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On the Arc Reversal Properties of Digraphs without Loops

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Abstract

Let G be a multidigraph without loops. Let l_i be the upper bounds for arcs $a_i \in A(G)$ to be visited by any closed directed walk in G. We prove that there exists a sequence of finite integers $\{l_i\}$ for which every arc (and every parallel number of arcs) reversal in G decreases the number of closed directed walks if and only if every arc belongs to an elementary directed cycle in G.

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1 Introduction

In this work we focus on closed directed walks in multidigraph G without loops with repeated vertices allowed and repeated arcs a_i allowed up to l_i times. So, every arc $a_i \in A(G)$ is assigned upper bound l_i on the number of visits allowed for any closed directed walk in G. Throughout the rest of this paper by *closed walks* we mean closed directed walks. We investigate impact of any arc reversal (first introduced by Ádám [1]) on the number of pairwise distinct closed walks in G. We consider two closed walks the same only if they can be expressed by the same sequence of arcs. In addition, arc $a_i \in A(G)$ is distinguishable from arc $a_j \in A(G)$ for the same ordered pair of vertices if $i \neq j$. In this note we prove that there exists a sequence of finite integers $\{l_i\}$ for which every arc (and every parallel number of arcs) reversal in G decreases the number of closed directed walks if and only if every arc belongs to an elementary directed cycle in G.

We say that G is *balanced* if $d^+_G(v) = d^-_G(v)$ for every $v \in V(G)$. In [4] we studied closed walks that were allowed to visit an arc at most once in balanced digraphs. In particular, we obtained the best upper bound on the ratio of the number of closed walks for opposite arcs in the balanced digraphs, which is $\frac{3}{2}$ and it is not attainable. In addition, in [2] we studied impact of arc reversal on the number of closed walks, where closed walks were again allowed to visit each arc at most once. The following was obtained based on this study:

Theorem 1.1 [2] *Let G be a balanced multidigraph without loops. Then the reversal of any arc in G decreases the number of closed walks.* □

Let $a_i^-(v)$ (respectively $a_i^+(v)$) be incoming (respectively outcoming) arc a_i in relation to vertex $v \in V(G)$. With every arc a_i we associate a positive integer l_i that provides an upper bound on the number of visits of a_i by any closed walk. Let $\sum_{a_i^-(v)} l_i$ (respectively $\sum_{a_i^+(v)} l_i$) be a summation of integers l_is over all incoming (respectively outcoming) arcs into/from vertex $v \in V(G)$.

The result of Theorem 1.1 was extended based on the above definitions as follows.

Theorem 1.2 [3] *Let $G = (V, A)$ be a multidigraph without loops. Let $l_i \geq 1$ be the maximum number of visits of arc $a_i \in A(G)$ allowed by a closed walk. Let $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ holds for every vertex v, $v \in V(G)$. Then the reversal of any arc in G decreases the number of closed walks.* □

In addition, proof of Theorem 1.2 in [3] implies the following more detailed property.

Lemma 1.3 [3] *Let $G = (V, A)$ be a multidigraph without loops. Let $l_i \geq 1$ be the maximum number of visits of arc $a_i \in A(G)$ allowed by a closed walk. Let $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ holds for every vertex v, $v \in V(G)$. Then the reversal of arc a_x in G does not increase the number of closed walks that visit a_x exactly k times for any integer k, and it decreases the number of closed walks that visit a_x at least once.* □

We will use Theorem 1.2 in the next section to prove our main result (i.e., Theorem 2.2), and we will use Lemma 1.3 in the next section to obtain Corollaries 2.1 and 2.3.

2 Main Results

We first extend Theorem 1.2 as follows.

Corollary 2.1 *Let $G = (V, A)$ be a multidigraph without loops and with at least one pair of parallel arcs. Let $l_i \geq 1$ be the maximum number of visits of arc $a_i \in A(G)$ allowed by a closed walk. Let $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ holds for every vertex v, $v \in V(G)$. Then the reversal of any parallel arcs for a given pair of vertices in G decreases the number of closed walks.*

Proof. Let G^1 be obtained from G by merging k parallel arcs $a_1, a_2, ..., a_k$ into a single a_x arc for some pair of vertices and $k \geq 2$. So, for $a_1, a_2, ..., a_k \rightarrow a_x$ G^1 contains arc a_x with $l_x = l_1 + l_2 + ... + l_k$, and $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ is satisfied. By Theorem 1.2 the reversal of arc a_x in G^1 decreases the number of closed walks. So, the rest of the proof follows by the property of Lemma 1.3. □

Next, we prove by construction and induction our main Theorem as follows.

Theorem 2.2 *Let $G = (V, A)$ be a multidigraph without loops. Let $l_i \geq 1$ denote the maximum number of visits of arc $a_i \in A(G)$ allowed by any closed walk in G. Then there exists a sequence of finite integers $l_1, l_2, ..., l_{|A(G)|}$ for which the reversal of any arc in G decreases the number of closed walks if and only if every arc in G belongs to at least one elementary cycle.*

Proof. Necessary condition is trivial since the reversal of any arc that does not belong to a cycle does not decrease the number of closed walks. So, we now consider sufficient condition. Let $V(G^1) = V(G)$, $A(G^1) = A(G)$, and $l_i = 0$ for every arc $a_i \in G^1$. Clearly, $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ is satisfied in G^1. We choose an arbitrary elementary cycle C_x in G^1 and increment l_i by one for every arc in C_x. So, we transform $G^1 \rightarrow G^2$ that preserves $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ in G^2. Suppose that $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ is satisfied in G^q for $|A(G)| > q \geq 2$. If there is no arc a_x in G^q with corresponding $l_x = 0$ then we are done. Otherwise, we choose arbitrary arc $a_x \in A(G^q)$ with a corresponding $l_x = 0$. For given a_x in G^q we choose an elementary cycle C_x that includes a_x and increment l_i by one for every arc a_i in C_x. This results in transformation $G^q \rightarrow G^{q+1}$ that preserves $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ in G^{q+1} and decreases the number of arcs a_j with corresponding $l_j = 0$. Hence, by induction for some finite positive integer r we obtain $G = G^r$ that satisfies $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$, and for every arc $a_i \in A(G)$ we have $l_i \geq 1$. So, by Theorem 1.2 the reversal of any arc in G decreases the number of closed walks for some sequence $\{l_i\}$ of finite positive integers. □

Consider as an example a non Eulerian digraph H in Figure 1 where every arc belongs to a cycle. By Theorem 2.2 there exist finite integers $\{l_i\}$ for which any arc reversal in H decreases the number of closed walks. In particular, for

$l_1 = l_2 = l_5 = 1$, $l_3 = 2$, $l_4 = 3$ there are the following 8 closed walks in H from Figure 1:

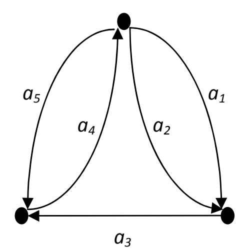

Figure 1: Multidigraph H

$$a_4a_5, a_1a_3a_4, a_2a_3a_4, a_1a_3a_4a_5a_4, a_2a_3a_4a_5a_4,$$

$$a_1a_3a_4a_2a_3a_4, a_1a_3a_4a_5a_4a_2a_3a_4, a_1a_3a_4a_2a_3a_4a_5a_4,$$

which exhaust all pairwise distinct closed walks in H.Since in this case $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ is satisfied for every vertex v in H then by Theorem 1.2 every arc reversal decreases the number of closed walks in H, which is easy to verify.

We can also extend result from Theorem 2.2 to parallel arcs based on Lemma 1.3 as follows.

Corollary 2.3 *Let $G = (V, A)$ be a multidigraph without loops and with at least one pair of parallel arcs. Let every arc in G belongs to at least one elementary cycle. Let $l_i \geq 1$ be the maximum number of visits of arc $a_i \in A(G)$ allowed by any closed walk in G. Then there exists a sequence of finite integers $l_1, l_2, ..., l_{|A(G)|}$ for which the reversal of any parallel arcs for a given pair of vertices in G decreases the number of closed walks.*

Proof. Let G^1 be obtained from G by merging k parallel arcs $a_1, a_2, ..., a_k$ into a single a_x arc for some pair of vertices and $k \geq 2$. By Theorem 2.2 $\sum_{a_i^-(v)} l_i = \sum_{a_j^+(v)} l_j$ is satisfied for some sequence $\{l_i\}$ in G^1, so the reversal of arc a_x in G^1 decreases the number of closed walks. Thus, the rest of the proof follows by the property of Lemma 1.3. □

If we take into consideration again H from Figure 1 then by Corollary 2.3 the reversal of arcs a_1, a_2 in H decreases the number of closed walks, which in this case results in a single closed walk a_4a_5.

References

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