0}\right)^{m_1} (x_2(t_1)^{m_2} - x_2^*(t_2)^{m_2}) \right|_2 \\
+\ & \left| \left(\left(\frac{x_1(t_1)}{x_1(t_1)+D_0}\right)^{m_1} - \left(\frac{x_1^*(t_2)}{x_1^*(t_2)+D_0}\right)^{m_1} \right) (x_2^*(t_2)^{m_2}) \right|_2 \\
\leq\ & C_1|x_1 - x_1^*|_2 + |(x_2(t_1)^{m_2} - x_2^*(t_2)^{m_2})|_2 + |x_2^*(t_2)^{m_2})|_2 \\
\leq\ & C_1|x_1 - x_1^*|_2 + C_2|x_2(t_1) - x_2^*(t_2)|_2 + |x_2^*(t_2)|^{m_2}_{2m_2} \\
\leq\ & C_1|x_1 - x_1^*|_2 + C_2|x_2(t_1) - x_2(t_2) + x_2(t_2) - x_2^*(t_2)|_2 + |\nabla x_2^*(t_2)|_2 \\
\leq\ & C_1|x_1 - x_1^*|_2 + C_2|x_2(t_1) - x_2(t_2)|_2 + |x_2(t_2) - x_2^*(t_2)|_2 + |\nabla x_2^*(t_2)|_2 \\
\leq\ & C_1|x_1 - x_1^*|_2 + C_2\left|\frac{\partial x_2}{\partial t}\right|_2 |t_1 - t_2| + C_3 \\
\leq\ & C|x_1 - x_1^*|_2 + C|t_1 - t_2|
\end{aligned} \tag{110}$$

This follows via the uniform H_0^1 *estimates of* x_2 *and* $\frac{\partial x_2}{\partial t}$ *and the compact Sobolev embedding of*

$$H_0^1(\Omega) \hookrightarrow L^6(\Omega) \hookrightarrow L^2(\Omega) \tag{111}$$

This proves the Lemma.

We can now state the following theorem concerning strong solution to the diffusive three species aquatic model

Theorem 4.8 *Consider the diffusive three species aquatic model,* (1)- (3). *Then for every* $u_0 \in H_0^1(\Omega)$*, there is a* $T > 0$ *such that the mild solution of the system in* $H_0^1(\Omega)$*, is a unique strong solution in* $H_0^1(\Omega)$ *with*

$$(x_1, x_2, x_3) \in C[t_0, T; H_0^1(\Omega)) \cap C_{loc}^{t_0,1-r}(0, T; H^{2-\delta}(\Omega)) \cap C(t_0, T; D(A)) \tag{112}$$

for all α *and* r *with* $0 \leq \alpha \leq \beta$ *and* $0 \leq r < 1$, $0 < \delta << 1$ *and* $t_0 > \max(t^*, t_1^*)$ [3].

Proof 4.9 *We have that with* $A = -\Delta$ *the standing hypothesis* A *is satisfied. Furthermore with* $W = L^2(\Omega)$, $\beta = \frac{1}{2}$ *thus* $V^{2\beta} = H_0^1(\Omega)$*, we can use the above estimates and apply the results of the Lemma 4.5 to prove the Theorem. Thus the Theorem is a direct consequence of Lemma 4.6 in conjunction with Lemma 4.5.*

Remark 4.10 *Existence of mild with further regularity, thus strong solutions guarantees uniqueness. See [SY02]*

5 References

[M87] R.M. May *Chaos and the dynamics of biological populations* Proc. R. Soc. Lond., **413** (1987), 2744

[M01] R.M.May, "Stability and Complexity in Model Ecosystems" Princeton University Press, Princeton,NJ, 2001

[BSM97] J.Bascompte, R.Sole and N.Martinez *Population cycles and spatial patterns in snowshoe hares: An individual-oriented simulation* J. Theor. Biol. **187** ,213222, 1997

[3] Here t^* is the same as appears in (44). t_1^* is the similar explicit time in the computations for $|\nabla x_{2n}|_2$

[MRRJ06] S. Mandal, S. Ray, S. Roy, S. E. Jorgensen *Order to chaos and vice versa in an aquatic ecosystem* Ecological Modelling, **197**, (2006), 498504 . 113

[UR97] R.K. Upadhyay, V. Rai Why chaos is rarely observed in natural populations? Chaos Solitons and Fractals, **8** (12), pp. 19331939, 1997.

[UKR09] R.K. Upadhyay, N.Kumari, V.Rai *Wave of chaos in a diffusive system: Generating realistic patterns of patchiness in plankton-fish dynamics* Chaos Solitons and Fractals Vol. **40**(1), 262-276, 2009

[UNK07] R.K. Upadhyay, R.K. Naji, N. Kumari, *Dynamical complexity in some ecological models: effect of toxin production by phytoplankton* Nonlinear Analysis: Modelling and Control, **12**(1), pp. 123138, 2007.

[U08] R.K. Upadhyay *Chaotic Dynamics in a Three Species Aquatic Population Model with Holling Type II Functional Response* Nonlinear Analysis: Modelling and Control, 2008, Vol. **13**, No. 1, 103115

[DGO75] D. L. DeAngelis, R.A. Goldstein, R.V. ONeill *A model for trophic interaction* Ecology **56** pp. 881892, 1975.

[H04] T.W. Hwang *Uniqueness of limit cycles of the predator-prey system with Beddington- DeAngelis functional response* J. Math. Anal. Appl. **290**, pp. 113122, 2004.

[PG11] R.D.Parshad, J.Gutierrez *On the existence of global attractor for the Trojan Y Chromosome model* Comm. on pure and Appl. Anal. Vol 10-no.1, pp 339-359 , January 2011.

[BMT96] M. Begon, M. Mortimer, D. J. Thompson, “Population Ecology: A unified study of Animals and Plants” 3rd ed., Blackwell Science, Oxford, U.K., 1996.

[T97] R.Temam, “Infinite-Dimensional Dynamical Systems in Mechanics and Physics”. 2^{nd} ed. Applied Mathematical Sciences, 68. Springer, New York, 1997.

[E98] L.Evans, “Partial Differential Equations”. 2^{nd} ed. Graduate Studies in Mathematics, 19. Springer, New York, 1998.

[SY02] G.Sell, Y.You, “Dynamics of evolutionary equations”. 2^{nd} ed. Applied Mathematical Sciences, 143. Springer, New York, 2002.

[UIR98] R.K. Upadhyay, S.R.K. Iyengar, V. Rai *Chaos: an ecological reality? Int. J. Bifurcation and Chaos* **8**(6), pp. 13251333, 1998.

[HP91] A. Hastings, T. Powell *Chaos in three species food-chain* Ecology **72**, pp. 896903, 1991.

[A01] D. Alstad "Basic Populas Models of Ecology" Prentice-Hall, Englewood Cliffs, NJ, 2001.

[AF90] W. G. Aiello and H. I. Freedman *A time-delay model of single-species growth with stage structure* Mathematical Biosciences, vol. **101**, no. 2, pp. 139153, 1990.

[ZCN00] X. Zhang, L. Chen, and A. U. Neumann, *The stage-structured predator-prey model and optimal harvesting policy* Mathematical Biosciences, vol. **168**, no. 2, pp. 201210, 2000.

[ZGF09] R. Zhang, L. guo, and S.Fu, *Global behavior for a diffusive predator prey model with stage structure and nonlinear density:The case in* $\mathbb{R}^n$ Boundary Value Problems, vol. **2009**, 2009.

[LCL02] S. Liu, L. Chen, and Z. Liu, *Extinction and permanence in nonautonomous competitive system with stage structure* Journal of Mathematical Analysis and Applications, vol. **274**, no. 2, pp. 667684, 2002.

[L06] Z. Lin, *Time delayed parabolic system in a two-species competitive model with stage structure* Journal of Mathematical Analysis and Applications, vol. **315**, no. 1, pp. 202215, 2006.

[LN06] Y. Lou and W.M. Ni, *Diffusion, self-diffusion and cross-diffusion* Journal of Differential Equations, vol. **131**, no. 1, pp. 79131, 1996.

[KR06] W. Ko and K. Ryu, *Qualitative analysis of a predator-prey model with Holling type II functional response incorporating a prey refuge* Journal of Differential Equations, vol. **231**, no. 2, pp. 534550, 2006.

[LNW98] Y. Lou, W.M. Ni, and Y. Wu, *On the global existence of a cross-diffusion system* Discrete and Continuous Dynamical Systems, vol. **4**, no. 2, pp. 193203, 1998.

[P92] C.V.Pao "Nonlinear Parabolic and Elliptic Equations" Plenum Press, New York, 1992.

[S02] S.A. Shim, *Uniform boundedness and convergence of solutions to cross-diffusion systems* Journal of Differential Equations, vol. **185**, no. 1, pp. 281305, 2002.

[S03] S.A. Shim, *Uniform boundedness and convergence of solutions to the systems with cross-diffusions dominated by self-diffusions* Nonlinear Analysis: RealWorld Applications, vol. 4, no. 1, pp. 6586, 2003.

[ZGF09] R. Zhang, L. Guo,2 and S. Fu *Global Behavior for a Diffusive Predator-Prey Model with Stage Structure and Nonlinear Density Restriction-I: The Case in R^n* Boundary Value Problems, Volume 2009

[YF08] F. Yang and S. Fu, *Global solutions for a tritrophic food chain model with diffusion* The Rocky Mountain Journal of Mathematics, vol. **38**, no. 5, pp. 17851812, 2008.

[C01] J.M. Cushing, S. M.Henson, R. A.Desharnais, B.Dennis, R. F. Costantino, and A. A.King, *A chaotic attractor in ecology: Theory and experimental data* Chaos Solit. Fract. **12**, 219234, 2001.

[C03] J.M.Cushing, R. F.Costantino, B.Dennis, R. A. Desharnais, and S. M.Henson, "Chaos in Ecology:Experimental Non-Linear Dynamics" Elsevier, San Diego, 2003.

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