Applied Mathematical Sciences, Vol. 6, 2012, no. 14, 667 - 674

The Invariants of 4-Moves

Noureen A. Khan

Department of Mathematics and Information Sciences
University of North Texas at Dallas, TX 75241 USA
noureen.khan@unt.edu

Abstract

This paper introduces $4-moves$, a class of tangle replacement moves called rational $\frac{p}{q}-moves$. We demonstrate the strength of invariants of $4-moves$ and make modification to some long standing $4-moves$ problems like Nakanishi's $4-moves$ conjecture[**11**]. We present the results and examples to show that Nakanishi's $4-moves$ conjecture is true for all knots upto 12 crossings. We suggest that the link $9^*.2:.2:.2$ as a counter example to answer his conjecture.

Mathematics Subject Classifications: 57M99; 55N20D

Keywords: knots; links; 4-moves; Nakanishi's conjecture; Kawauci's conjecture

1 Introduction

The simplest move that reduces every link in S^3 into a trivial link is a crossing change. It can be easily seen that rational $2-$move is equivalent to the crossing change. So, every link in $\mathbf{S^3}$ reduces to a trivial link via $2-$moves. It was proven that $3-$moves [1] and in general, rational $\frac{p}{q}$ moves (p-prime, $p \geq 5$ and q an arbitrary integer) are not unknotting operations.[2]

Therefore, it turned out that results concerning 4-moves have important implication in the knot theory and in particular, the theory of invariants based on deformation of rational moves. Our motivation of exploring 4-moves came from the theory of the Skein Module of S^3 as it was shown in [**4**] that Cubic Skein Module is not generated by trivial links in the case of $M = S^3$. It was

[1] Montessinos-Nakanishi $3-$move conjecture[**10**] states that every link in S^3 can be reduced into a trivial link.

[2] The result of [**4**] shows that the conjecture does not hold. In particular, the link obtained as the standard braid closure of $(\sigma_1\sigma_2\sigma_3\sigma_4)^{10}$ is not $3-$ move reducible into the trivial link of 5 components.

also proven that rational 4-moves are also not unknotting operation for the links of more than one component. It can easily be shown by using the linking matrix modulo 2 that the Hopf link cannot be reduced to a trivial link of two componants. In 1979, Nakanishi [**11**] proposed the following conjecture about $4-$moves.

1.1 Conjecture

Every knot is $4-$*moves equivalent to the trivial knot.*

The conjecture was estabalished for standard braid closures of $3-$braids and knots upto 9 crossings[**6**]. Later on, Askitos [**2**] suggested a counter example of a knot of $16-$crossing (Figure 1) to Nakanishi's $4-$moves conjecture. However, the conjecture is still an open problem.

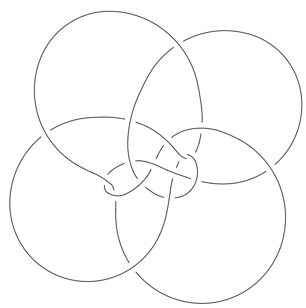

Figure 1: Askitas Knot

As mentioned above, the conjecture can not be extended to links of more than one component in S^3, not even to the class of links with trivial linking matrix modulo 2. In particular, Nakanishi proved that Borromean rings cannot be reduced by $4-$moves to the trivial link of $3-$components [**11**]. Afterwords, Kawauchi proposed a question that if links which are link-homotopically trivial can be reduced to trivial links via $4-$ moves. In turns, M.K. Dabkowski and J.H. Przytycki in [5] answered negative to this question in case of links of more than $2-$components. In particular, by using 4th Burnside group of a link, they showed that the link obtained as standard braid closure of the tangles can not be reduced to the trivial link of $3-$components. They also noted that the method of 4th Burside group of a link cannot be applied to homotopically trivial links of $2-$components. Therefore, the following problem remains an open:

1.2 Problem

Two link- homotopic links are $4-$*moves equivalent?*

In particular, every 2−component link is 4−moves equivalent to the trivial links of two components or to the Hopf link.

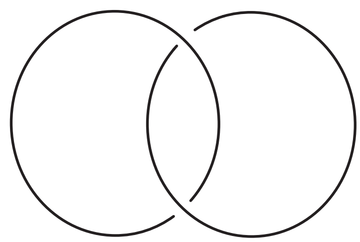

Figure 2: Hopf Link

The problem was answered positive for many classes of links of 2−components. In [**5**], it was shown that any 2−algebraic link of 2−components and any link of 2−components obtained as the standard braid closure of a 3−braid are 4−move equivalent to a trivial link or to the Hopf link. The results were extended to that all knots in the family of 6^* and all knots up to 10 crossings are 4−moves equivalent to the trivial knot and also that for 2-component links in the family of 6^*, given by Conway's symbol $6^*a_1.a_2.a_3.a_4.a_5.a_6$, whenever the link $6^*a_1.a_2.a_3.a_4.a_5.a_6$ is not equivalent to the link $6^*2.2.2.20.20.20$ (shown in Figure 3) and all 2-component links up to 10 crossings.

In section 2, we start with basic defination and then introduce 4−moves as a knot invariant. We demonstrate step by step reduction of Pretzel knot $P(5,1,5)$ of 11 crossings by 4−moves. Section 3 highlights important results about 4−moves as strong knot invariant and then present detailed reduction of the link $6^*2.2.2.20.20.20$ by 4−moves. In the last section, we present revised conjecture and a potential counter example to Nakanishi's 4−moves conjecture.

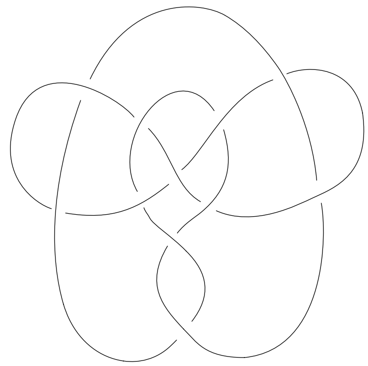

Figure 3: Link6*2.2.2.20.20.20

2 Introduction (4− moves)

A 4−moves is a class of tangle replacement moves called rational moves. In general, a 4−moves is just a local change in the link diagram as shown in the figures below (Figure 4).

Definition 2.1 *We say that a link $\mathcal{L}_1$ reduces to the link $\mathcal{L}_2$ by* 4−*moves if a diagram of $\mathcal{L}_2$ can be obtained from the diagram of $\mathcal{L}_1$ by a finite sequence of* 4−*moves and isotopy.*[3]

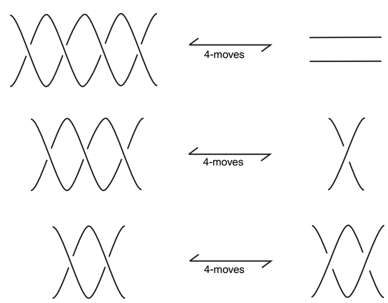

Figure 4: 4−moves

2.1 Example

We show the reduction of Pretzel knot $P(5, 1, 5)$. with 11 crossings.The dotted areas indicate where 4−moves are applied, the knot reduces to trivial knot of one component as shown below(Figure 5).

[3]The 4−move belongs to a class of more general tangle replacement moves called rational $\frac{p}{q}$-moves when the rational 2−tangle $\frac{p}{q}$ is replaced by the identity tangle.

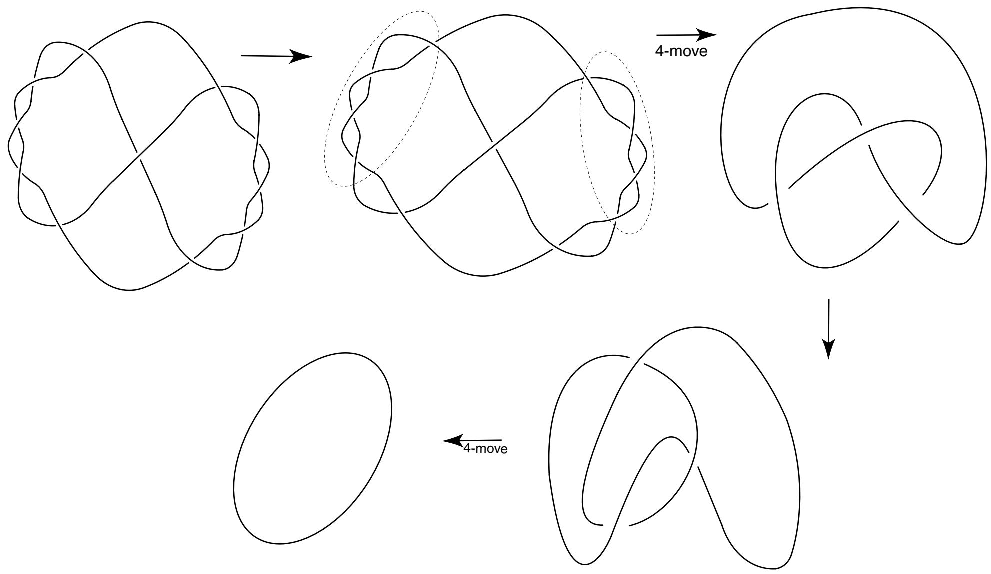

Figure 5: Reduction of Pretzel knot

3 Main Results about $4 - Moves$[7]

(i) Every knot $6^*a_1.a_2.a_3.a_4.a_5.a_6$ can be reduced to the trivial knot by 4-moves.

(ii) If $\mathfrak{L} = 6^*a_1.a_2.a_3.a_4.a_5.a_6$, is a 2-component link, where all $a_i's$ are 2-algebraic tangles with no closed components then $\mathfrak{L}$ reduceds to 2-component trivial link or to the Hopf link.

3.1 Theorem[9]

(i) Every knot or a link of 2-component links up to 11 crossings is $4 - moves$ equivalent to trivial knot, trivial link of two components or to the Hopf link.

(ii) All knots with 12 crossings are $4 - moves$ reducible to trivial knot.

3.2 Example

The reduction of link $6^*1.(2,2).20$ by $4-$moves and isotopy is shown below (Figure 6):

(iii) Let $\mathcal{L}$ be a knot or a link of two components described by $6^*a.b.c.d.e.f$, where a, b, c, d, e and f are $2-$algebraic tangles with no closed compo-

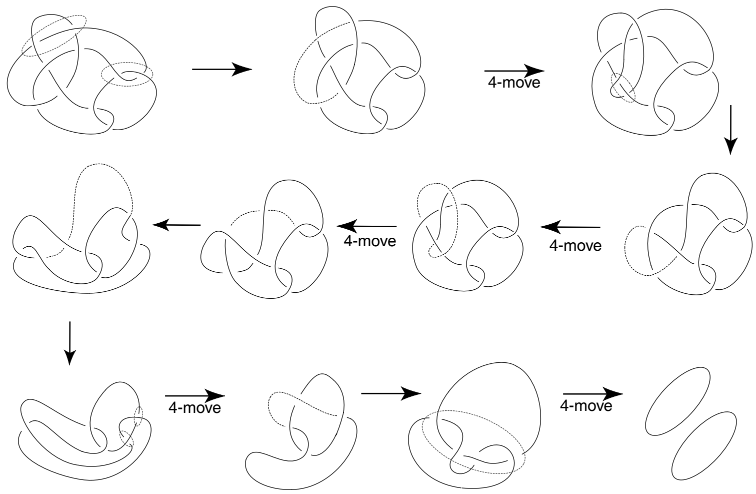

Figure 6: Reduction of Link $6^*1.(2,2).20$

nents. Then $\mathcal{L}$ is reduced by $4 - moves$ to trivial knot or trivial link of two components or to Hopf link.

(*iv*) The link, $9^*.2 : .2 : .2$ (see Figure 7) can't be reduced by 4−moves to trivial link of two components or to Hopf link, therefore serves as a counter example to Nakanishi's 4−moves conjecture.

The link, $9^*.2 : .2 : .2$ can only be reduced to the Hopf link, however we were not able to find its reduction neither using program LinKnot or by hands. There we suggest this link is a counter example for the problem 1.2.

4 Summary

In this paper, the results presnted are about the family of links in the family of 6^* defined in [**3**]. This family is obtained by substituting 2− algebraic tangles in to the vertives of the graph of an Octahedron. Therefore same techniques can be applied to other polyhedra graphs to obtain the famliy of links of $8^*, 9^*, 10^*, 11^*$ and 12^*. We have calculated and reduced all the links in family of 8^*, and the link $9^*.2 : .2 : .2$ is the only obsticle to complete the results about this family and at this point we leave is open with hope either someone will reduce it or will serve counter example to Problem 1.2 and in general, Nakanishi's 4−moves conjecture.

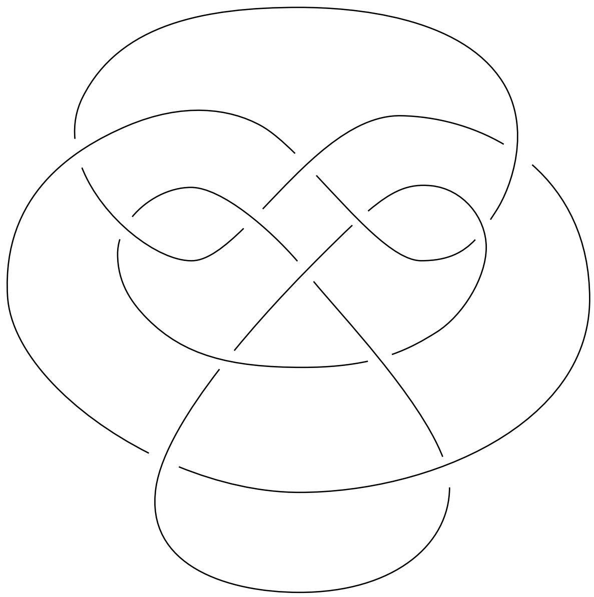

Figure 7: Link $9^*.2 : .2 : .2$

References

[1] S. Akbulut, H. King, Rational structures on 3-manifolds, *Pacific J. Math*, 150 (1991), 201-214.

[2] N. Askitas, A note on 4−equivalence, Journal of Knot Theory with Ramifications, 8 (1999), 261–263.

[3] J. H. Conway, An enumeration of knots and links, *Computational Problems in Abstract Algebra*,(J Leech, editor) Program Press (1969) 329-358

[4] M. K. Dabkowski, J. H. Przytycki, Burnside obstructions to the Montesinos-Nakanishi 3-move conjecture, Geometry and Topology, 6 (2002), 355-380.

[5] M. K. Dabkowski, J. H. Przytycki, Unexpected Connection between Burnside Groups and Knot Theory, Proc. Natl. Acad. Sci., (2004) ,17357-17360.

[6] M. K. Dabkowski, R. K. Sahi, New Invariant of 4-moves, Journal of Knot Theory with Ramifications, 16 (2007),1261-1282.

[7] N.A.Khan, New Invariants of Links in S^3 Preserved by 4−Moves, Ph.D Thesis, UTD, (2008).

[8] N. A. Khan, M. K. Dąbkowski, Derived Group as new Invariants of 4-moves, Journal of Knot Theory with Ramifications, (submitted).

[9] N. A. Khan, M. K. Dąbkowski, S. Jablan, R. K. Sahi, 4-moves equivalence classes of links in S^3, Journal of Knot Theory with Ramifications, 20 (2011), 47-90 .

[10] R. Kirby, Problems in Low-dimensional Topology; Geometric Topology (Proceedings of the Georgia International Topology Conference, 1993), Studies in Advanced Mathematics, Volume **2**, part 2., Ed. W.Kazez, AMS/IP, 1997, pp. 35-473.

[11] Y. Nakanishi, On Fox's Congruence Classes of Knots, Osaka J. Math. 24 (1987), 217–225.

[12] K. Reidemeister, Elementare Begründung der Knotentheorie, Abh. Math. Sem. Univ. Hamburg 5 (1926), 24-32.

Received: March, 2011