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A New Generalization of Lemma Gronwall-Bellman

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Abstract

In this paper, we present a new generalization of the Gronwall-Bellman lemma constitutes a key element in the stabilization of nonlinear systems considered. This new generalization can develop a simple command to exponentially stabilize a large class of nonlinear systems.

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1 Introduction

Gronwall-Bellman inequality, which is usually proved in elementary differential equations using continuity arguments (see [6], [7], [9]), is an important tool in the study of boundedness, uniqueness and other aspects of qualitative behavior of solutions of differential and stability.

In 1918, T. Gronwall gave the Gronwall-Bellman inequality (see [5]). After that, many authors gave a number of generalizations of this inequality and these generalizations had significant applications in differential and integral equations. In this paper, we give a new generalization of the Gronwall Bellman lemma and an application for a type of nonlinear dynamic systems.

2 Statement of results

Integral inequalities play an important role in the study of differential equations. In particular, there has been an increased interest in the following Gronwall-Bellman inequality.

Lemma 2.1 *(see [1], [5]) Let $z(t)$ and $f(t)$ be nonnegative continuous functions on $0 \leq t \leq T$, for which the inequality*

$$z(t) \leq c + \int_0^t f(s)z(s)ds, \quad t \in [0,T] \tag{1}$$

holds, where $c \geq 0$ is a constant.
Then

$$z(t) \leq c \exp\left(\int_0^t f(s)ds\right), \quad t \in [0,T] \tag{2}$$

With different motivations, many generalizations and applications of this lemma has been obtained and widely used, such as the study of the stability of solutions of nonlinear differential equations. The purpose of this work is to establish generalizations of Lemma Granwall-Bellman and some consequences of our results are also given.
Our main results are given in the following theorems:

Theorem 2.2 *(see [2], [3], [4]) Let $z(t)$ be a positive differentiable function satisfying the inequality:*

$$z(t) \leq c + \int_a^t (f(s)z(s) + g(s)z^n(s))ds, \quad t \in I = [a,b] \tag{3}$$

where $c \geq 0$, the functions $f(t)$ and $g(t)$ are continuous in I and $n > 1$ is a constant.
Then:

$$z(t) \leq \frac{c \exp\left(\int_a^t f(s)ds\right)}{\left(1 - (n-1)c^{n-1}\int_a^t g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right)ds\right)^{\frac{1}{n-1}}} \tag{4}$$

Under the assumption, for $t, s \in [a,b]$

$$1 - (n-1)c^{n-1}\int_a^t g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right)ds > 0 \tag{5}$$

Proof 2.3 *Expression (3) can be written: for all $t \in [a, b]$*

$$z(t) \leq c + \int_a^t (f(s) + g(s)z^{n-1}(s))z(s)ds, \tag{6}$$

using (2), we obtain

$$z(t) \leq c \exp\left(\int_a^t \left(f(s) + g(s)z^{n-1}(s)\right) ds\right), \tag{7}$$

the last inequality is equivalent to

$$z^{n-1}(t) \leq c^{n-1} \exp\left((n-1)\int_a^t \left(f(s) + g(s)z^{n-1}(s)\right) ds\right), \tag{8}$$

multiply by $-(n-1)g(t)\exp\left(-(n-1)\int_a^t g(s)z^{n-1}(s)ds\right)$, we obtain

$$\begin{aligned} - \quad & (n-1)g(t)\left(z(t)\right)^{n-1} \exp\left(-(n-1)\int_a^t g(s)z^{n-1}(s)ds\right) \\ \geq \quad & -(n-1)c^{n-1}g(t)\exp\left((n-1)\int_a^t f(s)ds\right), \end{aligned} \tag{9}$$

which implies that

$$\frac{d}{dt}\left(\exp\left(-(n-1)\int_a^t g(s)z^{n-1}(s)ds\right)\right) \geq -(n-1)c^{n-1}g(s)\exp\left((n-1)\int_a^t f(s)ds\right), \tag{10}$$

and integrating a to t, we obtain

$$\exp\left(-(n-1)\int_a^t g(s)z^{n-1}(s)ds\right) \geq 1-(n-1)c^{n-1}\int_a^t g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right) ds, \tag{11}$$

under the assumption (5), we get

$$\exp\left((n-1)\int_a^t g(s)z^{n-1}(s)ds\right) \leq \frac{1}{1-(n-1)c^{n-1}\int_a^t g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right) ds}, \tag{12}$$

Hence the inequality (8) becomes as:

$$z^{n-1}(t) \leq \frac{c^{n-1}\exp\left((n-1)\int_a^t f(s)ds\right)}{1-(n-1)c^{n-1}\int_a^t g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right) ds} \tag{13}$$

Therefore, taking into account that $n > 1$, we will

$$z(t) \leq \frac{c\exp\left(\int_a^t f(s)ds\right)}{\left(1-(n-1)c^{n-1}\int_a^t g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right) ds\right)^{\frac{1}{n-1}}} \tag{14}$$

This completes the proof of the theorem.

Remark 2.4 *If the assumption (5) is not verified, there is a subdivision* $(t_i)_{i=0}^{m-1}$ *defined by:* $t_0 = a$, $t_m = b$ *and for* $1 \leq i \leq m-1$*:*

$$\exp\left(-(n-1)\int_a^{t_i} z^{n-1}(s)g(s)ds\right) - (n-1)c^{n-1}\int_{t_i}^{t_{i+1}} g(s)\exp\left(\int_a^s (n-1)f(\tau)d\tau\right)ds = 0 \tag{15}$$

then in this case

$$z(t) \leq \frac{\exp\left(\int_a^t f(s)ds\right)}{\left((n-1)\int_{t_i}^{t_{i+1}} g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right)ds\right)^{\frac{1}{n-1}}} \tag{16}$$

Remark 2.5 *In case* $0 < n < 1$*, the hypothesis (5) is always satisfied. Which leads to:*

$$z(t) \leq c\exp\left(1-(n-1)c^{n-1}\int_a^t g(s)\exp\left((n-1)\int_a^s f(\tau)d\tau\right)ds\right)^{\frac{1}{n-1}} \tag{17}$$

Theorem 2.6 *Let* $z(t)$ *be a positive differentiable function satisfying the inequality:*

$$z(t) \leq c + \int_a^t \sum_{i=1}^{n}(f_i(s)z^i(s))ds, \quad t \in I = [a,b] \tag{18}$$

where $c \geq 0$ *is a constant, the functions* $f_i(t)$ *for* $i = 1, \ldots, n$ *are continuous in* I *and* $n > 1$ *is a constant.*
Then:

$$z(t) \leq \frac{c\exp\left(\int_a^t f_1(s)ds\right)}{\left(1-(n-1)\int_a^t \sum_{i=2}^{n} c^{i-1}f_i(s)\exp\left(\int_a^s (n-1)f_1(\sigma)d\sigma\right)ds\right)^{\frac{1}{n-1}}} \tag{19}$$

Under the assumption, for $t, s \in [a,b]$

$$1-(n-1)\int_a^t \sum_{i=2}^{n} c^{i-1}f_i(s)\exp\left(\int_a^s (n-1)f_1(\sigma)d\sigma\right)ds > 0 \tag{20}$$

Proof 2.7 *Inequality (18) is written as:*

$$z(t) \leq c + \int_a^t \left(f_1(s) + \sum_{i=2}^{n} f_i(s)z^{i-1}(s)\right)z(s)ds, \tag{21}$$

by applying the Gronwall-Bellman Lemma 2.1, we will

$$z(t) \leq c \exp\left(\int_a^t \left(f_1(s) + \sum_{i=2}^{n} f_i(s) z^{i-1}(s)\right) ds\right), \tag{22}$$

then, for all $i = 2, \ldots, n$

$$\begin{aligned} z^{i-1}(t) &\leq c^{i-1} \exp\left((i-1)\int_a^t \left(f_1(s) + \sum_{i=2}^{n} f_i(s) z^{i-1}(s)\right) ds\right), \\ &\leq c^{i-1} \exp\left((n-1)\int_a^t \left(f_1(s) + \sum_{i=2}^{n} f_i(s) z^{i-1}(s)\right) ds\right), \end{aligned} \tag{23}$$

multiply the last inequality by a negative term $-(n-1)f_i(t)$*, then:*

$$\begin{aligned} &- (n-1) f_i(t) z^{i-1}(t) \exp\left(-(n-1)\int_a^t \sum_{i=2}^{n} f_i(s) z^{i-1}(s) ds\right) \geq \\ &- (n-1) f_i(t) c^{i-1} \exp\left((n-1)\int_a^t f_1(s) ds\right) \end{aligned} \tag{24}$$

by summing the inequalities $i = 2$ *to* n*, we obtain:*

$$\begin{aligned} &- (n-1) \sum_{i=2}^{n} f_i(t) z^{i-1}(t) \exp\left(-(n-1)\int_a^t \sum_{i=2}^{n} f_i(s) z^{i-1}(s) ds\right) \geq \\ &- (n-1) \sum_{i=2}^{n} f_i(t) c^{i-1} \exp\left((n-1)\int_a^t f_1(s) ds\right) \end{aligned} \tag{25}$$

by integrating a *to* t*, we find*

$$\begin{aligned} &\exp\left(-(n-1)\int_a^t \sum_{i=2}^{n} f_i(s) z^{i-1}(s) ds\right) \geq \\ &1 - (n-1)\int_a^t \sum_{i=2}^{n} f_i(s) c^{i-1} \exp\left((n-1)\int_a^t f_1(\sigma) d\sigma\right) ds \end{aligned} \tag{26}$$

then

$$\begin{aligned} &\exp\left(\int_a^t \sum_{i=2}^{n} f_i(s) z^{i-1}(s) ds\right) \leq \\ &\frac{1}{\left(1 - (n-1)\int_a^t \sum_{i=2}^{n} f_i(s) c^{i-1} \exp\left((n-1)\int_a^t f_1(\sigma) d\sigma\right) ds\right)^{\frac{1}{n-1}}} \end{aligned} \tag{27}$$

where inequality (22) becomes

$$z(t) \leq \frac{c \exp\left(\int_a^t f_1(s) ds\right)}{\left(1 - (n-1)\int_a^t \sum_{i=2}^{n} c^{i-1} f_i(s) \exp\left(\int_a^s (n-1) f_1(\sigma) d\sigma\right) ds\right)^{\frac{1}{n-1}}} \tag{28}$$

3 Applications

There are many applications of the inequalities obtained in the previous section. in this section we study the output feedback stabilization of nonlinear dynamical systems that model many phenomena from various disciplines.

Now consider the following nonlinear system:

$$\begin{cases} \dot{x}(t) &= Ax(t) + \sum_{i=1}^{m} g_i\left(x(t)\right) u_i(t) + Bu(t) \\ y(t) &= Cx(t) \\ x(0) &= x_0 \end{cases} \tag{29}$$

where: $x(t) \in (R^+)^n$ is the vector of state, $x_0 \in (R^+)^n$ is the initial condition, $y(t) \in R^m$ is output, $u(t) \in R^p$ is the vector controls and A, B, C are constant matrices of appropriate size such that (A, B) stabilizable and (A, C) is detectable.
The nonlinear function $g_i(x(t))$ is measurable with $g_i(0) = 0$ and satisfies the following hypothesis: for all $i = 1, \dots, m$, there exists an integer $q > 1$ as:

$$\|g_i(x(t))\| \leq \alpha_i \|x(t)\|^q \tag{30}$$

where α_i are positive constants.

Remark 3.1 *We assume that the gain K exists because the fact that (A, B) stabilizable and (A, C) detectable provide necessary conditions but not sufficient for the existence of the gain K such that all eigenvalues of the matrix $(A - BKC)$ either negative real part.*
This implies that there exists $M > 0$ and $\omega < 0$ such that:

$$\| e^{(A-BKC)t} \| \leq Me^{\omega t} \quad \textit{for all } t > 0 \tag{31}$$

In the literature, many authors have proposed sufficient conditions for problem of stabilizing static output feedback for linear systems. Unfortunately for the nonlinear case the problem remains open. Exponential stabilization by output feedback static system (29) is a extension of Theorem 2.2, it is given by the following theorem.

Theorem 3.2 *Let $\alpha = \sum_{i=1}^{m} \alpha_i$, $R = \left(\dfrac{-\omega}{\alpha M^{q+1} \|KC\|} \right)^{\frac{1}{q}}$ and $M_0 = \dfrac{M}{\left(1 + \dfrac{\alpha M^{q+1} \|KC\| \|x_0\|^q}{\omega}\right)^{\frac{1}{q}}}$*
For $\|x_0\| < R$, the system (29) controlled by the linear state output feedback

$u(t) = -Ky(t)$ satisfies:

$$\|x(t)\| \leq M_0\|x_0\|e^{\omega t} \text{ and } \|u(t)\| \leq M_0\|KC\|\|x_0\|e^{\omega t}.$$

And then is exponentially stable.

Proof 3.3 *If $C \neq I_n$ and $p < n$, the solution of system (29) is controlled by the state output feedback $u(t) = -Ky(t)$ is given by:*

$$x(t) = e^{(A-BKC)t}x_0 + \int_0^t e^{(A-BKC)(t-s)} \sum_{i=1}^{m} g_i\left(x(s)\right)\left(KCx(s)\right)_i ds \tag{32}$$

where $\left(KCx(s)\right)_i$ is the i^{ieme} component of vector $KCx(s)$.

while taking into account (30) and (31) with $\alpha = \sum_{i=1}^{m} \alpha_i$, it will:

$$\|x(t)\| \leq M\|x_0\|e^{\omega t} + e^{\omega t}\int_0^t \alpha M\|KC\|\|x(s)\|^{q+1}ds \tag{33}$$

then

$$\|x(t)e^{-\omega t}\| \leq M\|x_0\| + \int_0^t \alpha M\|KC\|e^{\omega qs}\|x(s)e^{-\omega s}\|^{q+1}ds \tag{34}$$

where: $c = M\|x_0\|$, $f(s) = 0$, $g(s) = \alpha M\|KC\|e^{\omega qs}$ and $n = q+1$ the application of Theorem 2.2 gives:

$$\begin{aligned}
\|x(t)e^{-\omega t}\| &\leq \frac{M\|x_0\|}{\left(1 - \alpha q M^{q+1}\|KC\|\|x_0\|^q \int_0^t e^{\omega qs}ds\right)^{\frac{1}{q}}} \\
&\leq \frac{M\|x_0\|}{\left(1 - \dfrac{\alpha M^{q+1}\|KC\|\|x_0\|^q}{\omega}(e^{\omega qt} - 1)\right)^{\frac{1}{q}}} \\
&\leq \frac{M\|x_0\|}{\left(1 + \dfrac{\alpha M^{q+1}\|KC\|\|x_0\|^q}{\omega}\right)^{\frac{1}{q}}}
\end{aligned} \tag{35}$$

this proves that: for all $t \geq 0$

$$\|x(t)\| \leq M_0\|x_0\|e^{\omega t}$$

Under the assumption of Theorem 2.2:

$$1 - \alpha q M^{q+1}\|KC\|\|x_0\|^q \int_0^t e^{\omega qs}ds > 0 \tag{36}$$

We obtain:

$$1 + \frac{\alpha M^{q+1}\|KC\|\|x_0\|^q}{\omega} > 0$$

then:

$$\|x_0\| < \left(\frac{-\omega}{\alpha M^{q+1}\|KC\|}\right)^{\frac{1}{q}} \tag{38}$$

This completes the proof of the theorem.

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