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Bi-Ideals in Ordered Ternary Semigroups

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Abstract

In this note, we show that an ordered ternary semigroup is left and right simple if and only if it does not contain proper bi-ideals. The similar result is valid on ordered semigroups [7], on ordered Γ-semigroups [3] and on ternary semigroups [4].

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1 Introduction

An *ordered semigroup* [1] is a semigroup $(S, \cdot)$ together with a partial order $\leq$ on S that is compatible with the semigroup operation, meaning that $x \leq y$ implies $z \cdot x \leq z \cdot y$ and $x \cdot z \leq y \cdot z$ for all x, y, z in S. It is customary to write $(S, \cdot, \leq)$ rather than $(S, \cdot)$. Let $(S, \cdot, \leq)$ be an ordered semigroup. A nonempty subset A of S is called a *left (respectively, right, bi-) ideal* of S if (i) $SA \subseteq A$ (respectively, $AS \subseteq A$, $ASA \subseteq A$) and (ii) If $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$. A left (respectively, right, bi-) ideal of S is said to be *proper* if $A \neq S$. S is said to be *left (respectively, right) simple* if S does not contain proper left (respectively, right) ideals.

In [7], the authors proved that an ordered semigroup $(S, \cdot, \leq)$ is left and right simple if and only if $(S, \cdot, \leq)$ does not contain proper bi-ideals. The purpose of this note is to show that the result is also valid on ordered ternary semigroups.

2 Main Results

Let S be a nonempty set. Then S is called a *ternary semigroup* ([5], [9]) if there exists a ternary operation $S \times S \times S \to S$, written as $(x_1, x_2, x_3) \mapsto [x_1x_2x_3]$, such that

$$[[x_1x_2x_3]x_4x_5] = [x_1[x_2x_3x_4]x_5] = [x_1x_2[x_3x_4x_5]]$$

for all $x_1, x_2, x_3, x_4, x_5 \in S$.

Let $(S, [\ ])$ be a ternary semigroup. For nonempty subsets A_1, A_2 and A_3 of S, let

$$[A_1A_2A_3] = \{[x_1x_2x_3] \mid x_1 \in A_1, x_2 \in A_2, x_3 \in A_3\}.$$

For $x \in S$, let $[xA_1A_2] = [\{x\}A_1A_2]$. For any other cases can be defined analogously.

The notion of ternary semigroup was first introduced by Banach who showed by an example that a ternary semigroup does not necessary reduce to an ordinary semigroup [8]. Sioson studied ideals and radicals on ternary semigroups [10]. The authors in [5] investigated some properties of quasi-ideals and bi-ideals in ternary semigroups. Recently, the authors in [9] studied regular ternary semigroups; and the authors in [2] introduced and studied pure ideals in ternary semigroups.

A ternary semigroup $(S, [\ ])$ is called an *ordered ternary semigroup* [6] if there is a partial order $\leq$ on S such that

$$x \leq y \Rightarrow [xx_1x_2] \leq [yx_1x_2], [x_1xx_2] \leq [x_1yx_2], [x_1x_2x] \leq [x_1x_2y]$$

for all $x, y, x_1, x_2 \in S$.

Let $(S, [\ ], \leq)$ be an ordered ternary semigroup. For $A \subseteq S$, let

$$(A] = \{x \in S \mid x \leq a \text{ for some } a \in A\}.$$

In [6], the notion of ideal extensions of ordered ternary semigroups have been introduced and studied.

Hereafter, let S denote an ordered ternary semigroup $(S, [\ ], \leq)$.

A nonempty subset A of S is called a *left (respectively, right) ideal* [6] of S if

(i) $[SSA] \subseteq A$ (respectively, $[ASS] \subseteq A$).

(ii) If $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

S is said to be *left simple* if it does not contain proper left ideals of S. Dually, S is said to be *right simple* if it does not contain proper right ideals of S.

For an element x of S, we have

$$(\{x\} \cup [SSx]] \text{ and } (\{x\} \cup [xSS]]$$

are left ideal of S containing x and right ideal of S containing x, respectively.

A nonempty subset A of S is called a *bi-ideal* of S if

(i) $[ASASA] \subseteq A$.

(ii) If $x \in A$ and $y \in S$ such that $y \leq x$, then $y \in A$.

Note that every left and right ideal of S is a bi-ideal of S.

As usual, a left (respectively, right, bi-) ideal A of S is said to be *proper* if $A \neq S$.

A bi-ideal A of S is called a *subidempotent bi-ideal* if $[AAA] \subseteq A$.

S is said to be *regular* if for any $a \in S$, $a \in ([aSa]]$.

As in [7], [4] and [3], we have the following.

(1) S is left simple if and only if $([SSx]] = S$ for all $x \in S$.

Indeed: assume that $([SSx]] = S$ for all $x \in S$. Let A be a left ideal of S and $a \in A$. We have $([SSa]] = S$. Let $y \in S$. Then $y \in ([SSa]]$. Thus $y \leq [zwa]$ for some $z, w \in S$. Since $[zwa] \in A$, $y \in A$. Hence $A = S$. The opposite direction follows by $([SSx]]$ is a left ideal of S for all $x \in S$.

(2) S is right simple if and only if $([xSS]] = S$ for all $x \in S$.

This can be proved as (1).

(3) If S is left and right simple, then S is regular.

In fact: assume that S is left and right simple. To show that S is regular, let $a \in S$. By (1) and (2), $S = ([SSa]$ and $S = ([aSS]]$. Then

$$a \in S = ([aSS]] = ([aS([SSa]]]] = ([a[SSS]a]] \subseteq ([aSa]].$$

(4) If S is regular, then the bi-ideals and the subidempotent bi-ideals of S are the same.

To see this, assume that S is regular. Let A be a bi-ideal of S. Then $[ASASA] \subseteq A$. By assumption, $A \subseteq ([ASA]]$. Hence

$$\begin{aligned}[AAA] \subseteq [([ASA]]([ASA]]([ASA]]] &\subseteq ([ASA[ASAAS]A]] \\ &\subseteq ([ASASA]] \\ &\subseteq (A] \\ &= A.\end{aligned}$$

Theorem. Let $(S, [\ ], \leq)$ be an ordered ternary semigroup. $(S, [\ ], \leq)$ is left and right simple if and only if $(S, [\ ], \leq)$ does not contain proper bi-ideals.

Proof. Assume that $(S, [\ ], \leq)$ is left and right simple. Let A be a bi-ideal of $(S, [\ ], \leq)$. We shall show that $S \subseteq A$. Let $x \in S$ and $y \in A$. Since S is left simple, we have $S = (\{y\} \cup [SSy]]$. Then $x \leq y$ or $x \leq [zwy]$ for some $z, w \in S$. If $x \leq y$, then $x \in A$. Assume that $x \leq [zwy]$. Since S is right simple, $S = (\{y\} \cup [ySS]]$. Since $z \in S$, $z \leq y$ or $z \leq [yuv]$ for some $u, v \in S$. By (3), $(S, [\ ], \leq)$ is regular. Let $b \in S$ be such that $y \leq [yby]$. If $z \leq y$, then $x \leq [ywy] \leq [ybywy] \in [ASASA] \subseteq A$. If $z \leq [yuv]$, then $x \leq [[yuv]wy] \leq [[yuv]w[yby]] \in [ASASA] \subseteq A$. Hence $x \in A$. Thus $S \subseteq A$.

Conversely, assume that $(S, [\ ], \leq)$ does not contain proper bi-ideals. If A is a left ideal of $(S, [\ ], \leq)$, then A is a bi-ideal of $(S, [\ ], \leq)$. By assumption, $S = A$. Similarly, if A is a right ideal of $(S, [\ ], \leq)$, then A is a bi-ideal of $(S, [\ ], \leq)$, so $S = A$.

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