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Homotopy Perturbation Techniques for the Solution of Certain Nonlinear Equations

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Abstract

In this paper, we suggest and analyze a new class of iterative methods for solving nonlinear equations by using the homotopy perturbation method. Convergence of their method is also considered. Here we also discuss the efficiency index and computational order of convergence of new methods. Several numerical examples are given to illustrate the efficiency and performance of these new methods. These new iterative methods may be viewed as an extension and generalization of the existing methods for solving nonlinear equations.

Keywords: Nonlinear equations; Series solution; homotopy perturbation method; Convergence criteria; computational order of convergence, Numerical examples

1 Introduction

It is well known that a wide class of problem which arises in several branches of pure and applied science can be studied in the general framework of the nonlinear equations $f(x) = 0$, Due to their importance, several numerical methods have been suggested and analyzed under certain conditions. These numerical methods have been constructed using different techniques such as Taylor series, homotopy perturbation method and its variant forms, quadrature formula, variational iteration method, and decomposition method Chun [3, 5] , Javidi [9] Traub [25] and Noor [12, 18, 19]; Noor et al. [13- 16], Noor, et al. [24]. Using the technique of updating the solution and Taylor series expansion, Noor et al. [15] have suggested and analyzed a sixth-order predictor-corrector iterative type Halley method for solving the nonlinear equations free from second derivative. Chun [5] have also suggested a class of fifth-order and sixth-order iterative methods. In the implementation of the method of Noor et al. [15], one has to evaluate the second derivative of the function, which is a serious drawback of these methods. To overcome these drawbacks, we modify the predictor-corrector Halley method by replacing the second derivatives of the function by its suitable finite difference scheme. We prove that the new modified predictor-corrector method is of fifth-order convergence. In this paper, we apply the homotopy perturbation method to suggest and analyze some new iterative methods for solving nonlinear equations. The homotopy perturbation method was developed by He [6] and has been used to solve a wide class of problems arising in various branches of pure and applied sciences. Abbasbandy et al.[2] have used the homotopy perturbation method to derive some iterative methods for solving nonlinear equations, which were obtained using the decomposition method Abbasbandy, S. [1]. We also use this technique coupled with system of equations. It has been shown that these new iterative methods include a wide class of known and new iterative methods as special cases; we use the system of coupled equations to express the given nonlinear equations as a sum of linear and nonlinear operators. We then use the homotopy perturbation technique involving the auxiliary parameter $\hbar$.

2 Homotopy perturbation method

Consider the nonlinear equation of the type

$$f(x) = 0. \tag{2.1}$$

We assume that α is a simple root of (2.1) and γ is an initial guess sufficiently close to α. We can rewrite the nonlinear equation (2.1) by using Taylor's series as:

$$f(x) \cong f(\gamma) + \frac{(x-\gamma)}{1!} f'(\gamma) + \frac{(x-\gamma)^2}{2!} f''(\gamma) + \cdots. \tag{2.2}$$

where γ is the approximation solution of equation (2.1). Since $f(x) = 0,$ the equation (2.2) can be written as:

$$x = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{(x-\gamma)^2}{2!} \frac{f''(\gamma)}{f'(\gamma)}.$$

$$L(x) = c + \hbar N(x), \tag{2.3}$$

where

$$c = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \tag{2.4}$$

$$N(x) = -\frac{(x-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)}, \tag{2.5}$$

and $\hbar$ is the auxiliary parameter, c is a constant, L is a linear operator and N is a nonlinear operator.

We use the homotopy technique He [6] to develop a class of new iterative methods for solving nonlinear equations, we define the homotopy $H(v,p):(R\times[0,1]) \to R$ as:

$$\begin{aligned} H(v,p) &= (1-p)[L(v)-L(x_0)] + p[L(v) - c - \hbar N(v)], \\ &= L(v) - L(x_0) + pL(x_0) + p[-\hbar N(v) - c] = 0, \end{aligned} \tag{2.6}$$

where $p \in [0,1]$ is the embedding parameter and x_0 is an initial approximation.

From equation (2.6), we have

$$H(v,0) = L(v) - L(x_0) = v - c = 0, \tag{2.7}$$

$$H(v,1) = L(v) - c - N(v) = v - c - N(v) = 0. \tag{2.8}$$

Here p is called embedding parameter which monotonically increases from zero to 1. $H(v,0) = v - c = 0,$ is continuously deformed to original problem $H(v,1) = v - c - N(v) = 0.$ The changing process of p from zero to 1 is called deformation. $L(v) - L(x_0)$ and $L(v) - c - N(v)$ are homotopic. The basic assumption is that the solution v of (2.6) can be expressed as a power series in p in the form

$$v = x_0 + p^1 x_1 + p^2 x_2 + \cdots. \tag{2.9}$$

The approximate solution of (2.3), can be obtained as:

$$x = \lim_{p \to 1} v = x_0 + x_1 + x_2 + \cdots. \tag{2.10}$$

The convergence of the series (2.10) has been proved by He in this paper He, J. H. [7].

We can write equation (2.3) as follows by expanding $N(v)$ into a Taylor's series around x_0.

$$v = c + p\hbar\{N(x_0) + (v - x_0)N'(x_0) + (v - x_0)^2 \frac{N''(x_0)}{2!} + \cdots \tag{2.11}$$

From equation (2.9) and (2.11), we have

$$\begin{aligned} p^0 x_0 + p^1 x_1 + p^2 x_2 + \cdots &= p^0 c + p\hbar\{N(x_0) + (p^1 x_1 + p^2 x_2 + \cdots)\frac{N'(x_0)}{1!} \\ &\quad + (p^1 x_1 + p^2 x_2 + \cdots)^2 \frac{N''(x_0)}{2!} + \cdots\Big\}, \\ &= p^0 c + p\hbar\{N(x_0)\} + p^2\hbar\{x_1 N'(x_0)\} + p^3\hbar\left\{x_2 N'(x_0) + x_1^2 \frac{N''(x_0)}{2!}\right\} + \cdots. \end{aligned} \tag{2..12}$$

By equating the coefficients of the same powers of p on both sides, we have

$$x_0 = c, \tag{2.13}$$
$$x_1 = \hbar N(x_0), \tag{2.14}$$
$$x_2 = \hbar x_1 N'(x_0), \tag{2.15}$$
$$x_3 = \hbar\left\{x_2 N'(x_0) + x_1^2 \frac{N''(x_0)}{2!}\right\}. \tag{2.16}$$
$$\vdots$$

Combining (2.13)-(2.16), we have

$$x_0 + x_1 + x_2 + \cdots = c + \hbar\left\{N(x_0) + x_1 N'(x_0) + x_1^2 \frac{N''(x_0)}{2!} + \cdots = c + \hbar\sum_{m=1}^{\infty} x_m.\right.$$

From equation (2.5), we have

$$N(x_0) = -\frac{(x_0-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)}, \tag{2.17}$$
$$N'(x_0) = -(x_0-\gamma)\frac{f''(\gamma)}{f'(\gamma)}, \tag{2.18}$$
$$N''(x_0) = -\frac{f''(\gamma)}{f'(\gamma)}. \tag{2.19}$$

From equation (2.13)-(2.16), we have

$$x_0 = \gamma - \frac{f(\gamma)}{f'(\gamma)}, \tag{2.20}$$
$$x_1 = \hbar N(x_0) = -\frac{\hbar(x_0-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)}. \tag{2.21}$$
$$x_2 = \hbar x_1 N'(x_0) = \hbar^2 N(x_0) N'(x_0) = \frac{\hbar^2(x_0-\gamma)^3}{2!}\frac{f''^2(\gamma)}{f'^2(\gamma)}. \tag{2.22}$$
$$\begin{aligned} x_3 &= x_2 N'(x_0) + x_1^2 \frac{N''(x_0)}{2!}, \\ &= \hbar^3 N(x_0) N'^2(x_0) + \frac{\hbar^3 N^2(x_0) N''(x_0)}{2!} = -\frac{5(x_0-\gamma)^4}{8}\frac{f''^3(\gamma)}{f'^3(\gamma)}. \end{aligned} \tag{2.23}$$

Note that x in (2.10) is approximated by

$$X_n = x_0 + x_1 + x_2 \cdots,$$

where $x = \lim_{n\to\infty} X_n$.

For $n = 0$, from equation (2.20), we have

$$x \approx X_0 = x_0 = c = \gamma - \frac{f(\gamma)}{f'(\gamma)}.$$

This formulation allows us to suggest the following Algorithm for solving nonlinear equation (2.1).

Algorithm 2.1. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}.$$

This method is called Newton method with second-order convergence.

For $n=1,$ from equation (2.20) and (2.21), we have

$$x \approx X_1 = x_0 + x_1 = c + \hbar N(x_0) = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{\hbar(x_0-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)}.$$

This formulation allows us to suggest the following Algorithm for solving nonlinear equation (2.1).

Algorithm 2.2. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar(y_n-x_n)^2}{2!}\frac{f''(x_n)}{f'(x_n)}.$$

Algorithm 2.2 can be expressed in the following form.

Algorithm 2.3. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar f^2(x_n) f''(x_n)}{2! f'^3(x_n)},$$

which is called Householder method has cubic-order convergence at $\hbar = 1.$ One can obtain Algorithm 2.3 by using different techniques, see Abbasbandy, S. [1], Housholder, A. S. [8] and Noor, M. A. [17].

For $n=2,$ from equation (2.20)-(2.22), we have

$$\begin{aligned} x \approx X_2 &= x_0 + x_1 + x_2 \\ &= c + \hbar N(x_0) + \hbar^2 N(x_0) N'(x_0), \\ &= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{\hbar(x_0-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)} + \frac{\hbar^2(x_0-\gamma)^3}{2!}\frac{f''^2(\gamma)}{f'^2(\gamma)}. \end{aligned}$$

This formulation allows us to suggest the following Algorithm for solving nonlinear equation (2.1).

Algorithm 2.4. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar(y_n-x_n)^2}{2!}\frac{f''(x_n)}{f'(x_n)} + \frac{\hbar^2(y_n-x_n)^3}{2!}\frac{f''^2(x_n)}{f'^2(x_n)}.$$

Algorithm 2.4 can be expressed in the following form.

Algorithm 2.5. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar f^2(x_n) f''(x_n)}{2 f'(x_n)} - \frac{\hbar^2 f^3(x_n) f''^2(x_n)}{2 f'^5(x_n)}.$$

Abbasbandy, S. [1] derived Algorithm 2.5 by using the Adomian decomposition technique. This method has cubic-order convergence at $\hbar = 1$, otherwise this method has second-order convergence.

For $n = 3$, from equations (2.20)-(2.23), we have

$$x \approx X_3 = x_0 + x_1 + x_2 + x_3$$
$$= c + \hbar N(x_0) + \hbar x_1 N'(x_0) + \hbar\left\{x_2 N'(x_0) + x_1^2 \frac{N''(x_0)}{2!}\right\},$$
$$= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{\hbar(x_0-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)} + \frac{\hbar^2(x_0-\gamma)^3}{2!}\frac{f''^2(\gamma)}{f'^2(\gamma)} - \frac{5\hbar^3(x_0-\gamma)^4}{8}\frac{f''^3(\gamma)}{f'^3(\gamma)}.$$

This formulation allows us to suggest the following Algorithm for solving nonlinear equation (2.1).

Algorithm 2.6. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar(y_n-x_n)^2}{2!}\frac{f''(x_n)}{f'(x_n)} + \frac{\hbar^2(y_n-x_n)^3}{2!}\frac{f''^2(x_n)}{f'^2(x_n)} - \frac{5\hbar^3(y_n-x_n)^4}{8}\frac{f''^3(x_n)}{f'^3(x_n)}.$$

Algorithm 2.6 can be expressed in the following form.

Algorithm 2.7. For a given x_0, compute approximates solution x_{n+1} by the iterative scheme:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar f^2(x_n) f''(x_n)}{2f'^3(x_n)} - \frac{\hbar^2 f^3(x_n) f''^2(x_n)}{2f'^5(x_n)} - \frac{5\hbar^3 f^4(x_n) f''^3(x_n)}{8f'^7(x_n)}.$$

This new method has cubic convergence at $\hbar = 1$. If $f''(x_n) = 0$, Algorithm 2.7 is well-known second-order Newton's method.

3 Coupled System Technique

We can rewrite the nonlinear equation (2.1) as a coupled system, by using the Taylor's series,

$$f(\gamma) + \frac{(x-\gamma)}{1!}f'(\gamma) + \frac{(x-\gamma)^2}{2!}f''(\gamma) + g(x) = 0, \quad (2.24)$$

$$g(x) = f(x) - f(\gamma) - \frac{(x-\gamma)f'(\gamma)}{1!} - \frac{(x-\gamma)^2 f''(\gamma)}{2!}. \quad (2.25)$$

From equation (2.25), we have

$$x = \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{(x-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)} - \frac{g(x)}{f'(\gamma)}. \quad (2.26)$$

$$= c + \hbar N(x), \quad (2.27)$$

Where

$$c=\gamma-\frac{f(\gamma)}{f'(\gamma)}, \tag{2.28}$$

$$N(x)=-\frac{(x-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)}-\frac{g(x)}{f'(\gamma)}. \tag{2.29}$$

Here $\hbar$ is a auxiliary parameter, c is constant and N is called nonlinear operator.
Now we used the new ideas to derive some new iterative methods for solving nonlinear equation (2.1), from (2.20), (2.25) and (2.29), we obtains

$$g(x_0)=f(x_0)-\frac{1}{2}\left(\frac{f(\gamma)}{f'(\gamma)}\right)^2 f''^2(\gamma). \tag{2.30}$$

and

$$N(x_0)=-\frac{(x_0-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)}-\frac{g(x_0)}{f'(\gamma)}=-\frac{(x_0-\gamma)^2}{2!}\frac{f''(\gamma)}{f'(\gamma)}-\frac{\left[f(x_0)-\frac{1}{2!}\left(\frac{f(\gamma)}{f'(\gamma)}\right)^2 f''(\gamma)\right]}{f'(\gamma)}=-\frac{f(x_0)}{f'(\gamma)}. \tag{2.31}$$

$$N'(x_0)=1-\frac{f'(x_0)}{f'(\gamma)}, \tag{2.32}$$

$$N''(x_0)=-\frac{f''(x_0)}{f'(\gamma)}. \tag{2.33}$$

From equation (2.13)-(2.16), (2.20), (2.32), (2.33) and (2.31), we have

$$x_0=c=\gamma-\frac{f(\gamma)}{f'(\gamma)},$$

$$x_1=\hbar N(x_0)=-\frac{\hbar f(x_0)}{f'(\gamma)}, \tag{2.34}$$

$$x_2=\hbar x_1 N'(x_0)=\hbar^2 N(x_0)N'(x_0)=-\frac{\hbar^2 f(x_0)}{f'(\gamma)}+\frac{\hbar^2 f(x_0)f'(x_0)}{f'^2(\gamma)}. \tag{2.35}$$

$$x_3=x_2N'(x_0)+x_1^2\frac{N''(x_0)}{2!},$$

$$=-\frac{\hbar^3 f(x_0)}{f'(\gamma)}+\frac{2\hbar^3 f(x_0)f'(x_0)}{f'^2(\gamma)}-\frac{\hbar^3 f(x_0)f'^2(x_0)}{f'^3(\gamma)}-\hbar^3\left(\frac{f(x_0)}{f'(\gamma)}\right)^2\frac{f''(x_0)}{2f'(\gamma)}. \tag{2.36}$$

$$\vdots$$

For $n=1$, from equations (2.20) and (2.34), we have

$x\approx X_1=x_0+x_1=c+x_1=\gamma-\frac{f(\gamma)}{f'(\gamma)}-\frac{\hbar f(x_0)}{f'(\gamma)}$. This formulation allows us to suggest the following Algorithm for solving nonlinear equation (2.1).

Algorithm 2.8. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar f(y_n)}{f'(x_n)},$$

This is well-know iterative method, has cubic-order convergence at $\hbar = 1,$ see Chun, C [3].

For $n = 2,$ from equations (2.20), (2.34) and (2.35), we have

$$x \approx X_2 = x_0 + x_1 + x_2,$$
$$= c + \hbar N(x_0) + \hbar^2 N(x_0)N'(x_0),$$
$$= \gamma - \frac{f(\gamma)}{f'(\gamma)} - \frac{\hbar(1+\hbar)f(x_0)}{f'(\gamma)} + \frac{\hbar^2 f(x_0)f'(x_0)}{f'^2(\gamma)}.$$

This formulation allows us to suggest the following Algorithm for solving nonlinear equation (2.1).

Algorithm 2.9. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \frac{\hbar(1+\hbar)f(y_n)}{f'(x_n)} + \frac{\hbar^2 f(y_n)f'(y_n)}{f'^2(x_n)},$$

This method has fourth-order at $\hbar = 1,$ otherwise this Algorithm has quadratic convergence, which is mainly due to Noor [13] and Chun, C. [3, 4] by using different decomposition techniques.

Similarly for $n = 3,$

Algorithm 2.10. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$
$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \hbar(1+\hbar+\hbar^2)\frac{f(y_n)}{f'(x_n)} + \hbar^2(1+2\hbar)\frac{f(y_n)f'(y_n)}{f'^2(x_n)} - \frac{\hbar^3 f(y_n)f'^2(y_n)}{f'^3(x_n)}$$
$$-\hbar^3\left(\frac{f(y_n)}{f'(x_n)}\right)^2 \frac{f''(y_n)}{2f'(x_n)}.$$

This method has fifth-order convergence at $\hbar = 1$, see Chun, C.[3], Li, et al. [10]. For $f''(y_n) = 0,$ Algorithm 2.10 reduces the new iterative method for solving nonlinear equation (2.1).

Algorithm 2.11. For a given x_0, compute approximates solution x_{n+1} by the iterative schemes:

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)},$$

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)} - \hbar(1+\hbar+\hbar^2)\frac{f(y_n)}{f'(x_n)} + \hbar^2(1+\hbar)\frac{f(y_n)f'(y_n)}{f'^2(x_n)} - \frac{\hbar^3 f(y_n)f'^2(y_n)}{f'^3(x_n)}.$$

This new iterative method has fourth-order convergence at $\hbar = 1$, see detail in theorem 2.1.

4 Convergence criteria

Now we consider the convergence criteria of Algorithm 2.11. In a similar way, we can proof the convergence of all other Algorithms.

Theorem 2.1. Let $\alpha \in D$ be a simple zero of sufficiently differentiable function $f : D \subset R \to R$ for an open interval D and let x_0 be initial choice, then Algorithm 2.11 has fourth-order convergence.

Proof: If α is the root and e_n be the error at n^{th} iteration, then $e_n = x_n - \alpha$. Using Taylor's expansion, we have

$$f(x_n) = f'(\alpha)[e_n + c_2 e_n^2 + c_3 e_n^3 + c_4 e_n^4 + c_5 e_n^5 + O(e_n^6)]. \quad (2.37)$$

$$f'(x_n) = f'(\alpha)[1 + 2c_2 e_n + 3c_3 e_n^2 + 4c_4 e_n^3 + 5c_5 e_n^4 + 6c_6 e_n^5 + O(e_n^6)]. \quad (2.38)$$

where

$$c_k = \frac{1}{k!}\frac{f^{(k)}(\alpha)}{f'(\alpha)}, \quad k = 2, 3, \cdots, \quad \text{let} \quad e_n = x_n - \alpha,$$

and from (2.37) and (2.38), we have

$$\frac{f(x_n)}{f'(x_n)} = e_n - c_2 e_n^2 - (2c_3 - 2c_2^2)e_n^3 - (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + (-6c_3^2 + 20c_3c_2^2 - 10c_2c_4 + 4c_5 - 8c_2^4)e_n^5 + O(e_n^6). \quad (2.39)$$

From equation (2.39), we have

$$y_n = x_n - \frac{f(x_n)}{f'(x_n)} = \alpha + c_2 e_n^2 + (2c_3 - 2c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 4c_2^3)e_n^4 + O(e_n^5). \quad (2.40)$$

From equation (2.40), we obtain

$$f(y_n) = f'(\alpha)[c_2 e_n^2 + 2(c_3 - c_2^2)e_n^3 + (3c_4 - 7c_2c_3 + 5c_2^3)e_n^4 + O(e_n^5)], \quad 2.41)$$

$$f'(y_n) = f'(\alpha)[1 + c_2^2 e_n^2 + 4(c_2c_3 - c_2^3)e_n^3 + (8c_2^4 + 6c_2c_4 - 11c_2^2c_3)e_n^4 + O(e_n^5). \quad (2.42)$$

Combining equations (2.38), (2.41) and (2.42), we have

$$\frac{f(y_n)}{f'(x_n)} = c_2 e_n^2 + 2(c_3 - 2c_2^2)e_n^3 + (3c_4 - 14c_2c_3 + 13c_2^3)e_n^4 + O(e_n^5). \quad (2.43)$$

$$\frac{f'(y_n)}{f'(x_n)} = 1 - 2c_2 e_n + 3(2c_2^2 - c_3)e_n^2 + O(e_n^3). \quad (2.44)$$

$$\frac{f(y_n)f'(y_n)}{f'^2(x_n)} = c_2 e_n^2 + 2(c_3 - 3c_2^2)e_n^3 + (3c_4 - 14c_2c_3 + 13c_2^3)e_n^4 + O(e_n^5). \quad (2.45)$$

$$\left(\frac{f'(y_n)}{f'(x_n)}\right)^2 = 1-4c_2e_n+(16c_2^2-6c_3)e_n{}^2-12(-c_2c_3+2c_2^2)e_n{}^3+9(4c_2^4-4c_2^2c_3+c_3^2)e_n{}^4+O(e_n{}^5). \quad 2.46)$$

Using (2.40)-(2.46) in Algorithm 2.11, we have

Thus, for $e_n = x_n - \alpha$, we have

$$e_{n+1} = c_2(-1+\hbar)e_n^2 + (2c_3-2c_2^2-2\hbar^2c_2^2+4\hbar c_2^2-2\hbar c_3)e_n^3 + (-3\hbar c_4+4c_2^3-7\hbar^2c_2c_3$$
$$+14\hbar c_2c_3-13\hbar c_2^3+14\hbar^2c_2^3-4\hbar^3c_2^3+3c_4-7c_2c_3)(e_n^4)+O(e_n^5).$$

which shows that Algorithm 2.11 has fourth-order convergence at $\hbar = 1$.

5. Numerical examples

In this section, we present some numerical examples to illustrate the efficiency and the accuracy of the new developed iterative methods in this chapter (Table 2.1-Table 2.3). We compare our methods obtained in this chapter Algorithm 2.11 with the method of Javidi, M. ([9], JM1), method of Noor, M. A. ([18, 19]) and method of Chun, C. ([4], MC1). All computations have been done by using the Maple 11 package with 25 digit floating point arithmetic. We accept an approximate solution rather than the exact root, depending on the precision (ε) of the computer. As for the convergence criteria, it was required that the distance of two consecutive approximations δ. Also displayed are the number of iterations to approximate the zero (IT), the approximate root x_n, the value $f(x_n)$ and the computational order of convergence (COC) can be approximated using the formula,

$$COC \approx \frac{\ln\left|(x_{n+1}-x_n)/(x_n-x_{n-1})\right|}{\ln\left|(x_n-x_{n-1})/(x_{n-1}-x_{n-2})\right|}.$$

Example 2.1. Consider the equation $f_1(x) = x^3+4x^2+8x+8, \quad x_0 = -0.5.$

Table 2.1 (Approximate solution of example 2.1)

Methods	*IT*	x_n	$f(x_n)$	δ	*COC*
MC1	3	2	0	1.58962e-26	4.00
MN	4	2	0	4.82785e-28	4
JM1	4	2	2.00000e-59	2.18067e-22	3.99
Alg. 2.11	3	2	0	1.58962e-26	4.00

Example 2.2. Consider the equation $f_2(x) = x - 2 - \exp(-x), \quad x_0 = 2.$
Table 2.2 (Approximate solution of example 2.2)

Methods	*IT*	x_n	$f(x_n)$	δ	*COC*
MC1	3	2.120028238987641229484688 0	4.00000e-60	3.37934e-34	4.00
MN	3	2.120028238987641229484688 0	-7.00000e-60	3.07565e-31	4.01
JM1	4	2.120028238987641229484688 0	4.00000e-60	2.73489e-34	4.00
Alg. 2.11	3	2.120028238987641229484688 0	4.00000e-60	3.37934e-34	4.00

Example 2.3. Consider the equation $f_3(x) = x^2 - (1-x)^5, \quad x_0 = 0.2.$
Table 2.3 (Approximate solution of example 2.3)

Methods	*IT*	x_n	$f(x_n)$	δ	*COC*
MC1	3	0.3459548158482420179582044	0	2.54241e-44	4.01
MN	4	0.3459548158482420179582044	0	3.50492e-37	4
JM1	4	0.3459548158482420179582044	0	3.03791e-4	4.00
Alg. 2.11	3	0.3459548158482420179582044	0	2.54241e-44	4.01

Conclusion

In this chapter, we have suggested a family of one-step and two-step iterative methods for solving nonlinear equations by using homotopy perturbation technique involving the auxiliary parameter. We derived some new iterative methods such as the methods derived in Algorithm 2.7 and Algorithm 2.11 at $\hbar = 1$ has third-order and fourth-order convergence respectively. These methods have better efficiency index and performance as compared to Newton method and its variant. It is well known that the obstacle and unilateral problems can be studied in the general framework of the variational inequalities. These problems can be characterized by a system of variational equations, see Noor [11,17], Noor et al [20, 23]. In recent years, the homotopy perturbation method has been used to solve such type of system of boundary value problems, see Noor et al (2011, 2011a). We hope that the ideas and techniques used in this paper may be used to solve such type of problems, which is another direction for further research.

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