Applied Mathematical Sciences, Vol. 6, 2012, no. 114, 5689 - 5693

Hit Number of Directed Bipartite Graphs

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Abstract

A total coloring f of a directed graph G is called edge-irregular if for any two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ of G the associated ordered triples $(f(u_1), f(e_1), f(v_1))$ and $(f(u_2), f(e_2), f(v_2))$ are different. The problem is to determine the minimum number of colors used in such a coloring of G. This parameter of G is denoted by $hit(G)$. In this paper we determine $hit(G)$ for complete bipartite graphs G.

Mathematics Subject Classification: 05C15, 05C20, 05C78

Keywords: directed graph, graph coloring, graph labeling

1 Introduction

In a vertex coloring (labeling) of a graph G, each vertex of G is assigned a color (label). If distinct vertices are assigned distinct colors, then the coloring is called *vertex-distinguishing* or *vertex-irregular*. That is, each vertex of G is uniquely determined by its color. Similarly, an edge coloring of G is *edge-irregular* if distinct edges are assigned distinct colors.

A proper vertex coloring is called *harmonious coloring* if every pair of colors appears on at most one pair of adjacent vertices. Since every coloring that assigns distinct colors to distinct vertices in a graph is a harmonious coloring, it follows that every graph has at least one harmonious coloring. The minimum positive integer k for which a graph G has a harmonious coloring with k colors is denoted by $h(G)$. Clearly, $h(G) \geq \Delta(G) + 1$, where $\Delta(G)$ denotes the maximum degree of G. An upper bound for this parameter was given by Lee and Mitchem [3], namely $h(G) \leq (\Delta^2(G) + 1) \cdot \lceil n^{1/2} \rceil$, where n denotes the number of vertices of G. Lu [4] and McDiarmid and Xinhua [5] determined two very similar but improved upper bounds for the harmonious chromatic number of a graph, namely, if G is a nonempty graph, then $h(G) \leq 2 \cdot \Delta(G) \cdot \lfloor n^{1/2} \rfloor$ and $h(G) \leq 2 \cdot \Delta(G) \cdot (n-1)^{1/2}$, respectively. Every harmonious coloring f of G induces an edge labeling of G where the edge uv is assigned the label $\{f(u), f(v)\}$, which is then a 2-element subset of the set of colors assigned to the vertices of G. Since no two edges of G are labeled the same, this vertex coloring is edge-irregular. Duchet introduced a related edge-irregular vertex coloring of a graph in which adjacent vertices are permitted to be colored the same. A *harmonic coloring* of a graph G is a vertex coloring f of G (where adjacent vertices may be assigned the same color) that induces an edge-irregular labeling that assigns to each edge uv the label $\{f(u), f(v)\}$, which is either a 2-element subset or a 1-element subset of colors, depending on whether $f(u) \neq f(v)$ or $f(u) = f(v)$. The minimum positive integer k for which a graph G has a harmonic coloring with k colors is denoted by $h'(G)$. Clearly, $\Delta(G) \leq h'(G) \leq h(G)$. By Vizing's theorem, $\Delta(G) \leq \chi'(G) \leq \Delta(G) + 1$, where $\chi'(G)$ is the edge chromatic number (chromatic index) of G. Salvi [6] showed that $\chi'(G) = \Delta(G)$ whenever $h'(G) = \Delta(G)$. Jendroľ [2] generalized the harmonic coloring to *multiset coloring*. A (not necessarily proper) total coloring f (an assignment of colors to the vertices and the edges) of a graph G, is a multiset coloring if for any two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ the associated triples $\{f(u_1), f(e_1), f(v_1)\}$ and $\{f(u_2), f(e_2), f(v_2)\}$ are different. The minimum number of colors used in such a coloring of G is denoted by $ms(G)$. Clearly, every multiset coloring of G such that each edge has the same color is also a harmonic coloring of G, hence $ms(G) \leq h'(G)$.

Considering the edge-irregular property, we may restrict our attention to directed graphs. In this case we assign an ordered pair $(f(u), f(v))$ or an ordered triple $(f(u), f(e), f(v))$ to every edge $e = uv$ of a graph and we can minimize the corresponding graph parameter. Budajová [1] introduced precisely this notion. Let $G = (V, E)$ be a *directed graph*. A mapping $f : V \cup E \to \{1, \ldots, k\}$ is called *total coloring*. The total coloring f is called *edge-irregular* if for any two edges $e_1 = u_1v_1$ and $e_2 = u_2v_2$ the associated ordered triples $(f(u_1), f(e_1), f(v_1))$ and $(f(u_2), f(e_2), f(v_2))$ are different. The problem is to determine the minimum number of colors used in such a coloring of G. This parameter of G is denoted by $hit(G)$.

In this paper we determine $hit(G)$ for complete bipartite graphs G.

Note that if we have a graph G, then we can "extend" it to a directed graph G' – we assign a direction to every edge. Observe that every multiset coloring of G induces a total edge coloring of the corresponding directed graph

G' which is edge-irregular, therefore $ms(G) \geq hit(G')$. On the other hand, the difference between $ms(G)$ and $hit(G')$ can be arbitrarily large.

2 Results

Let $K_{m,n}$ denote the directed complete bipartite graph on $m+n$ vertices, let $A = \{u_1, \ldots, u_m\}$ and $B = \{v_1, \ldots, v_n\}$ be the parts of $K_{m,n}$ and let $u_i v_j$ be the edges, $i \in \{1, \ldots, m\}$, $j \in \{1, \ldots, n\}$. We will use this notation of vertices of the two parts of $K_{m,n}$ in the whole paper. Let $K^*_{m,n}$ be a graph obtained from $K_{m,n}$ by changing the directions of all edges.

Lemma 2.1 *$hit(K_{m,n}) = hit(K^*_{m,n})$ for any $m, n \geq 1$.*

Proof Clearly, if the ordered triples (x_1, x_2, x_3) and (y_1, y_2, y_3) are different, then (x_3, x_2, x_1) and (y_3, y_2, y_1) are also different. Therefore any edge-irregular coloring of $K_{m,n}$ is an edge-irregular coloring of $K^*_{m,n}$ and vice versa. □

Lemma 2.2 *$hit(K_{1,n}) = \lceil n^{1/2} \rceil$ for any $n \geq 1$.*

Proof The graph $K_{1,n}$ has n edges and each of them is incident with the same vertex u_1. If we have $\lceil n^{1/2} \rceil$ colors, then we can create $\lceil n^{1/2} \rceil^2 \geq n$ different pairs. Now we can define an edge-irregular coloring of $K_{1,n}$ in the following way: color the vertex u_1 with color 1 and assign different pairs of colors to uncolored edges and its endvertices.

Now assume that $hit(K_{1,n}) \leq \lceil n^{1/2} \rceil - 1$ for some n. In any edge-irregular coloring of $K_{1,n}$ all ordered triples are different. Since the first element of these triples is the same (the color of u_1), the pairs obtained from the triples by removing the first element must be different. It is impossible to create n different pairs from $\lceil n^{1/2} \rceil - 1$ colors, a contradiction. □

Lemma 2.3 *Let H be a subgraph of G. Then $hit(H) \leq hit(G)$.*

Proof Every edge-irregular coloring of G induces an edge-irregular coloring of H. □

In the rest of the paper we will assume that $m \leq n$ (see Lemma 2.1).

Lemma 2.4 *$hit(K_{m,n}) = \lceil n^{1/2} \rceil$ for any $m \leq \lceil n^{1/2} \rceil$, $n \geq 1$.*

Proof Since $K_{1,n}$ is a subgraph of $K_{m,n}$ Lemmas 2.2 and 2.3 imply that $hit(K_{m,n}) \geq \lceil n^{1/2} \rceil$. Hence it suffices to show that $K_{m,n}$ has an edge-irregular coloring with $\lceil n^{1/2} \rceil$ colors. Let $f(u_1) = 1$, $f(v_i) = c_i$ and $f(u_1 v_i) = c_{1i}$, $i \in \{1, \ldots, n\}$, be an edge-irregular coloring of $K_{1,n}$ with $\lceil n^{1/2} \rceil$ colors. Now we extend this coloring to an edge-irregular coloring of $K_{m,n}$ in the following way: $f(u_j) = j$, $f(v_i) = c_i$ and $f(u_j v_i) = c_{1i}$ for $j \in \{1, \ldots, m\}$, $i \in \{1, \ldots, n\}$. □

Corollary 2.1 *$hit(K_{m,m^2}) = m$ for any $m \geq 1$.*

Proof It immediately follows from Lemma 2.4. □

Since Lemma 2.4 gives the exactly value of $hit(K_{m,n})$ for $n \geq m^2$ in the following we will assume that $n < m^2$.

Corollary 2.2 *$hit(K_{m,n}) \leq m$ for any $n < m^2$.*

Proof It immediately follows from Lemma 2.3 and Corollary 2.1. □

Lemma 2.5 *If $hit(K_{m,n}) = k$, then $\lceil \frac{m}{k} \rceil \cdot \lceil \frac{n}{k} \rceil \leq k$.*

Proof If an edge-irregular coloring of $K_{m,n}$ uses k colors, then in the set $\{u_1, \ldots, u_m\}$ at least $\lceil \frac{m}{k} \rceil$ vertices have the same color, say c_1. Similarly in $\{v_1, \ldots, v_n\}$ at least $\lceil \frac{n}{k} \rceil$ vertices have the same color, say c_2. Consequently, we have at least $\lceil \frac{m}{k} \rceil \cdot \lceil \frac{n}{k} \rceil$ triples which have first element c_1 and third element c_2. Since these triples are different all the second elements must be different. We have only k colors for the second elements, therefore we can have at most k such triples. □

Lemma 2.6 *If $\lceil \frac{m}{k} \rceil \cdot \lceil \frac{n}{k} \rceil \leq k$, then $hit(K_{m,n}) \leq k$.*

Proof Let k be an integer which satisfies $\lceil \frac{m}{k} \rceil \cdot \lceil \frac{n}{k} \rceil \leq k$. Let $a = \lceil \frac{m}{k} \rceil$ and $b = \lceil \frac{n}{k} \rceil$. Let p, r, s, t be integers such that $m = p \cdot a + r$, $0 \leq r < a$ and $n = s \cdot b + t$, $0 \leq t < b$. Clearly, $a, b, p, s \geq 1$, $p, s \leq k$ and if $r \neq 0, t \neq 0$, then $p + 1 \leq k, s + 1 \leq k$, respectively.

Now we define a total coloring f of $K_{m,n}$ in the following way:

$f(u_{1+(i-1)\cdot a}) = \ldots = f(u_{a+(i-1)\cdot a}) = i$ for $i = 1, \ldots, p$, and $f(u_{1+p\cdot a}) = \ldots = f(u_{r+p\cdot a}) = p + 1$. $f(v_{1+(j-1)\cdot b}) = \ldots = f(v_{b+(j-1)\cdot b}) = j$ for $j = 1, \ldots, s$, and $f(v_{1+s\cdot b}) = \ldots = f(v_{t+s\cdot b}) = s + 1$. We color the edges joining $u_{1+(i-1)\cdot a}, \ldots, u_{a+(i-1)\cdot a}$ with $v_{1+(j-1)\cdot b}, \ldots, v_{b+(j-1)\cdot b}$ with different colors for $i = 1, \ldots, p + 1$, $j = 1, \ldots, s + 1$ (we have so many colors since $a \cdot b \leq k$).

Clearly, this coloring is edge-irregular, since if the first and the third elements of two triples are the same, then the second elements are different. □

Theorem 2.1 *$hit(K_{m,n}) = \min\{k : \lceil \frac{m}{k} \rceil \cdot \lceil \frac{n}{k} \rceil \leq k\}$.*

Proof Assume that $hit(K_{m,n}) = H$ and $\min\{k : \lceil \frac{m}{k} \rceil \cdot \lceil \frac{n}{k} \rceil \leq k\} = h$. Lemma 2.5 implies that $\lceil \frac{m}{H} \rceil \cdot \lceil \frac{n}{H} \rceil \leq H$. By the definition of h we have $h \leq H$.

From Lemma 2.6 (with $k = h$) it follows that $H = hit(K_{m,n}) \leq h$. □

References

[1] K. Budajová, Total edge irregular number of digraphs, Acta Avionica 21 (2011), 12–16.

[2] B. Jendroľ, Total edge-distinguishing number, Doctoral thesis, University of P.J. Šafárik (2002) (in Slovak).

[3] S.M. Lee and J. Mitchem, An upper bound for the harmonious chromatic number of a graph, J. Graph Theory 11 (1987), 565–567.

[4] Z. Lu, On an upper bound for the harmonious chromatic number of a graph, J. Graph Theory 15 (1991), 345–347.

[5] C. McDiarmid and L. Xinhua, Upper bounds for harmonious colorings, J. Graph Theory 15 (1991), 629–636.

[6] N.Z. Salvi, A note on the line-distinguishing chromatic number and the chromatic index of a graph, J. Graph Theory 17 (1993), 589–591.

Received: June, 2012