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Convergence Theorem of a New Iterative Method for Mixed Equilibrium Problems and Variational Inclusions: Approach to Variational Inequalities

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Abstract

The purpose of this work, we present a new iterative algorithm for finding a common of the set of solutions of a mixed equilibrium problem, the set of a variational inclusion and the set of fixed point of nonexpansive mapping in a real Hilbert space. Under suitable conditions, some strong convergence theorems for approximating a common element of the above three sets are obtained. The results presented in the paper improve some recent results of Y. C. Liou, [An Iterative Algorithm for Mixed Equilibrium Problems and Variational Inclusions Approach to Variational Inequalities, Fixed Point Theory and Applications, Volume 2010, Article ID 564361, 15 pages. doi:10.1155/2010/564361].

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1 Introduction

Let C be a closed convex subset of a real Hilbert space H. Let $F : C \to H$ be a nonlinear mapping, let $\varphi : C \to R$ be a function, and let Θ be a bifunction

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of $C \times C$ into R. Now we consider the following mixed equilibrium problem: to find $u \in C$ such that

$$\Theta(u, y) + \varphi(y) - \varphi(u) + \langle Fu, y - u\rangle \geq 0, \ \forall y \in C. \tag{1.1}$$

The set of solution of problem (1.1) is denoted by EP.

If $F = 0$, then the mixed equilibrium problem (1.1) becomes the followsing mixed equilibrium: to find $u \in C$ such that

$$\Theta(u, y) + \varphi(y) - \varphi(u) \geq 0, \forall y \in C, \tag{1.2}$$

which was considered by Ceng and Yao [3]. If $\varphi = 0$, then the mixed equilibrium problem (1.1) becomes the following equilibrium: to find $u \in C$ such that

$$\Theta(u, y) + \langle Fu, y - u\rangle \geq 0, \ \forall y \in C, \tag{1.3}$$

which was studied by S. Takahashi and W. Takahashi [16]. If $\varphi = 0$ and $F = 0$, then the mixed equilibrium problem (1.1) becomes the following problem: to find $u \in C$ such that

$$\Theta(u, y) \geq 0, \ \forall y \in C. \tag{1.4}$$

If $\Theta(x, y) = 0$ for all $x, y \in C$, the mixed equilibrium problem (1.1) becomes the following variational inequality problem: to find $u \in C$ such that

$$\varphi(y) - \varphi(u) + \langle Fu, y - u\rangle \geq 0, \ \forall y \in C. \tag{1.5}$$

The mixed equilibrium problems include fixed point problems, optimization problems, variational inequality problems, Nash equilibrium problems, and the equilibrium problems as special cases; see, for example, [2, 7, 19, 5, 4, 8] Some methods have been proposed to solve the mixed equilibrium problem and the equilibrium problem. In 1997, Flaim and Antipen [7] introduced an iterative method of finding the best approximation to the initial data and proved a strong convergence theorem. Subsequently, S. Takahashi and W. Takahashi [17] introduced another iterative scheme for finding a common element of the set of solutions of the equilibrium problem(1.2) and the set of fixed point points of a nonexpansive mapping. Furthermore, Yao et al. [18] introduced some new iterative schemes for finding a common element of the set of solutions of the equilibrium problem (1.2) and the set of common fixed points of finitely (infinitely) nonexpansive mappings. Very recently, Ceng and Yao [3] considered a new iterative scheme for finding a common element of the set of solutions of the mixed equilibrium problem and the set of common fixed points of finitely many nonexpansive mappings. Peng and Yao [12] developed a CQ method.

They obtained some strong convergence results for finding a common element of the set of solutions of the mixed equilibrium problem (1.1) and the set of the variational inequality and the set of fixed points of a nonexpansive mapping. Their results extend and improve the corresponding results in [3, 6, 9, 17].

Recall that a mapping $B : C \to C$ is said to be β-inverse strongly monotone if there exists a constant $\beta > 0$ such that $\langle Bx - By, x - y\rangle \geq \beta\|Bx - By\|^2$, for all $x, y \in C$. A mapping A is strongly positive on H if there exists a constants $\mu > 0$ such that $\langle Ax, x\rangle \geq \mu\|x\|^2$ for all $x \in H$. Let $B : H \to H$ be a single-valued nonlinear mapping and let $R : H \to 2^H$ be a set-valued mapping. Now we concern the following variational inclusion, which is to find a point $x \in H$ such that

$$\theta \in B(x) + R(x), \tag{1.6}$$

where θ is the zero vector in H. The set of solution of problem (1.6) is denoted by $I(B, R)$. If $H = R^m$, then problem (1.6) becomes the generalized equation introduced by Robinson [14]. If $B = 0$, then problem (1.6) becomes the inclusion problem introduced by Rockafellar [15].

In 2010, Y. C. Liou [11], introduce iterative algorithm generated by for $x_0 \in C$ the sequence $\{x_n\}$ and $\{u_n\}$ as follows:

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \ \forall y \in C \\ x_{n+1} = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)u_n], \end{cases} \tag{1.7}$$

where $\{\alpha_n\}$ is a real sequence in $[0, 1]$. Then $\{x_n\}$ strongly convergent to a common element of the set of solutions of a mixed equilibrium problem and the set of a variational inclusion in a real Hilbert space.

Inspired and motivate by the work in this paper, we present an iterative algorithm for finding a common element of the set of solutions of a mixed equilibrium problem, the set of variational inclusion and the set of fixed point of nonexpansive mappings in real Hilbert space.

2 Preliminary

Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a nonempty closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denoted by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\| \text{ for all } y \in C. \tag{2.1}$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\langle x-y, P_Cx - P_Cy\rangle \geq \|P_Cx - P_Cy\|^2,$$

for every $x, y \in H$. Further, for $x \in H$ and $x^* \in C$,

$$x^* = P_C(x) \Leftrightarrow \langle x - x^*, x^* - y\rangle \geq 0,\ \forall y \in C. \tag{2.2}$$

A set-valued mapping $T : H \to 2^H$ is called monotone if, for all $x, y \in H$, $f \in Tx$ and $g \in Ty$ imply $\langle x-y, f-g\rangle \geq 0$. A monotone mapping $T : H \to 2^H$ is maximal if its graph $G(T)$ is not properly contained in the graph of any other monotone mapping. It is known that a monotone mapping T is maximal if and only if for $(x, f) \in H \times H$, $\langle x-y, f-g\rangle \geq 0$ for every $(y, g) \in G(T)$ implies $f \in Tx$.

Let the set-valued mapping $R : H \to 2^H$ be maximal monotone. We define the resolvent operator

$$J_{R,\lambda}(x) = (I + \lambda R)^{-1}(x),\ x \in H, \tag{2.3}$$

where λ is a positive number. It is worth mentioning that the resolvent operator $J_{R,\lambda}$ is single valued, nonexpansive, and 1-inverse strong monotone and that a solution of problem (1.6) is a fixed point of the operator $J_{R,\lambda}(I - \lambda B)$ for all $\lambda > 0$, see, for instance, [9].

Throughout this paper, we assume that a bifunction $\Theta : C \times C \to \mathbb{R}$ and a convex function $\varphi : C \to \mathbb{R}$ satisfy the following condition:

$(H1)$ $\Theta(x, x) = 0$ for all $x \in C$;

$(H2)$ Θ is monotone, that is $\Theta(x, y) + \Theta(y, x) \leq 0$ for all $x, y \in C$;

$(H3)$ for each $y \in C, x \mapsto \Theta(x, y)$ is weakly upper semicontinuous;

$(H4)$ for each $x \in C, y \mapsto \Theta(x, y)$ is convex and lower semicontinuous;

$(H5)$ for each $x \in C$ and $r > 0$, there exists a bounded subset $D_x \subset C$ and $y_x \in C$ such that for any $z \in C \backslash D_x$,

$$\Theta(z, y_x) + \varphi(y_x) - \varphi(z) + \frac{1}{2}\langle y_x - z, z - x\rangle < 0. \tag{2.4}$$

Lemma 2.1. *[12] Let C be a nonempty closed convex subset of real Hilbert space H and let Θ be a bifunction of $C \times C$ into $\mathbb{R}$ and let $\varphi : C \to \mathbb{R}$ be a proper lower semicontinuous and convex function. For $r > 0$ anf $x \in C$, define a mapping $S_r : C \to C$ as follows:*

$$S_r(x) = \{z \in C : \Theta(z, y) + \varphi(y) - \varphi(z) + \frac{1}{2}\langle y - z, z - x\rangle \geq 0, \forall y \in C\} \tag{2.5}$$

for all $x \in C$. Assume that the condition $(H1)-(H5)$ hold. Then one has the following results:

(1) *for each $x \in C, S_r(x) \neq \emptyset$ and S_r is single valued;*

(2) *S_r is firmly nonexpansive, that is, for any $x, y \in C$,*

$$\|S_r(x) - S_r(y)\|^2 \leq \langle S_r(x) - S_r(y), x - y\rangle; \tag{2.6}$$

(3) *$F(S_r) = EP$;*

(4) *EP is closed and convex.*

Lemma 2.2. *[1] Let $R : H \to 2^H$ be a maximal monotone mapping and let $B : H \to H$ be a Lipschitz-continuous mapping. Then the mapping $(R + B) : H \to 2^H$ is maximal monotone.*

Lemma 2.3. *Let H be a real Hilbert space. Then for any $x, y \in H$ we have*

(i) $\|x + y\|^2 \leq \|x\|^2 + 2\langle y, x + y\rangle$

(ii) $\|x \pm y\|^2 = \|x\|^2 \pm 2\langle x, y\rangle + \|y\|^2$

(iii) $\|tx + (1-t)y\|^2 = t\|x\|^2 + (1-t)\|y\|^2 - t(1-t)\|x - y\|^2, \ \forall t \in [0, 1].$

Lemma 2.4. *[10] Let a_n, b_n, and c_n be three nonnegative real sequences satisfying*

$$a_{n+1} \leq (1 - t_n)a_n + b_n + c_n, \quad n \geq 0,$$

where $t_n \in [0, 1)$ with $\Sigma_{n=1}^{\infty} t_n = +\infty, b_n = o(t_n)$ and $\Sigma_{n=1}^{\infty} c_n < +\infty$, then $\lim_{n\to\infty} a_n = 0$.

Lemma 2.5. *[13] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

The following lemmas will be useful for proving the convergence result of this paper.

3 Main Results

In this section, we derive a strong convergence of an iterative algorithm which solves the problem of finding a common element of the set of solutions of a mixed equilibrium problem, the set of fixed points nonexpansive mapping of C into itself and the set of the variational inclusion in Hilbert space.

Theorem 3.1. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $T : C \to C$ be a nonexpansive mapping such that $F := F(T) \cap EP \cap I(B,R) \neq \emptyset$. Let $\Theta : C \times C \to \mathbb{R}$ be a bifunction satisfying $(H1)-(H4)$, let F, B be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let A be a strongly positive bounded linear operator with coefficient $0 < \mu < 1$ and $R : H \to 2^H$ be a maximal monotone mapping. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $x_0 = C$;*

$$\begin{cases} \Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \geq 0, \ \forall y \in C \\ y_n = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)u_n], \\ x_{n+1} = \beta_n u + (1-\beta_n)Ty_n, \ n \geq 0 \end{cases} \tag{3.1}$$

where $\{\alpha_n\}, \{\beta_n\} \subset [0,1]$ are satisfying: $\{r_n\} \subset (0,\infty)$. Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying:

(i) $\lim_{n\to\infty} \alpha_n = 0$, $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(ii) $\lim_{n\to\infty} \beta_n = 0$, $\Sigma_{n=1}^{\infty}\beta_n = \infty$, $\Sigma_{n=1}^{\infty}|\beta_{n+1} - \beta_n| < \infty$,

(iii) $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$,

(iv) $0 < r \leq 2\alpha, 0 < \lambda \leq 2\beta$,

Then $\{x_n\}$ converge strongly to $z_0 = P_F u$ which solves the following variational inequality

$$\langle (Ax, y - x\rangle \geq 0, \quad \forall y \in F. \tag{3.2}$$

Proof. Since F is α-inverse strongly monotone and B is β-inverse strongly monotone, we have

$$\|(I - rF)x - (I - rF)y\|^2 \leq \|x - y\|^2 + r(r - 2\alpha)\|Fx - Fy\|^2, \tag{3.3}$$

$$\|(I - \lambda B)x - (I - \lambda B)y\|^2 \leq \|x - y\|^2 + \lambda(\lambda - 2\beta)\|Bx - By\|^2. \tag{3.4}$$

It is clear that if $0 < r \leq 2\alpha$ and $0 < \lambda \leq 2\beta$, then $(I - rF)$ and $(I - \lambda B)$ are all nonexpansive. Set $w_n = J_{R,\lambda}(u_n - \lambda Bu_n), n \geq 0$. It follows that

$$\|w_n - x^*\| = \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\| \leq \|(u_n - \lambda Bu_n) - (x^* - \lambda Bx^*)\| \leq \|u_n - x^*\|. \tag{3.5}$$

By Lemma 2.3, we have $u_n = S_r(x_n - rFx_n)$ for all $n \geq 0$. Then, we have

$$\begin{aligned} \|u_n - x^*\|^2 &= \|S_r(x_n - rFx_n) - S_r(x^* - rFx^*)\|^2 \\ &\leq \|(x_n - rFx_n) - (x^* - rFx^*)\|^2 \\ &\leq \|(x_n - x^*) - r_n(Bx_n - Bx^*)\|^2 \\ &\leq \|x_n - x^*\|^2 - 2r\langle Fx_n - Fx^*, x_n - x^*\rangle + r\|Fx^* - Fx_n\|^2 \quad (3.6) \\ &\leq \|x_n - x^*\|^2 - 2r\alpha\|Fx_n - Fx^*\|^2 + r^2\|Fx^* - Fx_n\|^2 \\ &\leq \|x_n - x^*\|^2. \quad (3.7) \end{aligned}$$

Hence,

$$\|w_n - x^*\| \leq \|u_n - x^*\| \leq \|x_n - x^*\|. \quad (3.8)$$

Since A is linear bounded self-adjoint operator on H, then

$$\|A\| = \sup\{|\langle Au, u\rangle| : u \in H, \|u\| = 1\}.$$

Observe that

$$\langle (I - \alpha_n A)u, u\rangle = 1 - \alpha_n\langle Au, u\rangle \geq 1 - \alpha_n\|A\| \geq 0,$$

that is to say $I - \alpha_n A$ is positive operator. It follows that

$$\begin{aligned} \|(I - \alpha_n A)\| &= \sup\{|\langle (I - \alpha_n A)u, u\rangle| : u \in H, \|u\| = 1\} \\ &= \sup\{\langle (I - \alpha_n A)u, u\rangle : u \in H, \|u\| = 1\} \\ &= \sup\{1 - \alpha_n\langle Au, u\rangle : u \in H, \|u\| = 1\} \\ &\leq 1 - \alpha_n\mu. \end{aligned}$$

From (3.1), we deduce that

$$\begin{aligned} \|y_n - x^*\| &= \|P_C[(I - \alpha_n A)w_n] - x^*\| \\ &\leq \|(I - \alpha_n A)w_n - x^*\| \\ &= \|(I - \alpha_n A)(w_n - x^*) - \alpha_n Ax^*\| \\ &\leq \|I - \alpha_n A\|\|w_n - x^*\| + \alpha_n\|Ax^*\| \\ &\leq (1 - \alpha_n\mu)\|w_n - x^*\| + \alpha_n\|Ax^*\| \\ &= (1 - \alpha_n\mu)\|w_n - x^*\| + \alpha_n\mu\frac{\|Ax^*\|}{\mu} \\ &\leq (1 - \alpha_n\mu)\|x_n - x^*\| + \alpha_n\mu\frac{\|Ax^*\|}{\mu}, \end{aligned}$$

and

$$\begin{aligned}
\|x_{n+1}-x^*\| &= \|\beta_n u+(1-\beta_n)Ty_n-x^*\| \\
&= \|\beta_n(u-x^*)+(1-\beta_n)(Ty_n-x^*)\| \\
&\le \beta_n\|u-x^*\|+(1-\beta_n)\|Ty_n-x^*\| \\
&\le \beta_n\|u-x^*\|+(1-\beta_n)\|y_n-x^*\| \\
&\le \beta_n\|u-x^*\|+(1-\beta_n)[(1-\alpha_n\mu)\|x_n-x^*\|+\alpha_n\mu\frac{\|Ax^*\|}{\mu}] \\
&\le \beta_n\max\{\|u-x^*\|,\|x_0-x^*\|,\frac{\|Ax^*\|}{\mu}\} \\
&\quad +(1-\beta_n)\max\{\|u-x^*\|,\|x_0-x^*\|,\frac{\|Ax^*\|}{\mu}\} \\
&= \max\{\|u-x^*\|,\|x_0-x^*\|,\frac{\|\|Ax^*\|}{\mu}\}.
\end{aligned}$$

Therefore, $\{x_n\}$ is bounded. Hence, $\{u_n\},\{y_n\},\{Ty_n\}$ and $\{Ay_n\}$ are all bounded.

Step 2 We must show that $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$. From (3.1), we have

$$\begin{aligned}
\|x_{n+1}-x_n\| &= \|\beta_n u+(1-\beta_n)Ty_n-(\beta_{n-1}u+(1-\beta_{n-1})Ty_{n-1})\| \\
&= \|(1-\beta_n)(Ty_n-Ty_{n-1})+(1-\beta_n)Ty_{n-1}+(\beta_n-\beta_{n-1})u \\
&\quad +(1-\beta_{n-1})Ty_{n-1})\| \\
&= \|(1-\beta_n)(Ty_n-Ty_{n-1})+(\beta_n-\beta_{n-1})(u-Ty_{n-1})\| \\
&\le (1-\beta_n)\|y_n-y_{n-1}\|+|\beta_n-\beta_{n-1}|\|u-Ty_{n-1}\| \\
&\le \|y_n-y_{n-1}\|+|\beta_n-\beta_{n-1}|\|u-Ty_{n-1}\|. \qquad (3.9)
\end{aligned}$$

Note that,

$$\begin{aligned}
\|y_n-y_{n-1}\| &= \|P_C(I-\alpha_nA)w_n-P_C(I-\alpha_{n-1}A)w_{n-1}\| \\
&\le \|(I-\alpha_nA)w_n-(I-\alpha_{n-1}A)w_{n-1}\| \\
&= \|(I-\alpha_nA)(w_n-w_{n-1})+(I-\alpha_nA)w_{n-1}-(I-\alpha_{n-1}A)w_{n-1}\| \\
&\le (1-\alpha_n\mu)\|w_n-w_{n-1}\|+|\alpha_n-\alpha_{n-1}|\|Aw_{n-1}\|, \qquad (3.10)
\end{aligned}$$

and from $(I-\lambda B)$ and $(I-rF)$ are nonexpansive, we have

$$\begin{aligned}
\|w_n-w_{n-1}\| &= \|J_{R,\lambda}(u_n-\lambda Bu_n)-J_{R,\lambda}(u_{n-1}-\lambda Bu_{n-1})\| \\
&\le \|(I-\lambda B)u_n-(I-\lambda B)u_{n-1}\| \\
&= \|u_n-u_{n-1}\| \\
&= \|S_r(x_n-rFx_n)-S_r(x_{n-1}-rFx_{n-1})\| \\
&\le \|(I-rF)x_n-(I-rF)x_{n-1}\| \\
&\le \|x_n-x_{n-1}\|. \qquad (3.11)
\end{aligned}$$

Substituting (3.11) in (3.10), we get

$$\|y_n - y_{n-1}\| \le (1-\alpha_n\mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Aw_{n-1}\|, \tag{3.12}$$

and substituting (3.12) into (3.9), we get

$$\|x_{n+1} - x_n\| \le (1-\alpha_n\mu)\|x_n - x_{n-1}\| + |\alpha_n - \alpha_{n-1}|\|Aw_{n-1}\| + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|, \tag{3.13}$$

and we have

$$\|x_{n+1} - x_n\| \le (1-\alpha_n\mu)\|x_n - x_{n-1}\| + \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|\mu\frac{\|Aw_{n-1}\|}{\mu} + |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|. \tag{3.14}$$

Put $t_n := \alpha_n\mu$, $b_n := \alpha_n|1 - \frac{\alpha_{n-1}}{\alpha_n}|\mu\frac{\|Aw_{n-1}\|}{\mu}$ and $c_n := |\beta_n - \beta_{n-1}|\|u - Ty_{n-1}\|$ from $(i), (ii), (iii)$ and bounded of $\{\|u - Ty_{n-1}\|\}$ and by Lemma 2.4, we have $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0$.

Step 3 Prove that $\|Fx_n - F_{x^*}\| \to 0$ and $\|Bu_n - B_{x^*}\| \to 0$ as $n \to \infty$. Consider

$$\begin{aligned}
\|w_n - x^*\|^2 &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2 \\
&\le \|(I-\lambda B)u_n - (I-\lambda B)x^*\|^2 \\
&= \|u_n - x^*\|^2 + \lambda(\lambda - 2\beta)\|Bu_n - Bx^*\|^2 \\
&= \|x_n - x^*\|^2 + r(r-2\alpha)\|Fx_n - Fx^*\|^2 \\
&\quad + \lambda(\lambda - 2\beta)\|Bu_n - Bx^*\|^2, \qquad (3.15) \\
&\le \|x_n - x^*\|^2, (\text{ since } r < 2\alpha \text{ and } \lambda < 2\beta). \qquad (3.16)
\end{aligned}$$

and

$$\begin{aligned}
\|y_n - x^*\|^2 &= \|P_C(I-\alpha_nA)w_n - x^*\|^2 \\
&\le \|(I-\alpha_nA)w_n - x^*\|^2 \\
&= \|w_n - x^* - \alpha_nAw_n\|^2 \\
&= \|w_n - x^*\|^2 - 2\alpha_n\langle w_n - x^*, Aw_n\rangle + \alpha_n\|Aw_n\|^2 \qquad (3.17) \\
&= \|w_n - x^*\|^2 + \alpha_n(2\|w_n - x^*\|\|Aw_n\| + \|Aw_n\|^2) \\
&= \|w_n - x^*\| + d_n, \qquad (3.18)
\end{aligned}$$

where $d_n = \alpha_n(2\|w_n - x^*\|\|Aw_n\| + \|Aw_n\|^2)$. From $\lim_{n\to\infty}\alpha_n = 0$ and boundedness, we have $\lim_{n\to\infty} d_n = 0$, there exists $N \in \mathbb{N}$ such that

$$\|y_n - x^*\|^2 \le \|w_n - x^*\|^2 \le \|x_n - x^*\|^2, \forall n \ge N. \tag{3.19}$$

Note that from (3.15) and (3.18), we have

$$\begin{aligned}
\|x_{n+1}-x^*\|^2 &= \|\beta_n u+(1-\beta_n)Ty_n-x^*\|^2\\
&= \|\beta_n(u-x^*)+(1-\beta_n)(Ty_n-x^*)\|^2\\
&= \beta_n\|u-x^*\|^2+(1-\beta_n)\|Ty_n-x^*\|^2\\
&\leq \beta_n\|u-x^*\|^2+(1-\beta_n)\|y_n-x^*\|^2 \qquad (3.20)\\
&\leq \beta_n\|u-x^*\|^2+\|w_n-x^*\|^2+\alpha_n(2\|w_n-x^*\|\|Aw_n\|+\|Aw_n\|^2)\\
&\leq \|x_n-x^*\|^2+r(r-2\alpha)\|Fx_n-Fx^*\|^2+\lambda(\lambda-2\beta)\|Bu_n-Bx^*\|^2\\
&\quad +\beta_n\|u-x^*\|^2+d_n. \qquad (3.21)
\end{aligned}$$

It follows that

$$\begin{aligned}
r(2\alpha-r)\|Fx_n-Fx^*\|^2 &+ \lambda(2\beta-\lambda)\|Bu_n-Bx^*\|^2\\
&\leq \|x_n-x^*\|^2-\|x_{n+1}-x^*\|^2\\
&\quad +d_n+\beta_n\|u-x^*\|^2\\
&\leq \|x_{n+1}-x_n\|(\|x_n-x^*\|+\|x_n-x^*\|)\\
&\quad +d_n+\beta_n\|u-x^*\|^2\\
&.
\end{aligned}$$

From $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0$, $\lim_{n\to\infty}\alpha_n=0$ and $\lim_{n\to\infty}\beta_n=0$, we have

$$\lim_{n\to\infty}\|Fx_n-Fx^*\|=0=\lim_{n\to\infty}\|Bu_n-Bx^*\|.$$

Next, we show that $\lim_{n\to\infty}\|u_n-x_n\|=0=\lim_{n\to\infty}\|u_n-w_n\|$. Since S_r is a firmly nonexpansive, we have

$$\begin{aligned}
\|u_n-x^*\|^2 &= \|S_r(x_n-rFx_n)-S_r(x^*-rFx^*)\|^2\\
&\leq \langle x_n-rFx_n-(x^*-rFx^*),u_n-x^*\rangle\\
&= \frac{1}{2}(\|x_n-rFx_n-(x^*-rFx^*)\|^2+\|u_n-x^*\|^2\\
&\quad -\|x_n-rFx_n-(x^*-rFx^*)-(u_n-x^*)\|^2)\\
&\leq \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n-r(Fx_n-Fx^*)\|^2)\\
&\leq \frac{1}{2}(\|x_n-x^*\|^2+\|u_n-x^*\|^2-\|x_n-u_n\|^2\\
&\quad +2r\langle(Fx_n-Fx^*),x_n-u_n\rangle-r^2\|Fx_n-Fx^*\|^2)
\end{aligned}$$

It follows that

$$\|u_n-x^*\|^2\leq\|x_n-x^*\|^2-\|x_n-u_n\|^2+2r\|Fx_n-Fx^*\|\|x_n-u_n\|. \qquad (3.22)$$

Since $J_{R,\lambda}$ is 1−inverse strongly monotone, we have

$$\begin{aligned}\|w_n - x^*\|^2 &= \|J_{R,\lambda}(u_n - \lambda Bu_n) - J_{R,\lambda}(x^* - \lambda Bx^*)\|^2\\ &\leq \langle (u_n - \lambda Bu_n) - (x^* - \lambda Bx^*), w_n - x^*\rangle\\ &= \frac{1}{2}(\|u_n - \lambda Bu_n - (x^* - \lambda Bx^*)\|^2 + \|w_n - x^*\|^2\\ &\quad -\|u_n - \lambda Bu_n - (x^* - \lambda Bx^*) - (w_n - x^*)\|^2)\\ &\leq \frac{1}{2}(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n - \lambda(Bu_n - Bx^*)\|^2)\\ &\leq \frac{1}{2}(\|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n\|^2\\ &\quad +2\lambda\langle (Bu_n - Bx^*), u_n - w_n\rangle - \lambda^2\|Bu_n - Bx^*\|^2).\end{aligned}$$

Which implies that

$$\|w_n - x^*\|^2 \leq \|u_n - x^*\|^2 + \|w_n - x^*\|^2 - \|u_n - w_n\|^2 + 2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\|. \tag{3.23}$$

By (3.22) and (3.23), we have

$$\begin{aligned}\|w_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2\\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\|. \end{aligned}\tag{3.24}$$

Substituting (3.24) into (3.17), we have

$$\begin{aligned}\|y_n - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2\\ &\quad +2\lambda\|Bu_n - Bx^*)\|\|u_n - w_n\| + d_n\\ &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2\\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n. \end{aligned}\tag{3.25}$$

Substituting (3.25) into (3.20), we have

$$\begin{aligned}\|x_{n+1} - x^*\|^2 &\leq \|x_n - x^*\|^2 - \|x_n - u_n\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\| - \|u_n - w_n\|^2\\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n.\end{aligned}$$

It follows that

$$\begin{aligned}\|x_n - u_n\|^2 + \|u_n - w_n\|^2 &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + 2r\|Fx_n - Fx^*\|\|x_n - u_n\|\\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n + \beta_n\|u - x^*\|^2\\ &\leq \|x_{n+1} - x_n\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|)\\ &\quad +2r\|Fx_n - Fx^*\|\|x_n - u_n\|\\ &\quad +2\lambda\|Bu_n - Bx^*\|\|u_n - w_n\| + d_n + \beta_n\|u - x^*\|^2.\end{aligned}$$

From $\lim_{n\to\infty}\|x_{n+1}-x_n\|=0=\lim_{n\to\infty}\|Fx_n-Fx^*\|=0=\lim_{n\to\infty}\|Bu_n-Bx^*\|$, $\lim_{n\to\infty}\alpha_n=0=\lim_{n\to\infty}\beta_n$ and bounded of sequences, we get

$$\lim_{n\to\infty}\|u_n-x_n\|=0=\lim_{n\to\infty}\|u_n-w_n\|.$$

Step 4 Prove that $\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle\le 0$, where $z_0=P_Fu$. There exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ such that

$$\limsup_{n\to\infty}\langle u-z_0,x_n-z_0\rangle=\lim_{i\to\infty}\langle u-z_0,x_{n_i}-z_0\rangle.$$

Since $\{x_{n_i}\}$ is bounded, there exists a subsequence $\{x_{n_{i_k}}\}$ of $\{x_{n_i}\}$ such that $x_{n_{i_k}}\rightharpoonup w$. Without loss of generality, assume that $x_{n_i}\rightharpoonup w$. Consider, for all $x,y\in H$,

$$\begin{aligned}\|P_F(I-A)x-P_F(I-A)y\| &\le \|(I-A)x-(I-A)y\|\\ &\le \|I-A\|\|x-y\|\\ &\le (1-\mu)\|x-y\|.\end{aligned}$$

Hence $P_F(I-A)$ is contraction and has a unique fixed point, say $x^*\in F$. That is, $x^*=P_F(I-A)(x^*)$. We next prove that $w\in EP$. By $u_n=S_r(x_n-rFx_n)$, we know that

$$\Theta(u_n,y)+\varphi(y)-\varphi(u_n)+\frac{1}{r}\langle y-u_n,u_n-(x_n-rFx_n)\rangle\ge 0,\ \forall y\in C.$$

It follows from (H2) that

$$\varphi(y)-\varphi(u_n)+\frac{1}{r}\langle y-u_n,u_n-(x_n-rFx_n)\rangle\ge\Theta(y,u_n),\ \forall y\in C.$$

Hence,

$$\varphi(y)-\varphi(u_n)+\frac{1}{r}\langle y-u_{n_i},\frac{u_{n_i}-(x_{n_i}-rFx_{n_i})}{r}\rangle\ge\Theta(y,u_{n_i}),\ \forall y\in C. \tag{3.26}$$

For $t\in(0,1]$ and $y\in H$, let $y_t=ty+(1-t)w$. From (3.26) we have

$$\begin{aligned}\langle y_t-u_{n_i},Fy_t\rangle &\ge \langle y_t-u_{n_i},Fy_t\rangle-\varphi(y_t)+\varphi_{(u_{n_i})}\\ &\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-(x_{n_i}-rFx_{n_i})}{r}\rangle+\Theta(y,u_{n_i})\\ &= \langle y_t-u_{n_i},Fy_t-Fu_{n_i},Fu_{n_i}-Fx_{n_i}\rangle-\varphi(y_t)+\varphi(u_{n_i})\\ &\quad -\langle y_t-u_{n_i},\frac{u_{n_i}-x_{n_i}}{r}\rangle+\Theta(y,u_{n_i}).\end{aligned}$$

Since $\|u_{n_i}-x_{n_i}\|\to 0$, we have $\|Fu_{n_i}-Fx_{n_i}\|\to 0$. Further, from the inverse strongly monotonicity of φ, $\frac{u_{n_i}-x_{n_i}}{r}\to 0$ and $u_{n_i}\to w$ weakly, we have

$$\langle y_t-w,Fy_t\rangle\ge-\varphi(y_t)+\varphi(w)+\Theta(y_t,w),\ \forall y\in C. \tag{3.27}$$

From (H1), (H4), and (3.27), we also have

$$\begin{aligned} 0 &= \Theta(y_t, y_t) + \varphi(y_t) - \varphi(y_t) \\ &\leq t\Theta(y_t, y) + (1-t)\Theta(y_t, w) + t\varphi(y) + (1-t)\varphi(w) - \varphi(y_t) \\ &= t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)[\Theta(y_t, w) + \varphi(w) - \varphi(y_t)] \\ &\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y_t, w, Fy_t\rangle \\ &\leq t[\Theta(y_t, y) + \varphi(y) - \varphi(y_t)] + (1-t)\langle y, w, Fy_t\rangle \end{aligned}$$

and hence

$$0 \leq \Theta(y_t, y) + \varphi(y) - \varphi(y_t) + (1-t)\langle y - w, Fy_t\rangle.$$

Letting $t \to 0$, we have, for each $y \in C$

$$\Theta(w, y) + \varphi(y) - \varphi(w) + \langle y - w, Fw\rangle \geq 0.$$

This implies that $w \in EP$.

Next, we show that $w \in I(B, R)$. In fact, since B is β-inverse strongly monotone, B is Lipschitz continuous monotone mapping. It follows from Lemma 2.2 that $R + B$ is maximal monotone. Let $(v, g) \in G(R + B)$ that is, $g - Bv \in R(v)$. Again since $w_{n_i} = J_{R,\lambda}(u_{n_i} - \lambda Bu_{n_i})$, we have $u_{n_i} - \lambda u_{n_i} \in (I + \lambda R)(w_{n_i})$, that is, $(1/\lambda)(u_{n_i} - w_{n_i} - \lambda Bu_{n_i}) \in R(w_{n_i})$. By virtue of the maximal monotonicity of $R + B$, we have

$$\langle v - w_{n_i}, g - Bv - \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda Bu_{n_i})\rangle \geq 0,$$

and so

$$\begin{aligned} \langle v - w_{n_i}, g\rangle &\geq \langle v - w_{n_i}, Bv + \frac{1}{\lambda}(u_{n_i} - w_{n_i} - \lambda Bu_{n_i}\rangle \\ &= \langle v - w_{n_i}, Bv - Bw_{n_i} + Bw_{n_i} - Bu_{n_i} + \frac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle \\ &\geq \langle v - w_{n_i}, Bw_{n_i} - Bu_{n_i}\rangle + \langle v - w_{n_i}, \frac{1}{\lambda}(u_{n_i} - w_{n_i})\rangle. \end{aligned}$$

It follows from $\|u_n - w_n\| \to 0, \|Bu_n - Bw_n\| \to 0$ and $w_{n_i} \rightharpoonup w$ that

$$\lim_{n_i \to \infty}\langle v - w_{n_i}, g\rangle = \langle v - w, g\rangle \geq 0.$$

It follows from the maximal monotonicity of $B + R$ that $\theta \in (R + B)(w)$, that is $w \in I(B,R)$. Next, we can show that $w \in F(T)$. Consider,

$$\begin{aligned} \|y_n - x^*\|^2 &= \|P_C(I-\alpha_n A)w_n - P_C(I-\alpha_n A)x^*\|^2 \\ &\leq \langle (I-\alpha_n A)w_n - (I-\alpha_n A)x^*, y_n - x^*\rangle \\ &= \frac{1}{2}(\|(I-\alpha_n A)w_n - (I-\alpha_n A)x^*\|^2 + \|y_n - x^*\|^2 \\ &\quad -\|(I-\alpha_n A)w_n - (I-\alpha_n A)x^* - (y_n - x^*)\|^2) \\ &\leq \frac{1}{2}(\|w_n - x^*\|^2 + \|y_n - x^*\|^2 - \|w_n - y_n\|^2 \\ &\quad +2\alpha_n\langle w_n - y_n, Aw_n - Ay_n\rangle - \alpha_n^2\|Aw_n - Ay_n\|^2) \\ &\leq \frac{1}{2}(\|w_n - x^*\|^2 + \|y_n - x^*\|^2 - \|w_n - y_n\|^2 \\ &\quad +2\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|), \end{aligned}$$

it follows that

$$\|w_n - y_n\|^2 \leq \|w_n - x^*\|^2 - \|y_n - x^*\|^2 + \alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|. \quad (3.28)$$

From (3.20), we have

$$\begin{aligned} \|x_{n+1} - x^*\|^2 &\leq \beta_n\|u - x^*\|^2 + (1-\beta_n)\|y_n - x^*\|^2 \\ &\leq \beta_n\|u - x^*\|^2 + \|y_n - x^*\|^2 \end{aligned}$$

Then, we get

$$-\|y_n - x^*\|^2 \leq \beta_n\|u - x^*\|^2 - \|x_{n+1} - x^*\|^2. \quad (3.29)$$

Replace (3.8) and (3.29) into (3.28), we have

$$\begin{aligned} \|w_n - y_n\|^2 &\leq \|x_n - x^*\|^2 + (\beta_n\|u - x^*\|^2 - \|x_{n+1} - x^*\|^2) \\ &\quad +\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\| \\ &\leq \|x_n - x^*\|^2 - \|x_{n+1} - x^*\|^2 + \beta_n\|u - x^*\|^2 \\ &\quad +\alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\| \\ &\leq \|x_n - x_{n+1}\|(\|x_n - x^*\| + \|x_{n+1} - x^*\|) \\ &\quad +\beta_n\|u - x^*\|^2 + \alpha_n\|w_n - y_n\|\|Aw_n - Ay_n\|. \end{aligned}$$

From $\lim_{n\to\infty}\|x_{n+1} - x_n\| = 0, \lim_{n\to\infty}\alpha_n = 0, \lim_{n\to\infty}\beta_n = 0$ and boundedness, we obtain

$$\lim_{n\to\infty}\|w_n - y_n\| = 0.$$

Note that

$$\begin{aligned} \|Tu_n - u_n\| &\leq \|Tu_n - Tw_n\| + \|Tw_n - Ty_n\| + \|Ty_n - x_{n+1}\| \\ &\quad +\|x_{n+1} - x_n\| + \|x_n - u_n\| \\ &\leq \|u_n - w_n\| + \|w_n - y_n\| + \beta_n\|Ty_n - u\| + \|x_{n+1} - x_n\| \\ &\quad +\|x_n - u_n\| \to 0 \text{ as } n \to \infty. \end{aligned}$$

From $u_{n_i} \rightharpoonup w$ and H satisfying Opial's condition, it is easy to prove that $w \in F(T)$. Therefore, $w \in F$. It follows that

$$\begin{aligned}\limsup_{n\to\infty}\langle u - z_0, x_n - z_0\rangle &= \lim_{i\to\infty}\langle u - z_0, x_{n_i} - z_0\rangle \\ &= \langle u - z_0, w - z_0\rangle \le 0. \qquad (3.30)\end{aligned}$$

From (3.1), we have for any $n \ge \mathbb{N}$

$$\begin{aligned}\|x_{n+1} - z_0\|^2 &= \|\beta_n u + (1-\beta_n)Ty_n - z_0\|^2 \\ &= \|(1-\beta_n)(Ty_n - z_0) + \beta_n(u - z_0)\|^2 \\ &\le (1-\beta_n)\|Ty_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\beta_n)\|y_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \\ &\le (1-\beta_n)\|x_n - z_0\|^2 + 2\beta_n\langle u - z_0, x_{n+1} - z_0\rangle \qquad (3.31)\end{aligned}$$

Since $\Sigma_{n=1}^{\infty}\beta_n = \infty$, (3.30) and Lemma 2.4, we have $\lim_{n\to\infty}\|x_n - z_0\| = 0$, we get that $\{x_n\}$ converges strongly to $z_0 = P_F u$. □

Corollary 3.2. *[11] Let C be a nonempty closed convex subset of a real Hilbert space H. Suppose that $\Omega := \cap EP \cap I(B,R) \neq \emptyset$. Let $\Theta : H \times C \to \mathbb{R}$ be a bifunction satisfying $(H1) - (H4)$, let F, B be α-inverse strongly monotone and β-inverse strongly monotone, respectively. Let $r > 0$ and $\lambda > 0$ be two constants such that $r < 2\alpha$ and $\lambda < 2\beta$. Let A be a strongly positive bounded linear operator with coefficient $0 < \mu < 1$ and $R : H \to 2^H$ be a maximal monotone mapping. Let the sequences $\{x_n\}$ and $\{u_n\}$ be generated $x_0 = C$;*

$$\begin{cases}\Theta(u_n, y) + \varphi(y) - \varphi(u_n) + \frac{1}{r_n}\langle y - u_n, u_n - (x_n - rFx_n)\rangle \ge 0, \ \forall y \in C \\ x_{n+1} = P_C[(I - \alpha_n A)J_{R,\lambda}(I - \lambda B)u_n],\end{cases} \qquad (3.32)$$

where $\{\alpha_n\}[0,1]$ are satisfying: $\{r_n\} \subset (0,\infty)$. Assume that the control sequences $\{\alpha_n\}$ and $\{\beta_n\}$ satisfying:

(i) $\lim_{n\to\infty}\alpha_n = 0$,

(ii) $\Sigma_{n=1}^{\infty}\alpha_n = \infty$,

(iii) $\lim_{n\to\infty}(\frac{\alpha_{n+1}}{\alpha_n}) = 1$,

Then $\{x_n\}$ converge strongly to $x^* \in \Omega$ which solves the following variational inequality

$$\langle(Ax, y - x\rangle \ge 0, \quad \forall y \in \Omega. \qquad (3.33)$$

Proof. In Theorem 3.1, let $T = I$ and $\beta_n = 0$ for all $n \in \mathbb{N}$, then, we can obtain Corollary. This completes the proof. □

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