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Partial Sums of Certain Univalent Functions

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Abstract

The partial sums $f_3(z)$ of some extermal functions for various classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$ of starlike functions, convex functions and functions with positive real part in the open unit disk $\mathbb{U}$, respectively, are discussed. In general, the partial sums can not preserve the same character as the initial functions. The object of the present paper is to discuss the radius problems for partial sums of some extremal functions for the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$.

1 Introduction

Let $\mathcal{A}$ be the class of functions $f(z)$ of the form

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k \tag{1.1}$$

which are analytic in the open unit disk $\mathbb{U} = \{z \in \mathbb{C} : |z| < 1\}$.

If $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(\frac{zf'(z)}{f(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.2}$$

for some α $(0 \leqq \alpha < 1)$, then $f(z)$ is said to be starlike of order α in $\mathbb{U}$. We denote by $\mathcal{S}^*(\alpha)$ the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which are starlike of order α in $\mathbb{U}$. Similarly, we say that $f(z)$ is a member of the class $\mathcal{K}(\alpha)$ of convex functions of order α in $\mathbb{U}$ if $f(z) \in \mathcal{A}$ satisfies the following inequality

$$\operatorname{Re}\left(1 + \frac{zf''(z)}{f'(z)}\right) > \alpha \qquad (z \in \mathbb{U}) \tag{1.3}$$

for some α $(0 \leqq \alpha < 1)$. Also, let $\mathcal{R}(\alpha)$ be the subclass of $\mathcal{A}$ consisting of functions $f(z)$ which satisfy

$$\operatorname{Re}(f'(z)) > \alpha \quad (z \in \mathbb{U}) \tag{1.4}$$

for some α $(0 \leqq \alpha < 1)$. We note that

$$f(z) \in \mathcal{K}(\alpha) \quad \text{if and only if} \quad zf'(z) \in \mathcal{S}^*(\alpha) \tag{1.5}$$

and denote by $\mathcal{S}^*(0) \equiv \mathcal{S}^*$, $\mathcal{K}(0) \equiv \mathcal{K}$ and $\mathcal{R}(0) \equiv \mathcal{R}$. Furthermore, we remark that all functions $f(z) \in \mathcal{S}^*(\alpha)$, $\mathcal{K}(\alpha)$ or $\mathcal{R}(\alpha)$ are univalent in $\mathbb{U}$, respectively.

Example 1.1 We note thdat

$$f(z) = \frac{z}{(1-z)^2} = z + \sum_{k=2}^{\infty} k z^k \in \mathcal{S}^*, \tag{1.6}$$

$$f(z) = \frac{z}{1-z} = z + \sum_{k=2}^{\infty} z^k \in \mathcal{K} \tag{1.7}$$

and

$$f(z) = -z - 2\log(1-z) = z + \sum_{k=2}^{\infty} \frac{2}{k} z^k \in \mathcal{R}. \tag{1.8}$$

are extremal functions for the classes $\mathcal{S}^*$, $\mathcal{K}$ and $\mathcal{R}$.

In this paper, for $f(z) \in \mathcal{A}$, we introduce the partial sums $f_n(z)$ of $f(z)$ by

$$f_n(z) = z + \sum_{k=2}^{n} a_k z^k.$$

Remark 1.2 Generally, the partial sum cannot preserve the same character as the initial function. For example, we consider the partial sum $F(z) = \dfrac{z}{(1-z)^2} - 2z^2$ of the function $f(z)$ given by (1.6) is not univalent in $\mathbb{U}$ because $F(z_1) = F(z_2) = -1$ for two points $z_1 = \dfrac{1-i}{2} \in \mathbb{U}$ and $z_2 = \dfrac{1+i}{2} \in \mathbb{U}$.

For the partial sums $f_n(z)$ of $f(z) \in \mathcal{S}^*$, Szegö [4] proved the following result.

Theorem 1.3 *If $f(z) \in \mathcal{S}^*$, then $f_n(z) \in \mathcal{S}^*$ for $|z| < \dfrac{1}{4}$ and $f_n(z) \in \mathcal{K}$ for $|z| < \dfrac{1}{8}$. The result is sharp.*

Moreover, Padmanabhan [3] showed the following theorem.

Theorem 1.4 *If $f(z)$ is 2-valently starlike in $\mathbb{U}$, then $f_n(z)$ is 2-valently starlike for $|z| < \dfrac{1}{6}$. The result is sharp.*

The purpose of our systematic investigation is finding the radius r for the partial sums $f_n(z)$ of the extremal functions to be in the classes $\mathcal{R}(\alpha)$, $\mathcal{S}^*(\alpha)$ or $\mathcal{K}(\alpha)$, respectively.

2 The partial sums of some extremal functions to belong to the class $\mathcal{R}(\alpha)$

For the partial sum $f_3(z) = z + z^2 + z^3$ of the function $f(z)$ given by (1.7), we begin with considering the radius r for $f_3(z) \in \mathcal{R}(\alpha)$ which means that $f_3(z)$ belongs to the class $\mathcal{R}$ in $\mathbb{U}_r = \{z : |z| < r \ (0 < r \leqq 1)\}$.

Theorem 2.1 *Let $f_3(z) = z + z^2 + z^3$ be the partial sum of $f(z) = \dfrac{z}{1-z} \in \mathcal{K}$. Then,*

$$f_3(z) \in \mathcal{R} \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \dfrac{\sqrt{10}}{6} = 0.52704\cdots$. *Furthermore,*

$$f_3(z) \in \mathcal{R}(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} 1 - 2r + 3r^2 & \left(0 < r \leqq \dfrac{1}{6} = 0.16666\cdots\right) \\ \dfrac{5}{6} - 3r^2 & \left(\dfrac{1}{6} < r \leqq \dfrac{\sqrt{10}}{6} = 0.52704\cdots\right). \end{cases}$$

Proof. A simple computation gives us that

$$\mathrm{Re}(f_3'(z)) = 1 - 3r^2 + 2r\cos\theta + 6r^2\cos^2\theta$$

for $z = re^{i\theta}$. Letting

$$g(t) = 1 - 3r^2 + 2rt + 6r^2t^2 \quad (t = \cos\theta),$$

we see that $g'(t) = 2r(6rt+1) = 0$ for $t_1 = -\dfrac{1}{6r} < 0$. Thus,

$$g(t) \geqq g(-1) = 1 - 2r + 3r^2 =: \alpha(r) \geqq \alpha\left(\frac{1}{6}\right) = \frac{3}{4}$$

for the case $0 < r \leqq \dfrac{1}{6}$, and

$$g(t) \geqq g(t_1) = \frac{5}{6} - 3r^2 =: \alpha(r) \geqq \alpha\left(\frac{\sqrt{10}}{6}\right) = 0$$

for the case $\dfrac{1}{6} < r \leqq \dfrac{\sqrt{10}}{6}$. This completes the proof of the theorem. □

Next, we consider the partial sum $f_3(z) = z + z^3 + z^5$ of the odd starlike function $f(z)$.

Theorem 2.2 *Let* $f_3(z) = z + z^3 + z^5$ *be the partial sum of function*

$$f(z) = \frac{z}{1-z^2} = z + \sum_{k=2}^{\infty} z^{2k-1}$$

in the class $\mathcal{S}^*$. *Then,*

$$f_3(z) \in \mathcal{R} \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \dfrac{\sqrt[4]{1550}}{10} = 0.62745\cdots$. *Furthermore,*

$$f_3(z) \in \mathcal{R}(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} 1 - 3r^2 + 5r^4 & \left(0 < r \leqq \dfrac{\sqrt{15}}{10} = 0.38729\cdots\right) \\ \\ \dfrac{31}{40} - 5r^4 & \left(\dfrac{\sqrt{15}}{10} < r \leqq \dfrac{\sqrt[4]{1550}}{10} = 0.62745\cdots\right). \end{cases}$$

Proof. We see that

$$\mathrm{Re}((f_3'(z)) = 1 - 5R^2 + 3R\cos\varphi + 10R^2\cos^2\varphi$$

for $z^2 = r^2 e^{i2\theta} = Re^{i\varphi}$. Let the function $g(t)$ be given by

$$1 - 5r^4 + 3r^2 t + 10r^4 t^2 \quad (t = \cos\varphi).$$

Then, we know that $g'(t) = r^2(20r^2 t + 3) = 0$ for $t_1 = -\dfrac{3}{20r^2} < 0$. Therefore,

$$g(t) \geqq g(-1) = 1 - 3r^2 + 5r^4 =: \alpha(r) \geqq \alpha\left(\frac{\sqrt{15}}{10}\right) = \frac{53}{80}$$

for $0 < r \leqq \dfrac{\sqrt{15}}{10}$, and

$$g(t) \geqq g(t_1) = \frac{31}{40} - 5r^4 =: \alpha(r) \geqq \alpha\left(\frac{\sqrt[4]{1550}}{10}\right) = 0$$

for $\dfrac{\sqrt{15}}{10} < r \leqq \dfrac{\sqrt[4]{1550}}{10}$. The proof of the theorem is completed. □

In the same manner, the following results are obtained.

Theorem 2.3 *Let* $f_3(z) = z + z^2 + \dfrac{2}{3}z^3$ *be the partial sum of* $f(z) = -z - 2\log(1 - z) \in \mathcal{R}$. *Then,*

$$f_3(z) \in \mathcal{R} \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \dfrac{\sqrt{6}}{4} = 0.61237\cdots$. *Furthermore,*

$$f_3(z) \in \mathcal{R}(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} 1 - 2r + 2r^2 & \left(0 < r \leqq \dfrac{1}{4} = 0.25\right) \\[2ex] \dfrac{3}{4} - 2r^2 & \left(\dfrac{1}{4} < r \leqq \dfrac{\sqrt{6}}{4} = 0.61237\cdots\right). \end{cases}$$

Theorem 2.4 *Let* $f_3(z) = z+2z^2+3z^3$ *be the partial sum of* $f(z) = \dfrac{z}{(1-z)^2} \in \mathcal{S}^*$. *Then,*

$$f_3(z) \in \mathcal{R} \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \dfrac{\sqrt{7}}{9} = 0.29397\cdots$. *Furthermore,*

$$f_3(z) \in \mathcal{R}(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} 1 - 4r + 9r^2 & \left(0 < r \leqq \dfrac{1}{9} = 0.11111\cdots\right) \\[2ex] \dfrac{7}{9} - 9r^2 & \left(\dfrac{1}{9} < r \leqq \dfrac{\sqrt{7}}{9} = 0.29397\cdots\right). \end{cases}$$

3 Radius for the partial sums of some extremal functions to be in the class $\mathcal{S}^*(\alpha)$

Next, in this section, we first consider the partial sum $f_3(z) = z + z^2 + z^3$ of the function $f(z)$ given by (1.7) involving the results by Owa [1] and Owa, Srivasrava and Saito [2].

Theorem 3.1 *Let* $f_3(z) = z + z^2 + z^3$ *be the partial sum of* $f(z) = \dfrac{z}{1-z} \in \mathcal{K}$. *Then,*

$$f_3(z) \in \mathcal{S}^* \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \sqrt{\dfrac{23}{71}} = 0.56916\cdots$. *More strictly,*

$$f_3(z) \in \mathcal{S}^*(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{1-2r+3r^2}{1-r+r^2} & \left(0 < r \leqq \dfrac{7-3\sqrt{5}}{2} = 0.14589\cdots\right) \\ \dfrac{24(1+r^2+r^4)-(13+11r^2)\sqrt{3(1+r^2+r^4)}}{12(1+r^2+r^4)-6(1+r^2)\sqrt{3(1+r^2+r^4)}} & \left(\dfrac{7-3\sqrt{5}}{2} < r \leqq \sqrt{\dfrac{23}{71}} = 0.56916\cdots\right). \end{cases}$$

Proof. We consider α such that

$$\mathrm{Re}\left(\frac{zf_3'}{f_3(z)}\right) = 3 - \mathrm{Re}\left(\frac{2+z}{1+z+z^2}\right) > \alpha.$$

It follows that

$$\mathrm{Re}\left(\frac{2+z}{1+z+z^2}\right) = 1 + \frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+2r(1+r^2)\cos\theta+4r^2\cos^2\theta} < 3-\alpha$$

for $z = re^{i\theta}$, which implies that

$$\frac{(1-r^2)(1+r^2+r\cos\theta)}{1-r^2+r^4+2r(1+r^2)\cos\theta+4r^2\cos^2\theta} < 2-\alpha.$$

Now, setting the function $g(t)$ by

$$g(t) = \frac{(1-r^2)(1+r^2+rt)}{1-r^2+r^4+2r(1+r^2)t+4r^2t^2} \quad (t = \cos\theta),$$

we see that

$$g'(t) = \frac{-r(1+r)(1-r)\{1+5r^2+r^4+8r(1+r^2)t+4r^2t^2\}}{\{1-r^2+r^4+2r(1+r^2)t+4r^2t^2\}^2} = 0$$

for $t = t_1$ and $t = t_2$ $(t_1 > t_2)$, where $t_2 < -1$. Since

$$t_1 = \frac{-2(1+r^2)+\sqrt{3(1+r^2+r^4)}}{2r} < 0,$$

we discuss the following two cases (i) $t_1 \leqq -1$ and (ii) $-1 < t_1 < 1$. If $0 < r \leqq \dfrac{7-3\sqrt{5}}{2}$, then the case (i) holds true, so that

$$g(t) \leqq g(-1) = G_1(r) = \frac{1-r^2}{1-r+r^2}.$$

Therefore, we have that

$$\alpha(r) = 2 - G_1(r) = \frac{1-2r+3r^2}{1-r+r^2} \geqq \frac{4-\sqrt{5}}{2} = 0.88196\cdots$$

for $0 < r \leqq \dfrac{7-3\sqrt{5}}{2}$. Similarly, if $\dfrac{7-3\sqrt{5}}{2} < r \leqq \sqrt{\dfrac{23}{71}}$, then the case (ii) holds true, so that

$$g(t) \leqq g(t_1) = G_2(r) = \frac{(1-r^2)\sqrt{3(1+r^2+r^4)}}{12(1+r^2+r^4)-6(1+r^2)\sqrt{3(1+r^2+r^4)}}.$$

Therefore, we obtain that

$$\alpha(r) = 2 - G_2(r) = \frac{24(1+r^2+r^4)-(13+11r^2)\sqrt{3(1+r^2+r^4)}}{12(1+r^2+r^4)-6(1+r^2)\sqrt{3(1+r^2+r^4)}} \geqq 0$$

for $\dfrac{7-3\sqrt{5}}{2} < r \leqq \sqrt{\dfrac{23}{71}}$. □

Next, we derive

Theorem 3.2 *Let $f_3(z) = z + z^3 + z^5$ be the partial sum of $f(z) = \dfrac{z}{1-z^2} \in \mathcal{S}^*$. Then,*

$$f_3(z) \in \mathcal{S}^* \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \sqrt[4]{\dfrac{2}{11}} = 0.65299\cdots$. *More strictly,*

$$f_3(z) \in \mathcal{S}^*(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{1-3r^2+5r^4}{1-r^2+r^4} & \left(0 < r \leqq \dfrac{3-\sqrt{5}}{2} = 0.38196\cdots\right) \\[2ex] \dfrac{36(1+r^4+r^8)-4(5+4r^4)\sqrt{3(1+r^4+r^8)}}{12(1+r^4+r^8)-6(1+r^4)\sqrt{3(1+r^4+r^8)}} & \left(\dfrac{3-\sqrt{5}}{2} < r \leqq \sqrt[4]{\dfrac{2}{11}} = 0.65299\cdots\right). \end{cases}$$

Proof. We consider α such that

$$\operatorname{Re}\left(\frac{zf_3'(z)}{f_3(z)}\right) = 5 - \operatorname{Re}\left(\frac{2(2+z^2)}{1+z^2+z^4}\right) > \alpha$$

which is equivalent to

$$\operatorname{Re}\left(\frac{2+z^2}{1+z^2+z^4}\right) = 1 + \frac{(1-R^2)(1+R^2+R\cos\varphi)}{1-R^2+R^4+2R(1+R^2)\cos\varphi+4R^2\cos^2\varphi} < \frac{5-\alpha}{2}$$

for $z^2 = r^2 e^{i2\theta} = Re^{i\varphi}$. It follows that

$$\frac{(1-R^2)(1+R^2+R\cos\varphi)}{1-R^2+R^4+2R(1+R^2)\cos\varphi+4R^2\cos^2\varphi} < \frac{3-\alpha}{2}.$$

Letting

$$g(t) = \frac{(1-R^2)(1+R^2+Rt)}{1-R^2+R^4+2R(1+R^2)t+4R^2t^2} \quad (t=\cos\varphi),$$

and following the help of the proof of Theorem 3.1, we have

$$g(t) \leqq G_1(R) = \frac{1-R^2}{1-R+R^2},$$

that is,

$$\alpha(r) = 3 - 2G_1(R) = \frac{1-3R+5R^2}{1-R+R^2} \geqq 3-\sqrt{5} = 0.76393\cdots$$

for $0 < R = r^2 \leqq \dfrac{7-3\sqrt{5}}{2}$, and

$$g(t) \leqq G_2(R) = \frac{(1-R^2)\sqrt{3(1+R^2+R^4)}}{12(1+R^2+R^4)-6(1+R^2)\sqrt{3(1+R^2+R^4)}},$$

that is,

$$\alpha(r) = 3-2G_2(R) = \frac{36(1+R^2+R^4)-4(5+4R^2)\sqrt{3(1+R^2+R^4)}}{12(1+R^2+R^4)-6(1+R^2)\sqrt{3(1+R^2+R^4)}} \geqq 0$$

for $\dfrac{7-3\sqrt{5}}{2} < R = r^2 \leqq \sqrt{\dfrac{2}{11}}$. □

We also prove

Theorem 3.3 *Let $f_3(z) = z + z^2 + \frac{2}{3}z^3$ be the partial sum of $f(z) = -z - 2\log(1-z) \in \mathcal{R}$. Then,*

$$f_3(z) \in \mathcal{S}^* \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \sqrt{\dfrac{37}{78}} = 0.68873\cdots$. More strictly,

$$f_3(z) \in \mathcal{S}^*(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{3(1-2r+2r^2)}{3-3r+2r^2} & \left(0 < r \leqq \dfrac{13-\sqrt{145}}{4} = 0.23960\cdots\right) \\[2ex] \dfrac{80(9+3r^2+4r^4)-(69+34r^2)\sqrt{10(9+3r^2+4r^4)}}{40(9+3r^2+4r^4)-10(3+2r^2)\sqrt{10(9+3r^2+4r^4)}} \\[2ex] & \left(\dfrac{13-\sqrt{145}}{4} < r \leqq \sqrt{\dfrac{37}{78}} = 0.68873\cdots\right). \end{cases}$$

Proof. Let us consider α shch that

$$\mathrm{Re}\left(\frac{zf_3'(z)}{f_3(z)}\right) = 3 - \mathrm{Re}\left(\frac{2+z}{1+z+\frac{2}{3}z^2}\right) > \alpha$$

which implies that

$$\frac{9-4r^4+3r(3-2r^2)\cos\theta}{9-3r^2+4r^4+6r(3+2r^2)\cos\theta+24r^2\cos^2\theta} < 2-\alpha.$$

Thus, if we define the function $g(t)$ by

$$g(t) = \frac{9-4r^4+3r(3-2r^2)t}{9-3r^2+4r^4+6r(3+2r^2)t+24r^2t} \quad (t=\cos\theta),$$

then we see that

$$g'(t) = \frac{-3r(3-2r^2)\{24r^2t^2+16r(3+2r^2)+9+27r^2+4r^4\}}{\{9-3r^2+4r^4+6r(3+2r^2)t+24r^2t\}^2} = 0$$

for $t=t_1$ and $t=t_2$ $(t_1>t_2)$, where $t_2<-1$. Since

$$t_1 = \frac{-4(3+2r^2)+\sqrt{10(9+3r^2+4r^4)}}{12r},$$

we need to separate our problem in the following two cases (i) $t_1 \leqq -1$ and (ii) $-1<t_1<1$. If $0<r\leqq \dfrac{13-\sqrt{145}}{4}$, then $t_1 \leqq -1$, so that

$$g(t) \leqq g(-1) = G_1(r) = \frac{3-2r^2}{3-3r+2r^2}.$$

This means that

$$\alpha(r) = 2 - G_1(r) = \frac{3(1-2r+2r^2)}{3-3r+2r^2} \geqq \frac{20-\sqrt{145}}{10} = 0.79584\cdots$$

for $0 < r \leqq \dfrac{13-\sqrt{145}}{4}$. If $\dfrac{13-\sqrt{145}}{4} < r \leqq \sqrt{\dfrac{37}{78}}$, then $-1 < t_1 < 1$, so that

$$g(t) \leqq g(t_1) = G_2(r) = \frac{3(3-2r^2)\sqrt{10(9+3r^2+4r^4)}}{40(9+3r^2+4r^4)-10(3+2r^2)\sqrt{10(9+3r^2+4r^4)}}.$$

This shows that

$$\alpha(r) = 2 - G_2(r) = \frac{80(9+3r^2+4r^4)-(69+34r^2)\sqrt{10(9+3r^2+4r^4)}}{40(9+3r^2+4r^4)-10(3+2r^2)\sqrt{10(9+3r^2+4r^4)}} \geqq 0$$

for $\dfrac{13-\sqrt{145}}{4} < r \leqq \sqrt{\dfrac{37}{78}}$. Thus we complete the proof of the theorem. □

Further, we show

Theorem 3.4 *Let $f_3(z) = z+2z^2+3z^3$ be the partial sum of $f(z) = \dfrac{z}{(1-z)^2} \in \mathcal{S}^*$. Then,*

$$f_3(z) \in \mathcal{S}^* \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \sqrt{\dfrac{5}{47}} = 0.32616\cdots$. More strictly,

$$f_3(z) \in \mathcal{S}^*(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{1-4r+9r^2}{1-2r+3r^2} & \left(0 < r \leqq \dfrac{5-\sqrt{22}}{3} = 0.10319\cdots\right) \\[2ex] \dfrac{24(1+2r^2+9r^4)-3(3+7r^2)\sqrt{6(1+2r^2+9r^4)}}{12(1+2r^2+9r^4)-4(1+3r^2)\sqrt{6(1+2r^2+9r^4)}} & \left(\dfrac{5-\sqrt{22}}{3} < r \leqq \sqrt{\dfrac{5}{47}} = 0.32616\cdots\right). \end{cases}$$

Proof. We seek α shch that

$$\mathrm{Re}\left(\frac{zf_3'(z)}{f_3(z)}\right) = 3 - \mathrm{Re}\left(\frac{2(1+z)}{1+2z+3z^2}\right) > \alpha.$$

Then, it follows that

$$g(t) = \frac{1-9r^4+2r(1-3r^2)t}{1-2r^2+9r^4+4r(1+3r^2)t+12r^2t^2} < 2-\alpha \quad (t = \cos\theta)$$

for $z = re^{i\theta}$. It is easy to check that

$$g'(t) = \frac{-2r(1-3r^2)\{12r^2t^2 + 12r(1+3r^2)t + 1 + 14r^2 + 9r^4\}}{\{1 - 2r^2 + 9r^4 + 4r(1+3r^2)t + 12r^2t^2\}^2} = 0$$

for $t = t_1$ and $t = t_2$ ($t_1 > t_2$), where $t_2 < -1$. Since

$$t_1 = \frac{-3(1+3r^2) + \sqrt{6(1+2r^2+9r^4)}}{6r},$$

we consider the radius r in the two cases (i) $t_1 \leqq -1$ and (ii) $-1 < t_1 < 1$. Then, we obtain that

$$g(t) \leqq g(-1) = G_1(r) = \frac{1-3r^2}{1-2r+3r^2}$$

which implies that

$$\alpha(r) = 2 - G_1(r) = \frac{1-4r+9r^2}{1-2r+3r^2} \geqq \frac{8-\sqrt{22}}{4} = 0.82739\cdots$$

for $0 < r \leqq \dfrac{5-\sqrt{22}}{3}$, and

$$g(t) \leqq g(t_1) = G_2(r) = \frac{(1-3r^2)\sqrt{6(1+2r^2+9r^4)}}{12(1+2r^2+9r^4) - 4(1+3r^2)\sqrt{6(1+2r^2+9r^4)}}$$

which gives us that

$$\alpha(r) = 2 - G_2(r) = \frac{24(1+2r^2+9r^4) - 3(3+7r^2)\sqrt{6(1+2r^2+9r^4)}}{12(1+2r^2+9r^4) - 4(1+3r^2)\sqrt{6(1+2r^2+9r^4)}} \geqq 0$$

for $\dfrac{5-\sqrt{22}}{3} < r \leqq \sqrt{\dfrac{5}{47}}$. □

4 Radius for the partial sums of some extremal functions to be in the class $\mathcal{K}(\alpha)$

We discuss some radius problems for the partial sums for some extremal functions to be in the class $\mathcal{K}(\alpha)$.

Theorem 4.1 *Let $f_3(z) = z + z^2 + z^3$ be the partial sum of $f(z) = \dfrac{z}{1-z} \in \mathcal{K}$. Then,*

$$f_3(z) \in \mathcal{K} \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \sqrt{\dfrac{5}{47}} = 0.32616\cdots$. *In particular,*

$$f_3(z) \in \mathcal{K}(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{1-4r+9r^2}{1-2r+3r^2} & \left(0 < r \leqq \dfrac{5-\sqrt{22}}{3} = 0.10319\cdots\right) \\[2ex] \dfrac{24(1+2r^2+9r^4)-3(3+7r^2)\sqrt{6(1+2r^2+9r^4)}}{12(1+2r^2+9r^4)-4(1+3r^2)\sqrt{6(1+2r^2+9r^4)}} & \\ & \left(\dfrac{5-\sqrt{22}}{3} < r \leqq \sqrt{\dfrac{5}{47}} = 0.32616\cdots\right). \end{cases}$$

The proof is the same as Theorem 3.4 because we can apply the relation (1.5) to Theorem 3.4 and it follows that the function $zf_3'(z) = z + 2z^2 + 3z^3$ for the function $f_3(z)$ in Theorem 4.1.

Theorem 4.2 *Let* $f_3(z) = z + z^3 + z^5$ *be the partial sum of* $f(z) = \dfrac{z}{1-z^2} \in \mathcal{S}^*$. *Then,*

$$f_3(z) \in \mathcal{K} \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \sqrt[4]{\dfrac{2}{65}} = 0.41882\cdots$. *In particular,*

$$f_3(z) \in \mathcal{K}(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{1-9r^2+25r^4}{1-3r^2+5r^4} & \left(0 < r \leqq \sqrt{\dfrac{31-\sqrt{781}}{30}} = 0.31904\cdots\right) \\[2ex] \dfrac{660(1+r^4+25r^8)-6(7+20r^4)\sqrt{220(1+r^4+25r^8)}}{220(1+r^4+25r^8)-11(1+5r^4)\sqrt{220(1+r^4+25r^8)}} & \\ & \left(\sqrt{\dfrac{31-\sqrt{781}}{30}} < r \leqq \sqrt[4]{\dfrac{2}{65}} = 0.41882\cdots\right). \end{cases}$$

Proof. We examine α such that

$$\mathrm{Re}\left(1+\frac{zf_3''(z)}{f_3'(z)}\right)=5-\mathrm{Re}\left(\frac{2(2+3z^2)}{1+3z^2+5z^4}\right)>\alpha$$

that is, that

$$g(t)=\frac{1-25R^4+3R(1-5R^2)t}{1-R^2+25R^4+6R(1+5R^2)t+20R^2t^2}<\frac{3-\alpha}{2}\qquad(t=\cos\varphi)$$

for $z^2=r^2e^{i2\theta}=Re^{i\varphi}$. Then, it follows that

$$g'(t)=\frac{-R(1-5R^2)\{60R^2t^2+40R(1+5R^2)t+3(1+21R^2+25R^4)\}}{\{1-R^2+25R^4+6R(1+5R^2)t+20R^2t^2\}^2}=0$$

for $t=t_1$ and $t=t_2$ $(t_1>t_2)$, where $t_2<-1$. Since

$$t_1=\frac{-20(1+5R^2)+\sqrt{220(1+R^2+25R^4)}}{60R},$$

we have to discuss the following two cases (i) $t_1\leqq -1$ and (ii) $-1<t_1<1$. Then,

$$g(t)\leqq g(-1)=G_1(R)=\frac{1-5R^2}{1-3R+5R^2}$$

which shows that

$$\alpha(r)=3-2G_1(R)=\frac{1-9R+25R^2}{1-3R+5R^2}\geqq\frac{33-\sqrt{781}}{11}=0.45942\cdots$$

for $0<R=r^2\leqq\dfrac{31-\sqrt{781}}{30}$, and

$$g(t)\leqq g(t_1)=G_2(R)=\frac{9(1-5R^2)\sqrt{220(1+R^2+25R^4)}}{440(1+R^2+25R^4)-22(1+5R^2)\sqrt{1+R^2+25R^4}}$$

which means that

$$\alpha(r)=3-2G_2(R)=\frac{660(1+R^2+25R^4)-6(7+20R^2)\sqrt{220(1+R^2+25R^4)}}{220(1+R^2+25R^4)-11(1+5R^2)\sqrt{220(1+R^2+25R^4)}}\geqq 0$$

for $\dfrac{31-\sqrt{781}}{30}<R=r^2\leqq\sqrt{\dfrac{2}{65}}$. This completes the proof of the theorem. □

In precisely the same process as detailed above, the following results are established.

Theorem 4.3 *Let* $f_3(z)=z+z^2+\frac{2}{3}z^3$ *be the partial sum of* $f(z)=-z-2\log(1-z)\in\mathcal{R}$. *Then,*

$$f_3(z)\in\mathcal{K}\quad(z\in\mathbb{U}_r)$$

where $0 < r \leqq \sqrt{\dfrac{7}{46}} = 0.39009\cdots$. *In particular,*

$$f_3(z) \in \mathcal{K}(\alpha) \quad (z \in \mathbb{U}_r)$$

where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{1-4r+6r^2}{1-2r+2r^2} & \left(0 < r \leqq \dfrac{3-\sqrt{7}}{2} = 0.17712\cdots\right) \\[2ex] \dfrac{8(1+4r^4)-(5+6r^2)\sqrt{2(1+4r^4)}}{4(1+4r^4)-2(1+2r^2)\sqrt{2(1+4r^4)}} & \left(\dfrac{3-\sqrt{7}}{2} < r \leqq \sqrt{\dfrac{7}{46}} = 0.39009\cdots\right). \end{cases}$$

Theorem 4.4 *Let* $f_3(z) = z+2z^2+3z^3$ *be the partial sum of* $f(z) = \dfrac{z}{(1-z)^2} \in \mathcal{S}^*$. *Then,*

$$f_3(z) \in \mathcal{K} \quad (z \in \mathbb{U}_r)$$

where $0 < r \leqq \dfrac{1}{\sqrt{29}} = 0.18569\cdots$. *In particular,*

$$f_3(z) \in \mathcal{K}(\alpha) \quad (z \in \mathbb{U}_r)$$

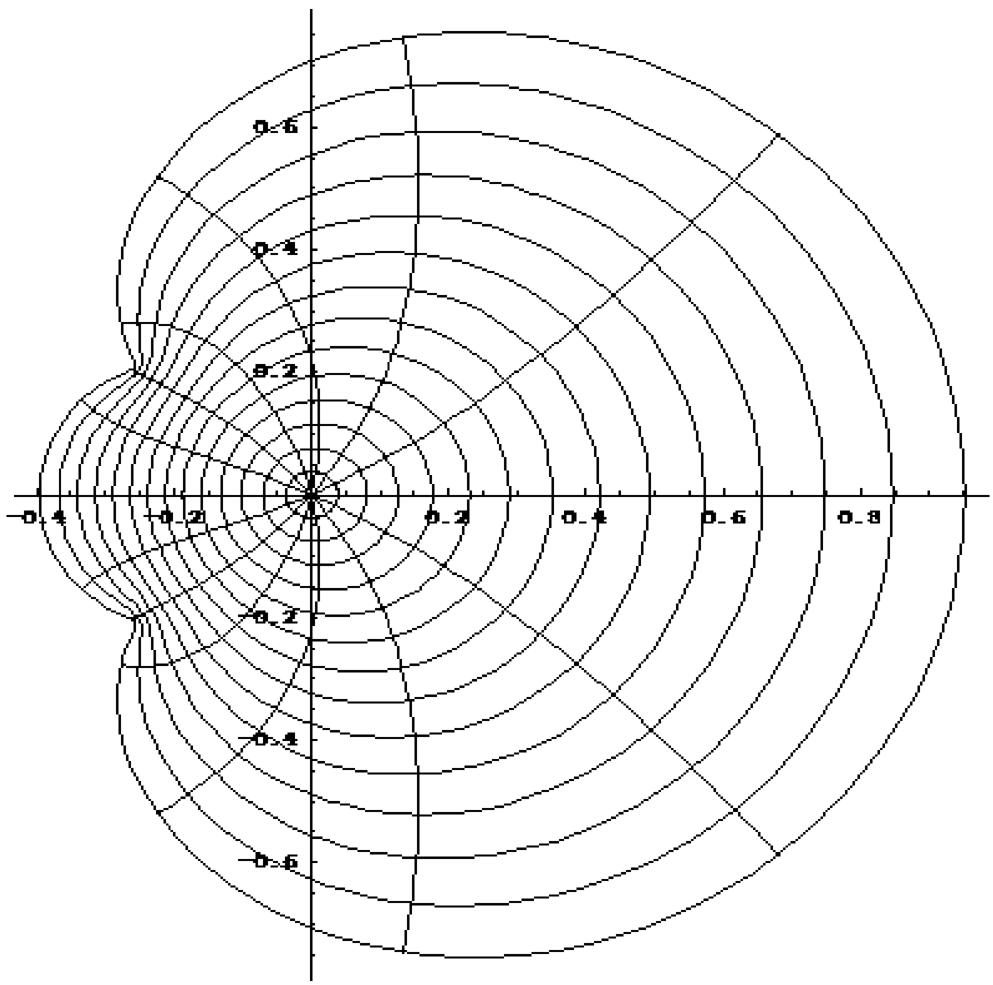
where

$$\alpha := \alpha(r) = \begin{cases} \dfrac{1-8r+27r^2}{1-4r+9r^2} & \left(0 < r \leqq \dfrac{7-2\sqrt{10}}{9} = 0.07504\cdots\right) \\[2ex] \dfrac{30(1+2r^2+81r^4)-12(1+6r^2)\sqrt{5(1+2r^2+81r^4)}}{15(1+2r^2+81r^4)-5(1+9r^2)\sqrt{5(1+2r^2+81r^4)}} & \\ \qquad\qquad\qquad\qquad \left(\dfrac{7-2\sqrt{10}}{9} < r \leqq \dfrac{1}{\sqrt{29}} = 0.18569\cdots\right). \end{cases}$$

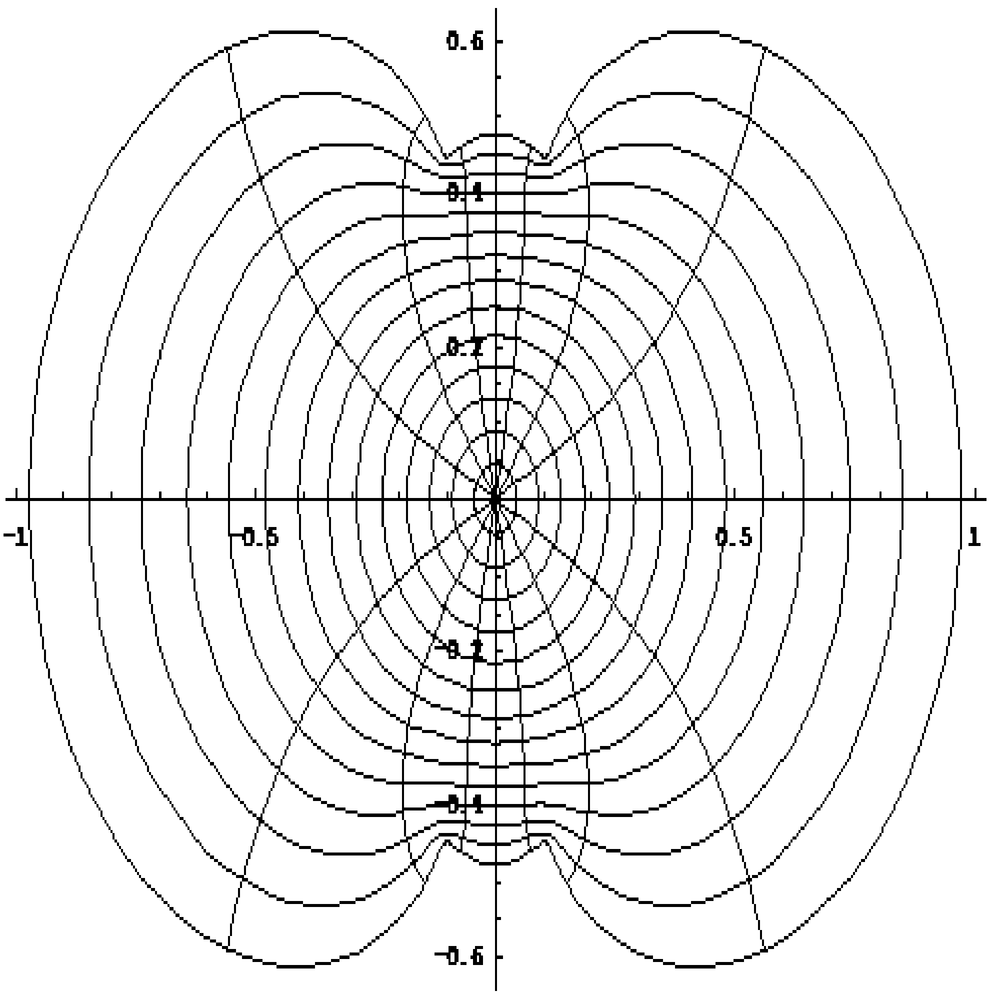
5 Some illustrative examples and image domains for the theorems

In this section, we enumerate several illustrative examples and the image domains of the appropriate radius r by the partial sums which belong to the class $\mathcal{R}$, $\mathcal{S}^*$ or $\mathcal{K}$ in $\mathbb{U}_r$ in the theorems in the previous sections.

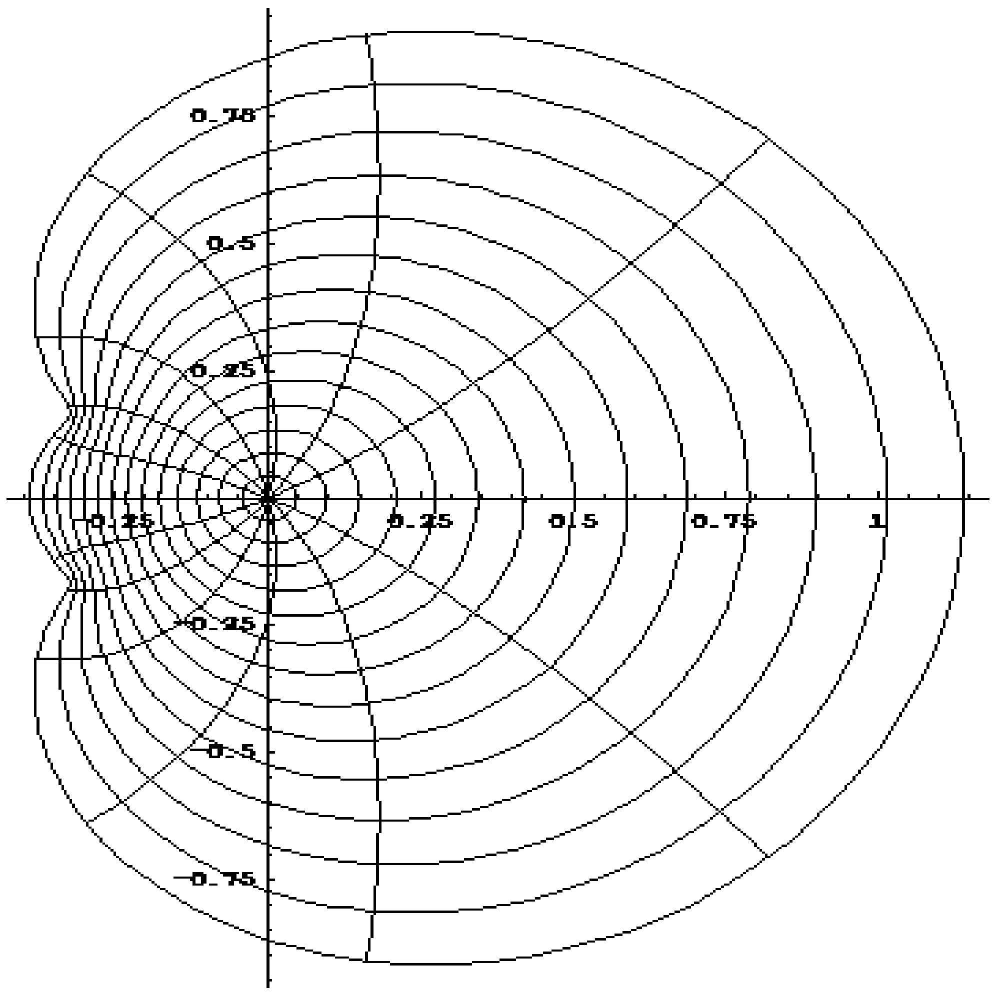
Example of Theorem 2.1 $f_3(z) = z+z^2+z^3 \in \mathcal{R} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \dfrac{\sqrt{10}}{6} = 0.52704\cdots\right).$



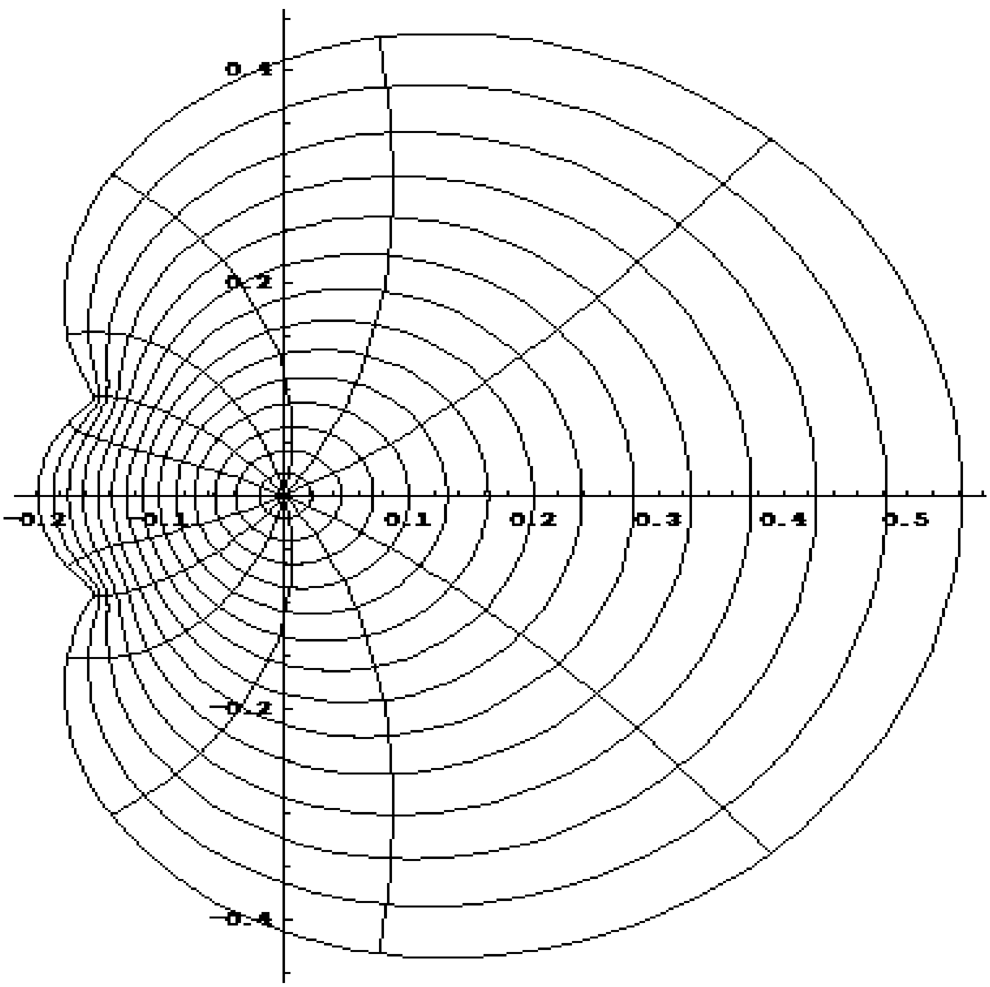
Example of Theorem 2.2 $f_3(z) = z+z^3+z^5 \in \mathcal{R} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \dfrac{\sqrt[4]{1550}}{10} = 0.62745\cdots \right).$



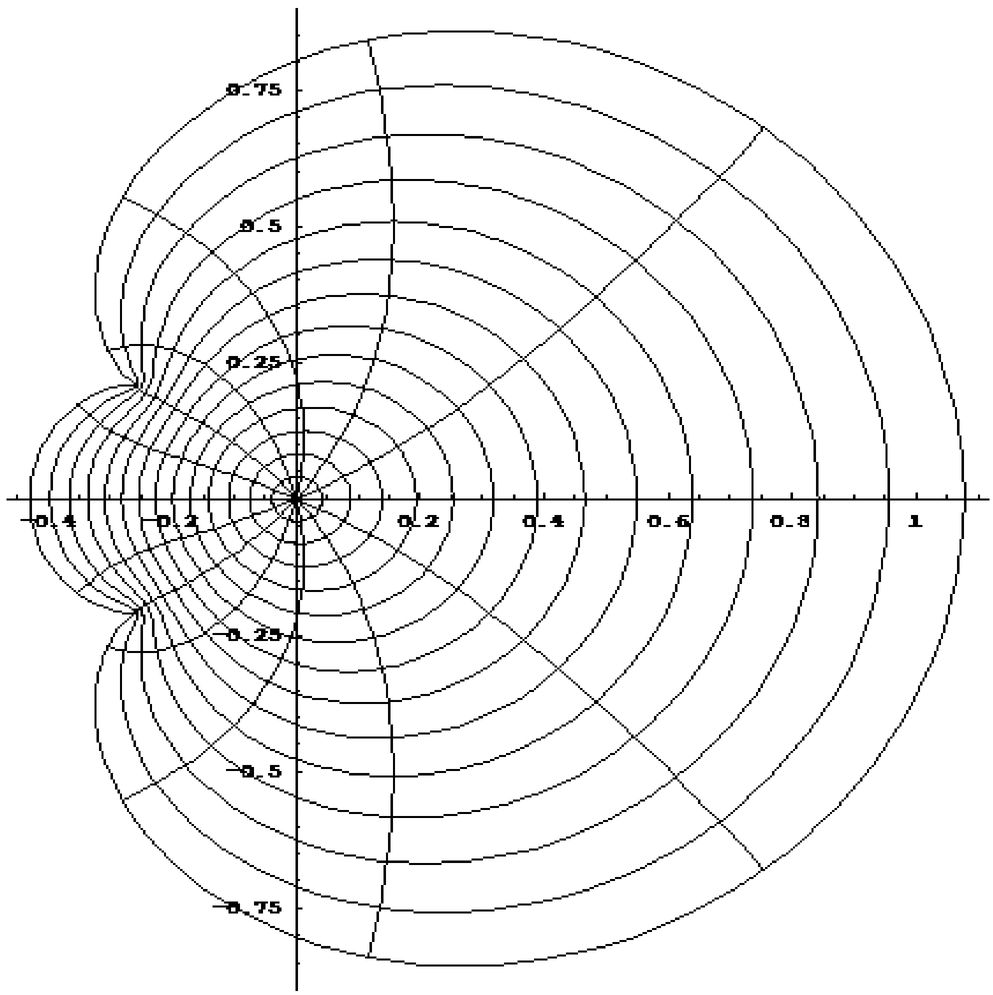
Example of Theorem 2.3 $f_3(z) = z+z^2+\frac{2}{3}z^3 \in \mathcal{R} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \frac{\sqrt{6}}{4} = 0.61237\cdots\right).$



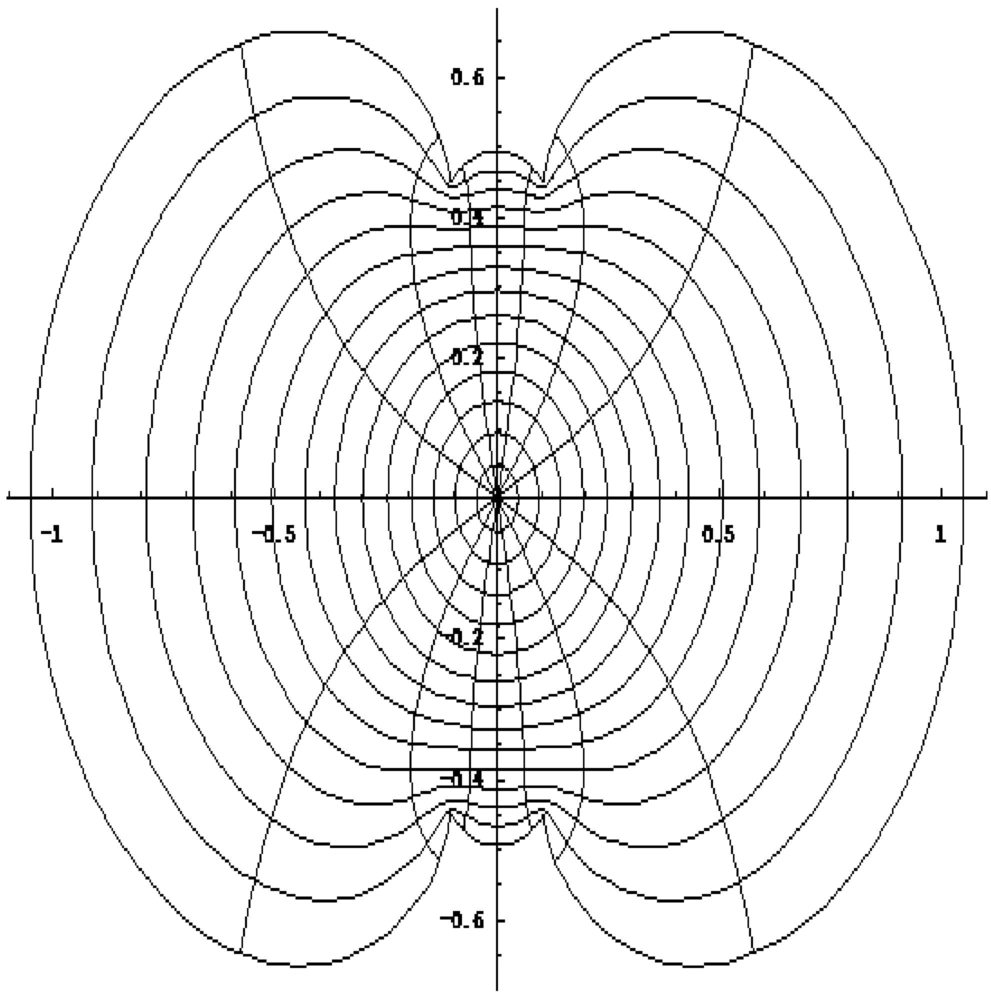
Example of Theorem 2.4 $f_3(z) = z+2z^2+3z^3 \in \mathcal{R} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \dfrac{\sqrt{7}}{9} = 0.29397\cdots\right).$



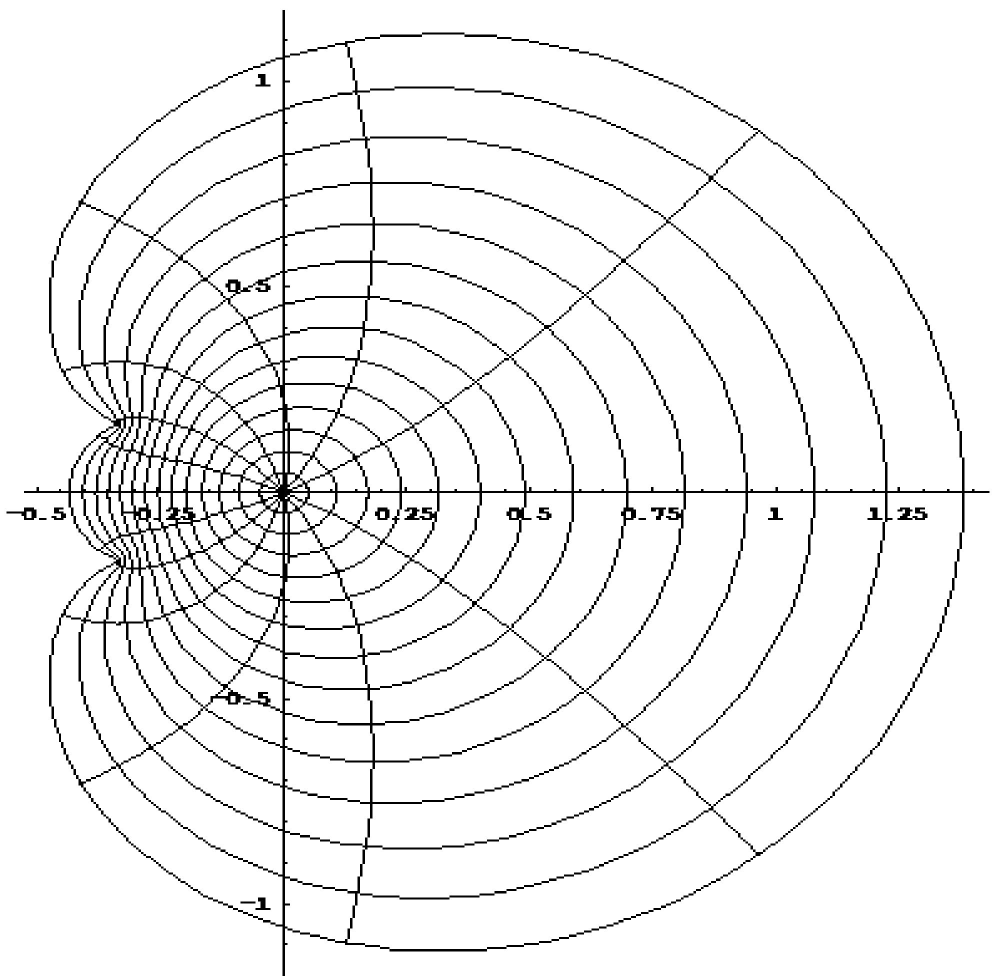
Example of Theorem 3.1 $f_3(z) = z+z^2+z^3 \in \mathcal{S}^* \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \sqrt{\dfrac{23}{71}} = 0.56916\cdots \right).$



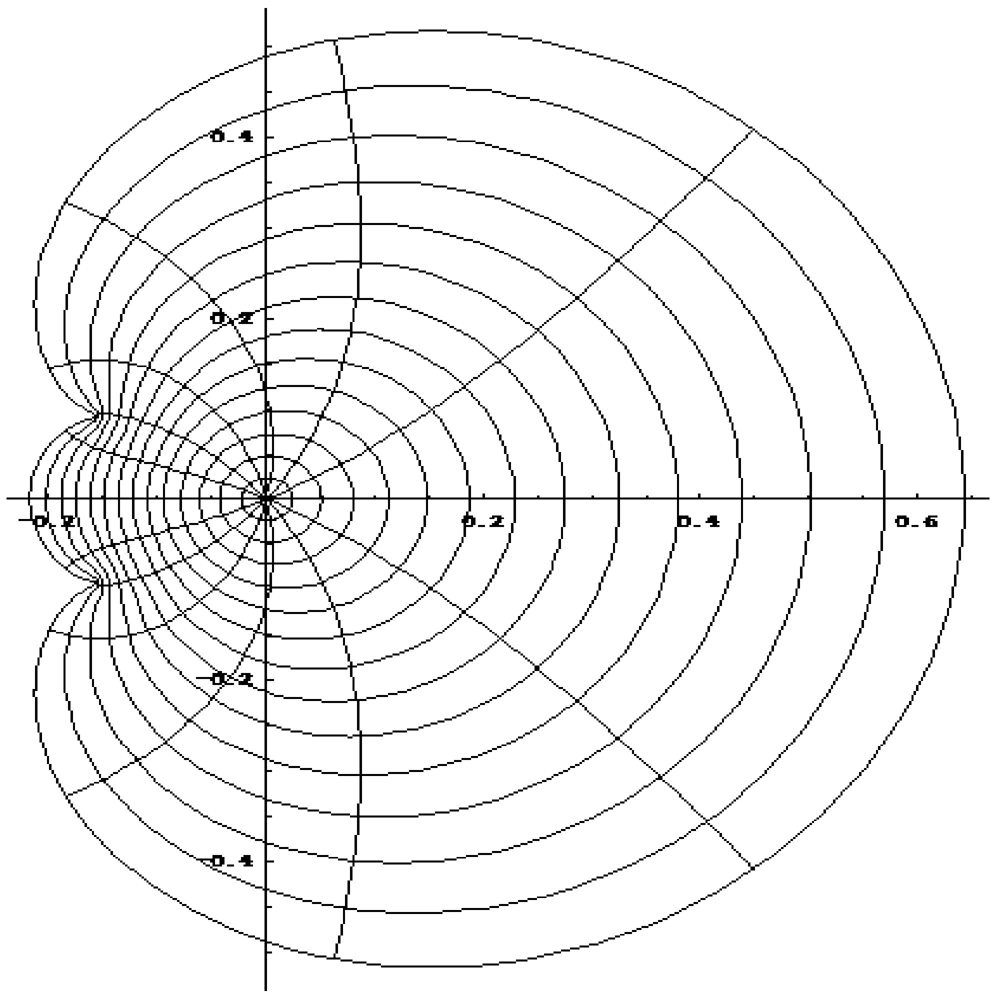
Example of Theorem 3.2 $f_3(z) = z+z^3+z^5 \in \mathcal{S}^* \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \sqrt[4]{\dfrac{2}{11}} = 0.65299\cdots\right).$



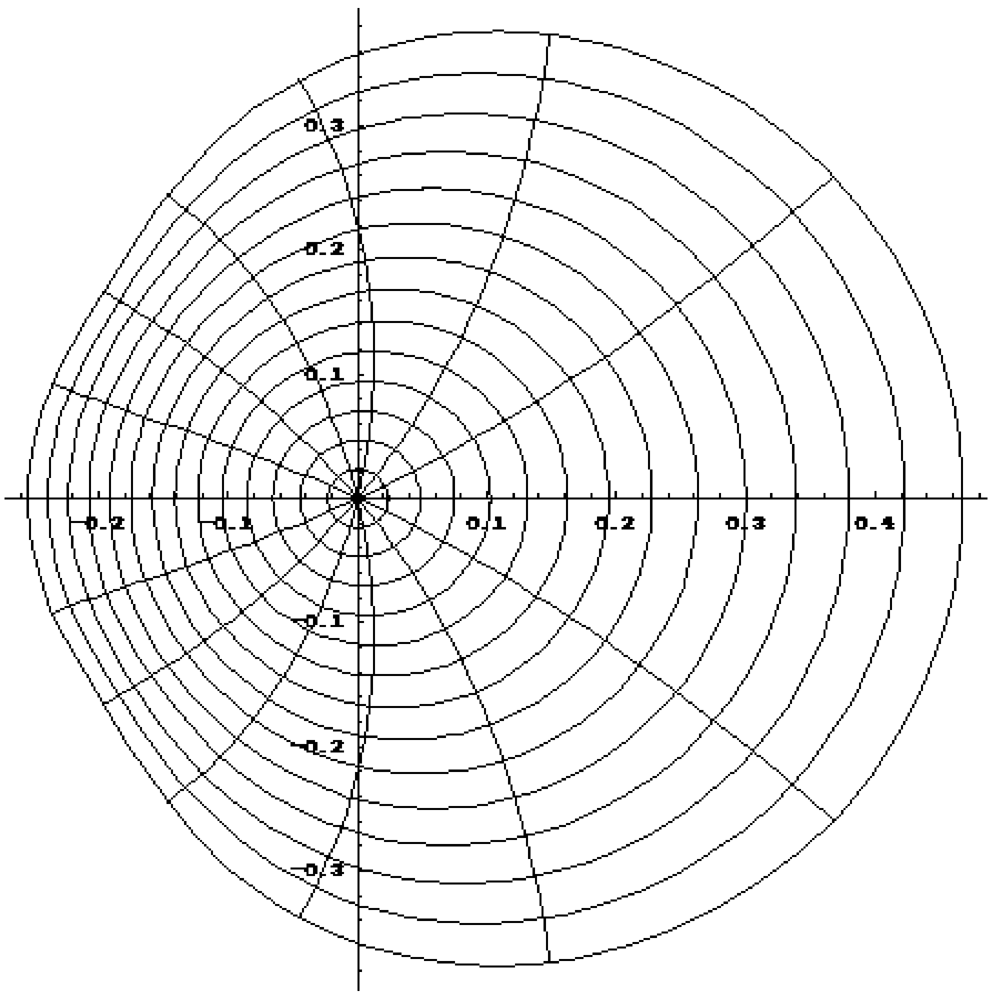
Example of Theorem 3.3 $f_3(z) = z+z^2+\frac{2}{3}z^3 \in \mathcal{S}^* \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \sqrt{\frac{37}{78}} = 0.68873\cdots\right).$



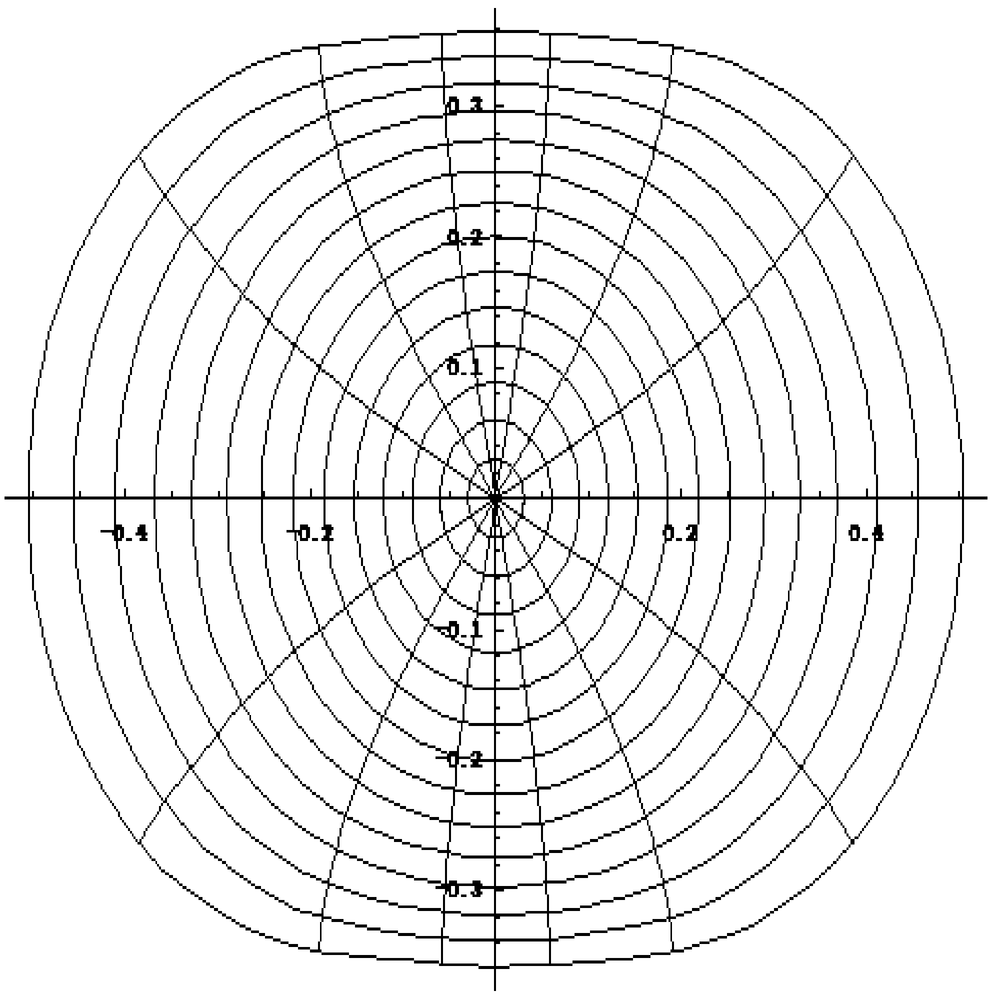
Example of Theorem 3.4 $f_3(z) = z + 2z^2 + 3z^3 \in \mathcal{S}^* \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \sqrt{\frac{5}{47}} = 0.32616\cdots \right).$



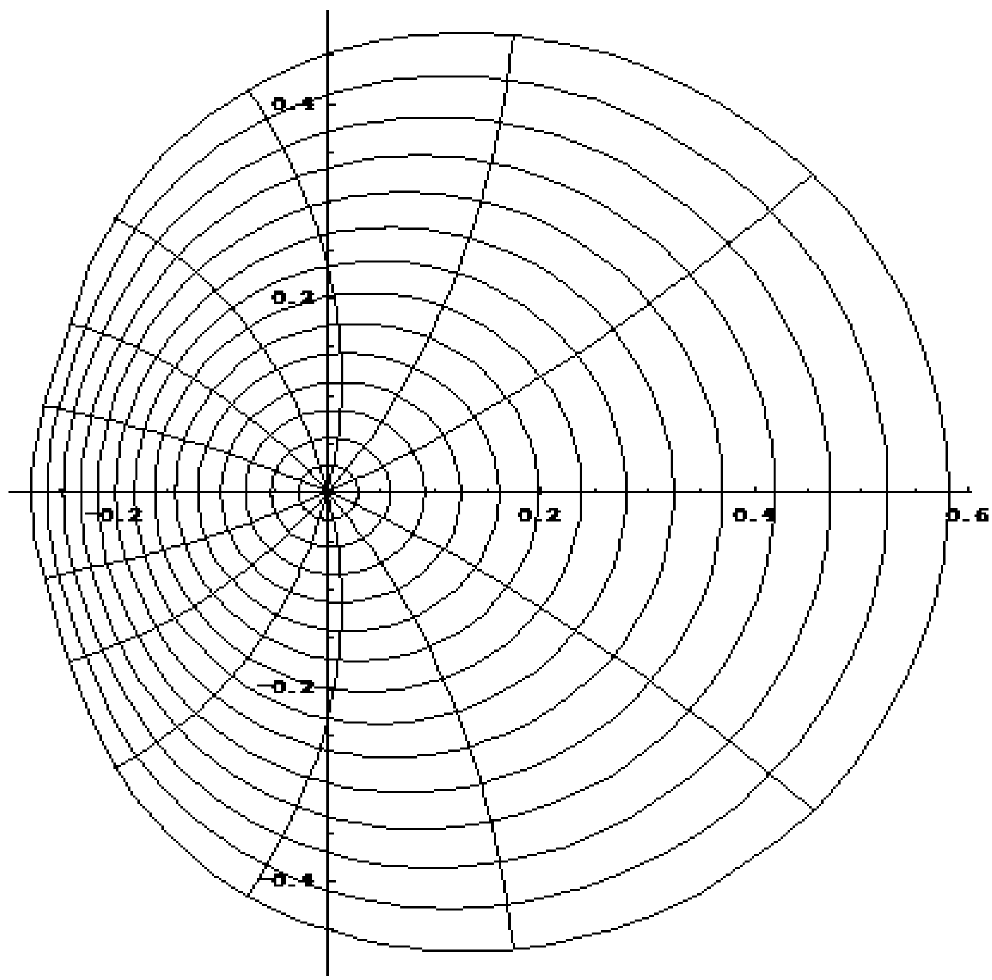
Example Theorem 4.1 $f_3(z) = z+z^2+z^3 \in \mathcal{K} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \sqrt{\dfrac{5}{47}} = 0.32616\cdots\right).$



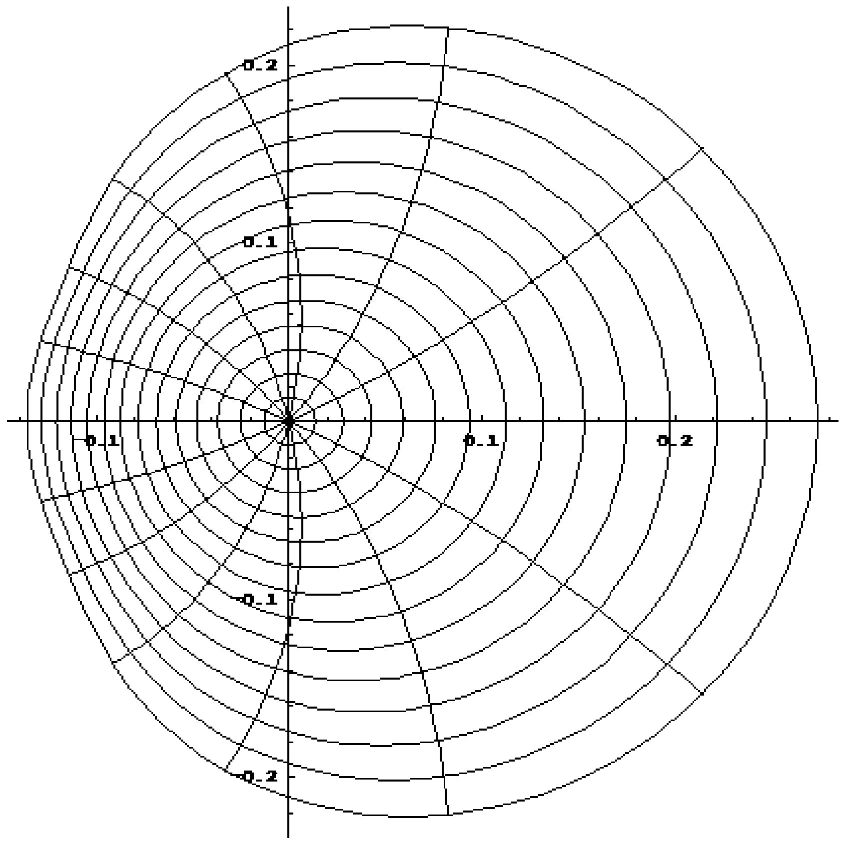
Example of Theorem 4.2 $f_3(z) = z+z^3+z^5 \in \mathcal{K} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \sqrt[4]{\dfrac{2}{65}} = 0.41882\cdots\right).$



Example of Theorem 4.3 $f_3(z) = z+z^2+\frac{2}{3}z^3 \in \mathcal{K} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \sqrt{\frac{7}{46}} = 0.39009\cdots \right).$



Example of Theorem 4.4 $f_3(z) = z+2z^2+3z^3 \in \mathcal{K} \quad \left(z \in \mathbb{U}_r\,;\ 0 < r \leqq \dfrac{1}{\sqrt{29}} = 0.18569\cdots\right).$



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