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# Analysis of $2^n$ Factorial Experiments with Exponentially Distributed Response Variable

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**Abstract**

This paper assesses and compares the performance of generalized linear models (GLM) with respect to log transformation and ANOVA approaches on the basis of coverage results and expected length of confidence intervals (CI) for the expected responses for exponentially distributed response variable. We also focus on the power functions of tests of hypothesis for testing significance of underlying factorial effects. Although, the comparison is mainly simulation based, whenever possible, theoretical investigation is carried out for saturated 2-level factorial experiments. The coverage results of log transformation approach are uniformly superior to GLM for two replications (very small sample) which is most frequently encountered situation in the industrial experiments. We also suggested a modification to GLM based confidence intervals that shows decisive improvement over the GLM. t-test under GLM approach exhibits best power function for testing significance of factorial effects but t-test under log transformation approach is also equally good.

**Mathematics Subject Classification:** 62K15
**Keywords:** confidence intervals; generalized linear models; log transformation; factorial experiments;

# 1 Introduction and Preliminaries

## 1.1 Introduction

The optimality properties of ANOVA based F-test heavily depend on normality and homogeneity assumptions. Certainly there are many situations, where the non-normal response and nonhomogeneous variance is a fact. These responses may be related non-linearly with the input factors and the variance is not constant but is a function of mean. Moreover, only very small number of replications is available due to stringent practical constraints. Traditionally in such situations, a suitably transformed response variable (most frequently a variance stabilizing transformation) is analyzed using ANOVA based F-test and the results are transformed back to the original scale. But this approach has many drawbacks, see for example, Myers and Montgomery [3].

GLM, introduced by Nelder and Wedderburn [1] can be used as an alternative approach to the above mentioned traditional approaches. Myers and Montgomery [3] gave a tutorial on GLM, Lewis et.al. [4] gave many examples of designed experiments with non-normal responses and Lewis et.al. [5] attempted a simulation study on analysis of designed experiments involving non-normal response variables using GLM. The use of GLM is suggested for the analysis of designed experiments involving non-normal response variables and small number of replications. All the above discussions are mainly concentrated on estimating CI for the expected responses. The authors compared all above mentioned approaches on the basis of response estimation and prediction limited to a sample of parameter combinations.

In industrial settings, sometimes the response variable could have an exponential distribution. The emphasis of this paper is to assess and compare the performance of GLM with respect to transformation and ANOVA approaches on the basis of coverage results and expected length of confidence intervals (E(LOCI))for the expected responses for exponentially distributed response variable. We also focus on the power functions of the tests of hypothesis for testing significance of underlying factorial effects. Although, the comparison is mainly simulation based, whenever possible, theoretical investigation is also carried out. We also suggest a modification to CI for GLM that shows decisive improvement over the GLM approach at the prescribed confidence coefficient. For transformation approach, we consider the log transformation, which is variance stabilizing transformation and is commonly used for lifetime data and refer it as 'LOG approach' henceforth.

## 1.2 Nature of the underlying design matrix

The set up discussed here is that of saturated 2-level factorial designs with $r(>1)$ replications. The number of parameters $(m)$ in this case is equal to the number of distinct treatment combinations ($k$) under which maximum likelihood estimators (m.l.e.), under the GLM setup take a closed form expression that enables the theoretical comparison of three approaches. Note that, the underlying model can be viewed as a regression model with design matrix (cf. Myers and Montgomery [3]),

$$X_{n\times k} = (\,A, ..., A\,)' \tag{1}$$

where $n = rk$ and $A$ is a $k \times k$ symmetric Hadamard matrix corresponding to a single replicate of the experiment.

From (1) and the properties of matrix $A$, it follows that, $X'X = rkI_k$ and $XX'$ is an $rk \times rk$ matrix consisting of $r \times r$ array of $k \times k$ submatrices, where each submatrix equals $kI_k$, $I_k$ being an identity matrix of order $k$. Throughout this paper, we assume that, the response variable $Y$ is an exponentially distributed random variable. $Y_{ij}, i = 1, 2, ..., k, j = 1, 2, ..., r$ are observations on $Y$. $W_{ij} = log(Y_{ij})$ is the log transformed response, $\bar{Y_{i.}} = \frac{1}{r}\sum_{j=1}^{r} Y_{ij}, S_y^2 = \frac{1}{k}\sum_{i=1}^{k} S_i^2$ where $S_i^2 = \frac{1}{r-1}\sum_{j=1}^{r}(Y_{ij} - \bar{Y_{i.}})^2$ and $\bar{W_{i.}}$ and $S_w^2$ are similarly defined.

The remainder of the paper is organized as follows: In Section 2, our discussion is concerned about the theoretical details of above three approaches. Section 3 reports the simulation based comparison regarding the performance of the above discussed methods in terms of E(LOCI) and coverage results for expected response. Here we also propose a modification to GLM based CI, that shows decisive improvement. In Section 4, we compare the three approaches based on the power functions of tests for testing significance of factorial effects and Section 5 gives concluding remarks.

# 2 Theoretical Details of the Three Approaches

## 2.1 The underlying model and related formulae for various approaches

Table 1 summarizes the underlying model, estimated responses, confidence limits and LOCI around expected responses for the three approaches. Here $\overset{\wedge}{Y}$ denotes the estimated response, $\mu_i$ denotes the expected response corresponding to the $i^{th}$ regressor vector $X_i$, which is $i^{th}$ row of the coefficient matrix $X$. $\sigma_y^2$ and $\sigma_w^2$ are the unknown constant variances of $Y$ and $W$ respectively. The components of the vector of parameters $\underset{\sim}{\beta}$ correspond to various factorial effects. $t_{(n-k),\alpha/2}$ is the upper $100\alpha/2^{th}$ percentile of the t-distribution with $n - k$ degrees of freedom (d.f.) and $Z_{\alpha/2}$ is the upper $100\alpha/2^{th}$ percentile of the standard normal distribution. For the illustration, the confidence coefficient $1 - \alpha = 0.95$ is used throughout the paper.

Note that, for ANOVA and LOG approach the formulae in the Table 1 follow easily using standard results and properties of Hadamard matrix.

**Table 1:** Underlying model and related formulae for the three approaches.

| Method | Model | Response distribution and estimated response under the respective model | Confidence Limits [a] and LOCI [b] |
|---|---|---|---|
| ANOVA | $E\underline{Y} = X\underline{\beta}$ | $Y_{ij} \sim N(\mu_i, \sigma_y^2)$ | $\bar{Y}_{i.} \pm t_{(n-k),\alpha/2}\sqrt{S_y^2/r}^{\,a}$ |
| | $\mu_i = (X_i'\underline{\beta})$ | $\hat{Y}_{ij} = \bar{Y}_{i.}$ | $2t_{(n-k),\alpha/2}\sqrt{S_y^2/r}^{\,b}$ |
| LOG | $W_{ij} = \log(Y_{ij})$ | $W_{ij} \sim N(\mu_i, \sigma_w^2)$ | $exp(\bar{W}_{i.} \pm t_{(n-k),\alpha/2}\sqrt{S_w^2/r})^a$ |
| | $E\underline{W} = X\underline{\beta}$ | $\hat{Y}_{ij} = \exp(\bar{W}_{i.})$ | $((v^2-1)/v)\exp(\bar{W}_{i.})^b$ |
| | $\mu_i = (X_i'\underline{\beta})$ | $= (\prod Y_{ij})^{1/r} = GM$ | $v = \exp(t_{(n-k),\alpha/2}\sqrt{s_w^2/r})$ |
| GLM | $E\underline{Y} = \exp(X'\underline{\beta})$ | $Y_{ij} \sim Exponential(\mu_i)$ | $\bar{Y}_{i.}\exp(\pm Z_{\alpha/2}/\sqrt{r})^a$ |
| | $\mu_i = \exp(X_i'\underline{\beta})$ | $\hat{Y}_{ij} = \bar{Y}_{i.}$ | $\bar{Y}_{i.}\Big(\exp\{Z_{\alpha/2}/\sqrt{r})\}$ $-\exp\{-Z_{\alpha/2}/\sqrt{r})\}\Big)^b$ |

For GLM approach, the model is $E\underline{Y} = \underline{\mu} = g^{-1}(\underline{\eta})$; $\underline{\eta} = X'\underline{\beta} = g(\underline{\mu})$ where $g(.)$ is an appropriate monotonic and differentiable link function. Two types of links exist; canonical and non-canonical link. The distribution of response variable $\underline{Y}$ is assumed to be a member of the exponential family. We refer to McCullagh and Nelder [2] for more theoretical details of GLM.

It is well known that, for exponentially distributed response $\underline{Y}$, the canonical link selects the reciprocal link $\eta_i = \mu_i^{-1}$. Due to some drawbacks of this link, for example, negative values of the estimated response may be encountered, we choose the non-canonical 'log link', which overcomes these drawbacks. For this link, the underlying model is

$$E(Y_{ij}) = \mu_i = exp(X_i'\underline{\beta}) \quad i = 1, ..., k, j = 1, ..., r. \tag{2}$$

where $X_i'$ is the $i^{th}$ row of $A$ in the design matrix (1) and $\underline{\mu}$ corresponds to a single replicate.

Parameter estimation in the GLM is performed using the method of maximum likelihood estimation. Under the full model, since, the matrix $A$ is nonsingular, $\underline{\beta}$ is just a reparameterization of $\underline{\mu}$, giving the estimated response in closed form; $\hat{Y}_{ij} = \hat{\mu}_i = exp(X_i'\hat{\underline{\beta}}) = \bar{Y}_{i.}$. Consequently $\hat{\underline{\beta}} = A^{-1}log(\bar{Y}_{i.})$ is also available in closed form, which is not the case in the non-full models.

The CI for mean response $\mu_i$ at the point $X_i$, is given by $exp(X_i'\hat{\underline{\beta}} \pm Z_{\alpha/2}\hat{\sigma}_i)$ where for the non-canonical link, $\hat{\sigma}_i = \sqrt{(a(\phi))^2 X_i'(X'\Delta\hat{V}\Delta X)^{-1}X_i}$ (cf. Myers and Montgomery [3]), $\hat{V}$ is the diagonal matrix whose $i^{th}$ diagonal element is estimate of $Var(Y_i)$ and $\Delta$ is the diagonal matrix whose $i^{th}$ diagonal element $\Delta_i$ is the derivative of $(-1/\mu_i)$ with respect to $X_i'\underline{\beta}$. For the design matrix (1), noting that, for exponential response variable $a(\phi) = -1$, $\Delta_i = exp(-X_i' \overset{\wedge}{\underline{\beta}})$ and $Var(Y_i) = exp(2X_i' \overset{\wedge}{\underline{\beta}})$, this simplifies to $\overset{\wedge}{\sigma}_i = 1/\sqrt{r}$. A further simplification leads to the CI given by $\bar{Y}_{i.}exp(\pm Z_{\alpha/2}/\sqrt{r})$ with length $\bar{Y}_{i.}(exp(Z_{\alpha/2}/\sqrt{r}) - exp(-Z_{\alpha/2}/\sqrt{r}))$ which are reported in the last column of Table 1. The E(LOCI) is

$$E(LOCI) = \mu_i\Big(exp(Z_{\alpha/2}/\sqrt{r}) - exp(-Z_{\alpha/2}/\sqrt{r})\Big)$$

which at $\alpha = 0.05$, for r=2 and 3 replications equals,

$$E(LOCI) = \begin{cases} 3.7484\mu_i & \text{for} \quad \text{r=2} \\ 2.7781\mu_i & \text{for} \quad \text{r=3} \end{cases} \tag{3}$$

Note that, both ANOVA and GLM estimate the expected response by $\bar{Y}_{i.}$ which is an unbiased estimator of $\mu_i$. But the estimated response for LOG approach is nothing but the geometric mean (GM) of the relevant group of observations, which underestimates the expected responses. In some situations, for example, when the response variable is time to recurrence of a defective item, this could be quit undesirable.

## 2.2 Approximate E(LOCI) for ANOVA and LOG

**i) E(LOCI) for ANOVA approach**

For ANOVA approach, $LOCI = 2t_{(n-k),\alpha/2}\sqrt{S_y^2/r}$ where $S_y^2 = (1/k)\sum_{i=1}^{k} S_i^2$, $S_i^2 = \frac{1}{r-1}\sum_{j=1}^{r}(Y_{ij} - \bar{Y}_{i.})^2$. Noting that, $S_i^2$ is an unbiased estimator of $\mu_i^2$, it follows that, $E(LOCI) \approx 2t_{(n-k),\alpha/2}\sqrt{\bar{\mu}^*/r}$, where $\bar{\mu}^*$ is the average of $\mu_i^2, i = 1, 2, ..., k$. At $\alpha = 0.05$, for r=2 and 3 replications this equals

$$E(LOCI) = \begin{cases} 3.9266\sqrt{\bar{\mu}^*} & \text{for} \quad \text{r=2} \\ 2.6627\sqrt{\bar{\mu}^*} & \text{for} \quad \text{r=3} \end{cases} \tag{4}$$

**ii) E(LOCI) for LOG approach**

For LOG approach, as reported in Table 1, $LOCI = ((v^2 - 1)/v)exp(\bar{W}_{i.})$ where $v = \exp(t_{(n-k),\alpha/2}\sqrt{s_w^2/r})$. The delta method leads to $E(S_w^2) = 1$ and as noted earlier, $exp(\bar{W}_{i.})$ is the GM of the underlying group of observations $E(\exp(\bar{W}_{i.})) = (\mu_i^{\frac{1}{r}}\frac{1}{r}\Gamma\frac{1}{r})^r$. Although $v$ and $\bar{W}_{i.}$ need not be independently distributed, we can have approximately

$$E(LOCI) \approx E((v^2 - 1)/v)E(\exp(\bar{W}_{i.}))$$

$$= \mu_i\Big(\exp(t_{(n-k),\alpha/2}/\sqrt{r}) - \exp(-t_{(n-k),\alpha/2}/\sqrt{r})\Big)\Big(\frac{1}{r}\Gamma(1/r)\Big)^r.$$

In particular at $\alpha = 0.05$, r=2 and 3 replications this gives,

$$E(LOCI) \approx \begin{cases} 5.4836\mu_i & \text{for} \quad \text{r=2} \\ 2.5080\mu_i & \text{for} \quad \text{r=3} \end{cases} \tag{5}$$

Note that, for GLM and LOG approach the expected lengths depend only on the particular $\mu_i$ under consideration while for ANOVA, they depend on all components of $\underline{\mu}$ through $\bar{\mu}^*$ and are same for all components of $\underline{\mu}$.

In the next section we discuss the expected coverage probability of the above approaches.

## 2.3 Coverage probability for three approaches

**i) Coverage probability for GLM approach**

For GLM approach, the desired coverage probability for CI is

$$Pr\Big(\bar{Y}_i.\exp(-Z_{\alpha/2}/\sqrt{r}) \leq \mu_i \leq \bar{Y}_i.\exp(Z_{\alpha/2}/\sqrt{r})\Big)$$

$$= Pr\Big(r\exp(-Z_{\alpha/2}/\sqrt{r}) \leq \sum_{j=1}^{r} Y_{ij}/\mu_i \leq r\exp(Z_{\alpha/2}/\sqrt{r})\Big)$$

Since, $\sum_{j=1}^{r} Y_{ij}/\mu_i$ follows Gamma$(1, r)$ distribution, at $\alpha = 0.05$ this equals

$$G\Big(r(7.0993)^{1/\sqrt{r}}, 1, r\Big) - G\Big(r(0.1409)^{1/\sqrt{r}}, 1, r\Big) \tag{6}$$

where G$(x, 1, r)$ is the cumulative distribution function at $x$ for Gamma$(1, r)$ distribution. For r=2 it is 0.9067 and for r=3 it equals 0.9207.

**ii) Lower Bound (L.B.) for coverage probability of ANOVA approach**

The desired coverage probability for ANOVA approach is

$$Pr\Big(\bar{Y}_{i.} - t_{(n-k),\alpha/2}\sqrt{S_y^2/r} \leq \mu_i \leq \bar{Y}_{i.} + t_{(n-k),\alpha/2}\sqrt{S_y^2/r}\Big)$$

Noting that, $Var(\bar{Y}_{i.}) = \mu_i^2/r$, replacing $S_y^2$ by $E(S_y^2) = (1/k)\sum_{i=1}^{k}\mu_i^2 = \bar{\mu}^*$, Chebychev's inequality gives

$$Pr(|\mu_i - \bar{Y}_{i.}| \leq t_{(n-k),\alpha/2}\sqrt{S_y^2/r}) \geq 1 - \{Var(\bar{Y}_{i.})(t^2{}_{(n-k),\alpha/2}(S_y^2/r))^{-1}\}$$

$$\cong 1 - ({\mu_i}^2/t^2{}_{(n-k),\alpha/2}\bar{\mu}^*)$$

This gives an approximate L.B. for coverage of ANOVA approach

$$L(\underline{\mu}) = 100\Big(1 - ({\mu_i}^2/t^2{}_{(n-k),\alpha/2}\bar{\mu}^*)\Big) \tag{7}$$

The L. B. in (7) for the vectors $\underline{\mu}$ used in the simulation study are listed in Tables 2 and 4, columns 6 and 11.

**iii) L. B. for Coverage probability of LOG approach**

For LOG approach, referring to Table 1, the desired coverage probability can be approximated as

$$Pr\Big(\bar{W}_{i.} - t_{(n-k),\alpha/2}\sqrt{S_w^2/r} \le log(\mu_i) \le \bar{W}_{i.} + t_{(n-k),\alpha/2}\sqrt{S_w^2/r}\Big)$$

Noting that, for LOG approach, $Var(\bar{W}_{i.}) = 1/r$ and $E(S_w^2) = 1$ arguments similar to those for ANOVA lead to, the approximate lower bound for coverage of LOG approach

$$L(\underline{\mu}) = 100\Big(1 - (1/t^2{}_{(n-k),\alpha/2})\Big) \tag{8}$$

Unfortunately for fixed $\alpha$, noting that $n = rk$, the L.B. in (8) is decreasing in the number of replications $r$ and the number of treatment combinations $k$ through the degrees of freedom associated with the t-quantile. At $\alpha = 0.05$, this equals 87.03 for $r = 2, k = 4$; 81.2 for $r = 3, k = 4$ as well as for $r = 2$, $k = 8$ and 77.75 for $r = 3, k = 8$. This is also reflected in simulation study reported in the next section.

# 3 Simulation Study

## 3.1 Back-ground and Details of the simulation study

Noting the dependence of E(LOCI) for the three approaches on the expected responses, $\underline{\mu}$ rather than on the individual parameters $\underline{\beta}$, we compare the above approaches for various values of $\underline{\mu}$. Since a small number of replications is a common practice in industrial experiments, the simulation study is carried out for $2^2$ and $2^3$ experiments with r=2 and 3 replications for exponentially distributed response $\underline{Y}$. A few sets of the mean response corresponding to a single replicate are selected for simulation study which are given in the first column of Tables 2 and 4. The coefficient matrix is as defined in (1). Observing that, coverage results and E(LOCI) for ANOVA approach depend heavily on ${\mu_i}^2$ and $\bar{\mu}^*$, (cf. equations (4) and (7)), a few sets of $\underline{\mu}_1$ with $\mu_i$ ranging between 0 to 115, values of $\bar{\mu}^* = \sum_{i=1}^{k} \mu_i^2/4$ ranging between 0 to 3000 and $\mu_i^2$ spread around $\bar{\mu}^*$, in various patterns are selected for simulation study. For each selected $\underline{\mu}$ the simulated E(LOCI) and coverage results reported in Tables 2, 3, 4, 5 are based on 5000 simulations from the appropriate model described in Section 1. The observations based on these tables are reported next.

**Table 2 Simulated coverage results for ANOVA, GLM, M-GLM and LOG approaches and theoretical L.B. for ANOVA approach for $2^2$ factorial experiment with $r$=2, 3 replications**

| | $r=2$ | | | | | $r=3$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{\mu}_1$ | ANOVA | GLM | M-GLM | LOG | L.B. ANOVA | ANOVA | GLM | M-GLM | LOG | L.B.ANOVA |
| 0.0003 | 100 | 91 | 95.12 | 92.86 | 100 | 100 | 92.06 | 95.02 | 90.08 | 100 |
| 0.0183 | 100 | 90.74 | 95 | 93.54 | 99.99 | 100 | 92.24 | 95.4 | 91 | 99.99 |
| 1 | 76.34 | 90.36 | 94.66 | 93.3 | 74.06 | 78.32 | 92.44 | 95.34 | 91.02 | 62.39 |
| 1 | 76.8 | 90.1 | 95.18 | 93.24 | 74.06 | 78.28 | 91.88 | 94.8 | 91.14 | 62.39 |
| 0.2865 | 98.7 | 90.2 | 95.06 | 93.38 | 99.95 | 99.88 | 92.06 | 94.78 | 91.26 | 99.93 |
| 0.0235 | 99.98 | 90.56 | 95.08 | 93.48 | 100 | 100 | 91.98 | 95.1 | 90.94 | 100 |
| 0.0019 | 100 | 90.18 | 94.42 | 93.48 | 100 | 100 | 92.24 | 95.08 | 91.16 | 100 |
| 9.4877 | 55.42 | 90.74 | 94.72 | 92.7 | 48.16 | 58.48 | 92.42 | 95.74 | 91.1 | 24.85 |
| 7.3891 | 87.44 | 91 | 95.14 | 94.48 | 82.71 | 87.7 | 92.68 | 95.08 | 90.24 | 74.93 |
| 7.3891 | 87.58 | 91.2 | 95.64 | 93.66 | 82.71 | 88.62 | 92.94 | 95.58 | 90.74 | 74.93 |
| 7.3891 | 87.96 | 90.24 | 94.84 | 93.5 | 82.71 | 88.22 | 92.66 | 95.36 | 90.88 | 74.93 |
| 0.1353 | 100 | 90.88 | 95.08 | 93.36 | 99.99 | 100 | 92.42 | 95.22 | 90.34 | 99.99 |
| 25.79 | 58.5 | 90.42 | 94.74 | 92.72 | 50.89 | 60.96 | 92.1 | 94.98 | 91.02 | 28.81 |
| 2.117 | 99.02 | 90.36 | 95.1 | 93.44 | 99.67 | 99.8 | 92.02 | 95.24 | 90.82 | 99.52 |
| 5.7546 | 93.44 | 90.52 | 94.64 | 93.28 | 97.55 | 96.9 | 92.68 | 95.5 | 90.74 | 96.46 |
| 0.1738 | 99.98 | 91 | 95.18 | 93.2 | 100 | 100 | 92.54 | 95.32 | 91.68 | 100 |
| 12.183 | 90.28 | 91.52 | 95.72 | 93.62 | 93.91 | 93.84 | 92.28 | 95.34 | 91.5 | 91.18 |
| 4.4817 | 98.68 | 91.32 | 95.36 | 93.16 | 99.18 | 99.56 | 92.3 | 94.82 | 91.08 | 98.81 |
| 33.116 | 64.26 | 90.72 | 95.3 | 93.46 | 55.02 | 64.88 | 92.52 | 95.02 | 91.12 | 34.8 |
| 0.0388 | 100 | 91.62 | 95.6 | 93.46 | 100 | 100 | 91.6 | 94.64 | 91.62 | 100 |
| 115.58 | 59.7 | 90.9 | 95.36 | 93.52 | 50.57 | 61.02 | 93 | 95.7 | 90.02 | 28.35 |
| 0.4724 | 100 | 91.58 | 95.66 | 92.2 | 100 | 100 | 92.52 | 95.9 | 90.42 | 100 |
| 25.79 | 92.24 | 91.12 | 95.08 | 93.66 | 97.54 | 96.1 | 91.76 | 94.58 | 91.32 | 96.43 |
| 0.0388 | 100 | 89.8 | 95.04 | 92.88 | 100 | 100 | 92.3 | 95.28 | 90.92 | 100 |
| 0.9375 | 93.68 | 90.8 | 95.14 | 92.82 | 88.71 | 95.28 | 92.6 | 95.52 | 91.7 | 83.63 |
| 0.875 | 94.84 | 90.96 | 95.2 | 93.7 | 90.16 | 95.54 | 92.08 | 95 | 90.84 | 85.74 |
| 1.0625 | 91.6 | 90.44 | 95.1 | 93.06 | 85.5 | 92.9 | 92.42 | 95.34 | 91.4 | 78.98 |
| 1.125 | 90.6 | 90.48 | 94.98 | 93.42 | 83.74 | 92.02 | 92.1 | 94.96 | 90.7 | 76.43 |
| 15 | 94.66 | 90.98 | 95 | 93.38 | 88.71 | 95.16 | 91.88 | 94.86 | 90.8 | 83.63 |
| 14 | 95.36 | 91.18 | 95.22 | 93.52 | 90.16 | 95.98 | 92.02 | 95 | 91.2 | 85.74 |
| 17 | 92.68 | 90.92 | 95.08 | 93.34 | 85.5 | 92.92 | 92.14 | 95.24 | 90.46 | 78.98 |
| 18 | 91.72 | 90.84 | 95.22 | 93.58 | 83.74 | 91.04 | 92.1 | 95.06 | 91.34 | 76.43 |
| 26 | 98.78 | 90.68 | 95.16 | 93.58 | 97.2 | 99.68 | 92.16 | 95.18 | 91.14 | 95.93 |
| 39 | 96.38 | 90.46 | 95.14 | 93 | 93.69 | 97.78 | 92.54 | 95.42 | 90.16 | 90.85 |
| 65 | 88.34 | 91.16 | 95.56 | 93.78 | 82.47 | 89.1 | 91.72 | 94.8 | 91.06 | 74.59 |
| 78 | 83.76 | 90.9 | 95.3 | 93.58 | 74.76 | 83.12 | 91.28 | 94.42 | 91.2 | 63.41 |
| **Average** | 89.96 | 90.77 | 95.13 | 93.34 | | 90.92 | 92.24 | 95.16 | 90.95 | |

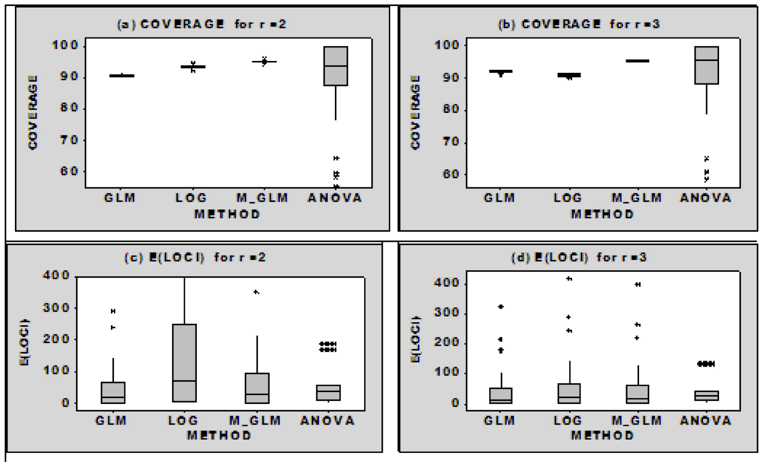


**Fig. 1: Box Plots for Coverage results and E(LOCI) for a $2^2$ factorial experiments for ANOVA, GLM, M-GLM and LOG Approaches**

**Table 3 Simulated E(LOCI) for ANOVA, GLM, M-GLM and LOG approaches for $2^2$ factorial experiment with $r$=2, 3 replications**

| | $r=2$ | | | | $r=3$ | | | |
|---|---|---|---|---|---|---|---|---|
| $\underline{\mu}_1$ | ANOVA | GLM | M-GLM | LOG | ANOVA | GLM | M-GLM | LOG |
| 0.0003 | 2.265 | 0.001 | 0.002 | 0.004 | 1.636 | 0.001 | 0.001 | 0.001 |
| 0.0183 | 2.265 | 0.069 | 0.1 | 0.279 | 1.636 | 0.051 | 0.063 | 0.07 |
| 1 | 2.265 | 3.681 | 5.363 | 12.944 | 1.636 | 2.762 | 3.412 | 3.686 |
| 1 | 2.265 | 3.724 | 5.426 | 12.586 | 1.636 | 2.766 | 3.417 | 3.657 |
| 0.2865 | 12.98 | 1.06 | 1.55 | 3.8 | 10.22 | 0.8 | 0.99 | 1.1 |
| 0.0235 | 12.98 | 0.09 | 0.13 | 0.31 | 10.22 | 0.07 | 0.08 | 0.09 |
| 0.0019 | 12.98 | 0.01 | 0.01 | 0.02 | 10.22 | 0.01 | 0.01 | 0.01 |
| 9.4877 | 12.98 | 35.44 | 51.64 | 125.78 | 10.22 | 26.49 | 32.72 | 36.25 |
| 7.3891 | 21.57 | 27.69 | 40.35 | 99.32 | 15.51 | 20.8 | 25.69 | 27.82 |
| 7.3891 | 21.57 | 27.67 | 40.31 | 104.59 | 15.51 | 20.23 | 24.99 | 27.39 |
| 7.3891 | 21.57 | 27.65 | 40.28 | 100.18 | 15.51 | 20.42 | 25.22 | 27.53 |
| 0.1353 | 21.57 | 0.51 | 0.74 | 1.92 | 15.51 | 0.38 | 0.46 | 0.51 |
| 25.79 | 38.18 | 95.46 | 139.08 | 334.66 | 28.99 | 71.91 | 88.83 | 94.77 |
| 2.117 | 38.18 | 7.9 | 11.52 | 27.29 | 28.99 | 5.89 | 7.28 | 8.03 |
| 5.7546 | 38.18 | 21.42 | 31.21 | 80.53 | 28.99 | 16.04 | 19.82 | 21.27 |
| 0.1738 | 38.18 | 0.65 | 0.95 | 2.16 | 28.99 | 0.49 | 0.6 | 0.64 |
| 12.183 | 54.67 | 46.93 | 68.38 | 151.52 | 40.61 | 33.91 | 41.89 | 45.15 |
| 4.4817 | 54.67 | 16.66 | 24.27 | 56.75 | 40.61 | 12.44 | 15.37 | 16.62 |
| 33.116 | 54.67 | 125.40 | 182.70 | 423.58 | 40.61 | 92.87 | 114.72 | 125.09 |
| 0.0388 | 54.67 | 0.15 | 0.21 | 0.48 | 40.61 | 0.11 | 0.13 | 0.12 |
| 115.58 | 170.43 | 438.96 | 639.53 | 1633.55 | 129.41 | 323.07 | 399.07 | 420.07 |
| 0.4724 | 170.43 | 1.79 | 2.6 | 5.6 | 129.41 | 1.3 | 1.61 | 1.74 |
| 25.79 | 170.43 | 99.42 | 144.85 | 329.46 | 129.41 | 71.36 | 88.14 | 97.88 |
| 0.0388 | 170.43 | 0.15 | 0.21 | 0.53 | 129.41 | 0.11 | 0.13 | 0.14 |
| 0.9375 | 3.49 | 3.51 | 5.11 | 12.46 | 2.49 | 2.66 | 3.29 | 3.56 |
| 0.875 | 3.49 | 3.29 | 4.8 | 12.5 | 2.49 | 2.44 | 3.01 | 3.33 |
| 1.0625 | 3.49 | 4.02 | 5.86 | 14.51 | 2.49 | 2.95 | 3.64 | 3.97 |
| 1.125 | 3.49 | 4.16 | 6.05 | 14.55 | 2.49 | 3.11 | 3.84 | 4.22 |
| 15 | 56.64 | 55.29 | 80.56 | 232.46 | 39.36 | 41.37 | 51.1 | 55.86 |
| 14 | 56.64 | 52.61 | 76.65 | 180.52 | 39.36 | 38.67 | 47.77 | 52.67 |
| 17 | 56.64 | 64.12 | 93.41 | 224.95 | 39.36 | 47.16 | 58.25 | 62.62 |
| 8 | 56.64 | 68.19 | 99.35 | 255.04 | 39.36 | 49.93 | 61.68 | 67.26 |
| 26 | 190.66 | 96.41 | 140.46 | 335.27 | 135.44 | 72.05 | 89 | 96.17 |
| 39 | 190.66 | 148.18 | 215.89 | 570.71 | 135.44 | 107.33 | 132.57 | 144.11 |
| 65 | 190.66 | 242.92 | 353.93 | 923.4 | 135.44 | 179.54 | 221.77 | 244.39 |
| 78 | 190.66 | 290.86 | 423.77 | 1108.59 | 135.44 | 214.86 | 265.4 | 290.41 |

**Table 4 Simulated coverage results for ANOVA, GLM, M-GLM and LOG approaches and theoretical L.B. for ANOVA approach for $2^3$ factorial experiment with $r$=2, 3 replications**

| | $r=2$ | | | | | $r=3$ | | | | |
|---|---|---|---|---|---|---|---|---|---|---|
| $\underline{\mu}_1$ | ANOVA | GLM | M-GLM | LOG | L.B.ANOVA | ANOVA | GLM | M-GLM | LOG | L.B.ANOVA |
| 0.05 | 100 | 91.38 | 95.8 | 92.08 | 99.96 | 100 | 91.8 | 95.12 | 89.38 | 99.95 |
| 0.65 | 98.2 | 91.24 | 95.72 | 91.28 | 93.36 | 98.92 | 92.36 | 95.18 | 89.92 | 92.14 |
| 0.75 | 97.6 | 90.4 | 94.9 | 91.78 | 91.15 | 98.18 | 92.5 | 95.46 | 89.5 | 89.53 |
| 0.85 | 96.76 | 91.46 | 95.64 | 92.2 | 88.64 | 97.12 | 91.92 | 94.9 | 89.62 | 86.56 |
| 1.25 | 90.06 | 90.4 | 95.12 | 91.3 | 75.43 | 90.72 | 91.98 | 95.34 | 89.6 | 70.93 |
| 1.15 | 92.5 | 90.78 | 95.42 | 91.54 | 79.2 | 92.76 | 91.78 | 94.74 | 89.3 | 75.39 |
| 1.5 | 84.84 | 90.94 | 95.36 | 91.4 | 64.62 | 85.1 | 92.6 | 95.72 | 89.2 | 58.13 |
| 1.65 | 81.7 | 90.5 | 95.26 | 91.88 | 57.19 | 81.68 | 91.5 | 94.92 | 89.7 | 49.34 |
| 0.9375 | 95.4 | 90.64 | 95.46 | 91.78 | 85.04 | 96.3 | 92.4 | 95.06 | 90.02 | 82.3 |
| 0.875 | 96.08 | 90.56 | 95.26 | 91.56 | 86.97 | 97.28 | 92.22 | 94.94 | 88.96 | 84.58 |
| 1.0625 | 92.68 | 90.84 | 95.22 | 91.54 | 80.78 | 94.52 | 92.72 | 95.22 | 89.58 | 77.26 |
| 1.125 | 92.54 | 91.44 | 95.76 | 92.16 | 78.46 | 92.82 | 92.22 | 94.88 | 89.72 | 74.51 |
| 0.8756 | 96.08 | 90.12 | 95.26 | 91.5 | 86.95 | 97.2 | 92.16 | 95.08 | 89 | 84.56 |
| 0.0678 | 100 | 90.5 | 95.24 | 92.06 | 99.92 | 100 | 92.16 | 95.18 | 89.74 | 99.91 |
| 1.567 | 83.48 | 91.44 | 95.86 | 91.78 | 58.2 | 82.2 | 92.16 | 95.42 | 90.06 | 50.54 |
| 1.254 | 89.82 | 90.68 | 95.46 | 92.1 | 73.23 | 90.78 | 91.88 | 94.66 | 88.7 | 68.32 |
| 12.1825 | 93.76 | 90.48 | 95.36 | 91.76 | 93.03 | 95.4 | 92 | 94.9 | 89.44 | 86.66 |
| 15.2345 | 89.08 | 90.78 | 95.3 | 91.82 | 93.02 | 91.36 | 91.5 | 94.9 | 89.26 | 79.13 |
| 5.3274 | 99.44 | 90.54 | 95.32 | 91.74 | 93.03 | 99.72 | 92.18 | 95.38 | 90.6 | 97.45 |
| 4.4817 | 99.44 | 90.16 | 95.14 | 91.42 | 93.05 | 99.84 | 92.3 | 95.26 | 89.04 | 98.19 |
| 33.1155 | 63.56 | 91.2 | 95.58 | 91.74 | 93.03 | 62.34 | 91.98 | 94.7 | 89.02 | 1.39 |
| 21.3042 | 80.1 | 90.62 | 95.42 | 92.18 | 93 | 81.4 | 92.42 | 95.16 | 89.42 | 59.19 |
| 0.0302 | 100 | 91.08 | 95.52 | 92.4 | 92.98 | 100 | 92.08 | 95.08 | 90.08 | 100 |
| 0.5439 | 100 | 90.72 | 95.22 | 91.06 | 93.08 | 100 | 91.4 | 94.66 | 89.18 | 99.97 |
| 0.0003 | 100 | 90.62 | 95.14 | 91.64 | 100 | 100 | 92.28 | 95.18 | 90.16 | 100 |
| 2.5346 | 85 | 91.1 | 95.48 | 91.54 | 76.4 | 87.56 | 92.62 | 95.48 | 89.28 | 72.07 |
| 1.2354 | 97.04 | 89.7 | 95.06 | 91.24 | 94.39 | 98.8 | 91.7 | 94.7 | 89.04 | 93.37 |
| 4 | 69.66 | 90.9 | 95.78 | 92.04 | 41.22 | 70.8 | 92.7 | 95.28 | 89.82 | 30.45 |
| 0.0183 | 100 | 90.16 | 95.28 | 91.32 | 100 | 100 | 91.98 | 94.78 | 88.48 | 100 |
| 1 | 98.52 | 90.74 | 95.48 | 92.4 | 96.33 | 99.18 | 92.32 | 95.18 | 89.58 | 95.65 |
| 4 | 69.44 | 90.8 | 95.54 | 91.84 | 41.22 | 69.84 | 92.06 | 95.06 | 89.36 | 30.45 |
| 0.0023 | 100 | 91.14 | 95.48 | 91.28 | 100 | 100 | 91.92 | 94.98 | 89.4 | 100 |
| Average | 91.65 | 90.75 | 95.4 | 91.73 | | 92.24 | 92.12 | 95.08 | 89.47 | |

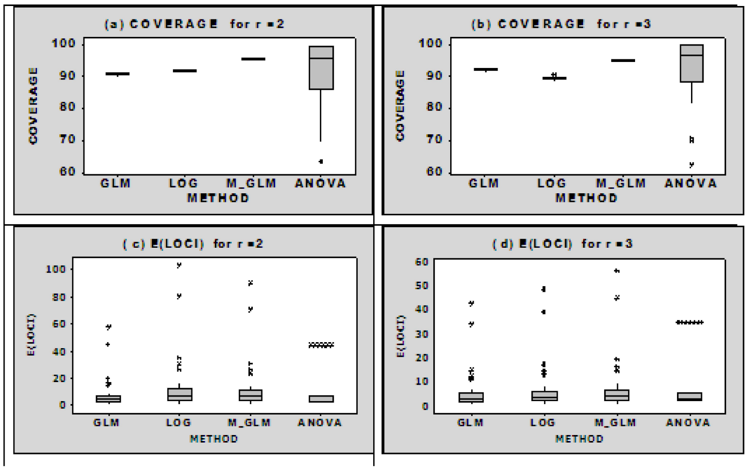


**Fig. 2: Box Plots for Coverage results and E(LOCI) for a $2^3$ factorial experiment for ANOVA, GLM, M-GLM and LOG Approaches**

**Table 5 Simulated E(LOCI) for ANOVA, GLM, M-GLM and LOG approaches for $2^3$ factorial experiment with $r$=2, 3 replications**

| | $r = 2$ | | | | $r = 3$ | | | |
|---|---|---|---|---|---|---|---|---|
| $\underline{\mu}_1$ | ANOVA | GLM | M-GLM | LOG | ANOVA | GLM | M-GLM | LOG |
| 0.05 | 3.25 | 0.19 | 0.29 | 0.32 | 2.54 | 0.14 | 0.18 | 0.16 |
| 0.65 | 3.25 | 2.43 | 3.79 | 4.24 | 2.54 | 1.78 | 2.35 | 2.1 |
| 0.75 | 3.25 | 2.82 | 4.41 | 4.9 | 2.54 | 2.09 | 2.75 | 2.43 |
| 0.85 | 3.25 | 3.19 | 4.98 | 5.79 | 2.54 | 2.36 | 3.11 | 2.76 |
| 1.25 | 3.25 | 4.6 | 7.18 | 8.1 | 2.54 | 3.5 | 4.62 | 4.07 |
| 1.15 | 3.25 | 4.35 | 6.79 | 7.59 | 2.54 | 3.24 | 4.28 | 3.72 |
| 1.5 | 3.25 | 5.66 | 8.84 | 9.94 | 2.54 | 4.12 | 5.43 | 4.82 |
| 1.65 | 3.25 | 6.24 | 9.74 | 10.68 | 2.54 | 4.6 | 6.07 | 5.28 |
| 0.9375 | 3.12 | 3.53 | 5.52 | 6.43 | 2.44 | 2.59 | 3.42 | 3.03 |
| 0.875 | 3.12 | 3.26 | 5.1 | 6.08 | 2.44 | 2.41 | 3.18 | 2.81 |
| 1.0625 | 3.12 | 3.99 | 6.23 | 7.17 | 2.44 | 2.99 | 3.94 | 3.45 |
| 1.125 | 3.12 | 4.32 | 6.74 | 7.62 | 2.44 | 3.11 | 4.11 | 3.68 |
| 0.8756 | 3.12 | 3.32 | 5.18 | 5.78 | 2.44 | 2.43 | 3.21 | 2.83 |
| 0.0678 | 3.12 | 0.25 | 0.4 | 0.46 | 2.44 | 0.19 | 0.25 | 0.22 |
| 1.567 | 3.12 | 5.91 | 9.22 | 10.45 | 2.44 | 4.35 | 5.74 | 5.04 |
| 1.254 | 3.12 | 4.69 | 7.31 | 8.29 | 2.44 | 3.48 | 4.59 | 4.02 |
| 12.1825 | 44.43 | 45.45 | 70.95 | 80.56 | 34.85 | 34.02 | 44.87 | 39.11 |
| 5.2345 | 44.43 | 57.69 | 90.04 | 103.24 | 34.85 | 42.7 | 56.3 | 48.78 |
| 5.3274 | 44.43 | 20.04 | 31.28 | 35.19 | 34.85 | 14.84 | 19.57 | 17.26 |
| 4.4817 | 44.43 | 16.83 | 26.28 | 30.24 | 34.85 | 12.4 | 16.35 | 14.29 |
| 33.1155 | 44.43 | 126.64 | 197.68 | 217.79 | 34.85 | 91.16 | 120.22 | 107.48 |
| 21.3042 | 44.43 | 79.54 | 124.16 | 140.35 | 34.85 | 59.13 | 77.98 | 68.48 |
| 0.0302 | 44.43 | 0.11 | 0.18 | 0.19 | 34.85 | 0.08 | 0.11 | 0.1 |
| 0.5439 | 44.43 | 2.03 | 3.17 | 3.51 | 34.85 | 1.52 | 2 | 1.76 |
| 0.0003 | 6.35 | 0.001 | 0.002 | 0.002 | 5.08 | 0.001 | 0.001 | 0.001 |
| 2.5346 | 6.35 | 9.43 | 14.72 | 17.28 | 5.08 | 7 | 9.24 | 8.04 |
| 1.2354 | 6.35 | 4.6 | 7.19 | 8.21 | 5.08 | 3.43 | 4.53 | 3.96 |
| 4 | 6.35 | 15.08 | 23.53 | 26.33 | 5.08 | 11.12 | 14.66 | 13.28 |
| 0.0183 | 6.35 | 0.07 | 0.11 | 0.12 | 5.08 | 0.05 | 0.07 | 0.06 |
| 1 | 6.35 | 3.71 | 5.79 | 6.49 | 5.08 | 2.81 | 3.71 | 3.26 |
| 4 | 6.35 | 14.89 | 23.25 | 26.85 | 5.08 | 11.21 | 14.78 | 13.09 |
| 0.0023 | 6.35 | 0.01 | 0.01 | 0.01 | 5.08 | 0.01 | 0.01 | 0.01 |

## 3.2 General observations regarding coverage results

**a) GLM approach:** The simulated coverage results for GLM approach exactly conform with theoretical ones given in (6) and are below the desired level of significance. These are increasing in the number of replications as desired.
**b) ANOVA approach:** The simulated coverage results for ANOVA approach follow the L.B. given in (7) except for a few cases for $r = 2$ replications. The coverages are poor if ${\mu_i}^2 > \bar{\mu}^*$, and good otherwise. Coverage results are very good, almost 100% when ${\mu_i}^2$ is quite less than $\bar{\mu}^*$ but in such cases, E(LOCI) tend to be unduly large.
**c) LOG approach:** Simulated coverage results for LOG approach follow the L.B. in (8) but the L. B. is not sharp due to the approximations employed for its computation. The coverages are reasonable and well concentrated but little below the desired level. But most undesirablly, as noted in Section 2 these are decreasing in the number of replications $r$ and the number of treatment combinations $k$. The decreasing pattern with respect to $r$ and $k$ is visible from the columns 5 and 10 of the Tables 2 and 4.

## 3.3 General observations regarding E(LOCI)

**a) GLM approach:** The simulated E(LOCI) for GLM exactly match with those given in (3). As expected they depend only on the value of the particular component $\mu_i$ of $\underline{\mu}$ are increasing in $\mu_i$ and decreasing in the number of replications $r$ but do not depend on the number of factors.
**b) ANOVA approach:** The simulated E(LOCI) for ANOVA approach depend on all components of $\underline{\mu}$, are same for all components of $\underline{\mu}$ and are smaller than the approximation obtained in (4). The major drawback of ANOVA approach is that, large components of $\underline{\mu}$ greatly inflate E(LOCI) for other components through $\bar{\mu}^*$. For example, for a $2^2$ experiment with r=2, the same value of $\mu_i = 25.79$, for $\underline{\mu}_1 = (25.79, 2.117, 5.7546, 0.1738)$ with $\bar{\mu}^* = 175.69$ has E(LOCI)=38.1786 while for $\underline{\mu}_1 = (115.584, 0.4724, 25.79, 0.0388)$ with $\bar{\mu}^* = 3506.27$ has E(LOCI)=170.4324. In exactly similar manner, for $\underline{\mu}_1 = (7.3891, 7.3891, 7.3891, 0.1353)$, the observation 0.1353, pulls down E(LOCI) for 7.3891 but results in poor coverage probability 87.44. The problem basically arises, since, ANOVA uses a single pooled estimate $S^2$ of variability for all components. A very good approximation, $E(LOCI) \approx 3.12\sqrt{\bar{\mu}^*}$ is obtained through regression of simulated E(LOCI), that is fairly consistent with the expression for E(LOCI) obtained in (4).
**c) LOG approach:** The simulated E(LOCI) for LOG approach are larger than the ones obtained in (5) for $r = 2$ replications and smaller for $r = 3$ replications. Like GLM these also depend only on the value of the particular component $\mu_i$ of $\underline{\mu}$ and are increasing in $\mu_i$. Regression of simulated E(LOCI) on $\mu_i$ for a $2^2$ experiment with $r$=2 gave the relation E(LOCI)=13 $\mu_i$ for $\mu_i \leq 5$

and E(LOCI)=-11.8 + 14.2$\mu_i$ for $\mu_i > 5$ but this is not very close to the expression obtained in (5) because of the approximations used therein. Similar linear relationships were observed for r=3 and $2^3$ experiments.

## 3.4 Comparison of three approaches

1. The coverage results of the ANOVA approach are most unstable while for both GLM and LOG approach they are below the desired level. The coverage results for GLM are uniformly inferior to those of LOG for two replications and superior for three replications.
2. The E(LOCI) for GLM are uniformly smaller than LOG approach.

Although Lewis et.al. [5] advocated the use of GLM in the analysis of factorial designs, the above observations indicate that, the LOG approach outperforms GLM for two replications, which is most typically encountered situation in industrial experiments but the improvement is not up to the desired level.

## 3.5 Proposed modification to GLM approach

The proposed modification to GLM based CI is deliberately designed to achieve the coverage probability exactly equal to the desired confidence coefficient. For the given level $\alpha$, a number $'\delta'$ can be selected so that,

$$Pr\Big(r\ exp(-\delta Z_{\alpha/2}/\sqrt{r}) \leq \sum_{j=1}^{r} Y_{ij}/\mu_i \leq r\ exp(\delta Z_{\alpha/2}/\sqrt{r})\Big) = 100(1-\alpha) \quad (7)$$

It is found that, at $\alpha = 0.05$, $\delta = 1.248$ for r=2 and 1.1566 for r=3. The values of $'\delta'$ can be well approximated as a function of $r$ as $\delta = 0.9865 + 0.5252\frac{1}{r}$. Consequently the CI can be obtained using this value of $\delta$. In particular for $r = 2$ and 3 these are given by $\bar{Y}_{i.}exp(\pm 1.7296)$ and $\bar{Y}_{i.}exp(\pm 1.3088)$ respectively with corresponding expected lengths

$$E(LOCI) = \begin{cases} 5.4613\mu_i & \text{for} \quad \text{r=2} \\ 3.4316\mu_i & \text{for} \quad \text{r=3} \end{cases} \quad (8)$$

Although, these expected widths are a little larger than those of GLM approach, they are much smaller than those of LOG approach and maintain the coverage probability exactly equal to 0.95 and hence the proposed modification to GLM outperforms the above methods and hence recommended for the use. This modification will be denoted by M-GLM henceforth.

Panels (a), (b) of Figures 1 and 2, display the box plots of coverage results for ANOVA, GLM, M-GLM and LOG approaches with r=2 and r=3 respectively based on Tables 2 and 4. The box plots of the corresponding E(LOCI) based on Tables 3 and 5 are displayed in the panels (c) and (d) of these figures.

# 4 Comparison of Three Approaches Based on the Powers of the Tests for Significance of Factorial Effects

In this section, we compare the three approaches based on the power functions of tests for testing significance of factorial effects.

## 4.1 The tests under consideration

Note that, for ANOVA and LOG approaches, the regression coefficients in the respective models are just half of the factorial effects that are commonly used in the analysis of factorial experiments, so that, testing significance of the regression coefficients using t-test is equivalent to testing significance of the underlying factorial effect with usual F-test. For GLM approach two types of tests can be employed: t-test and deviance test.

In all the three approaches, t statistic for the hypothesis $H_0 : \beta_i = 0$ is given by, $t_0 = \hat{\beta}_i / SE(\hat{\beta}_i)$, carries $n-k$ d.f. and $H_0$ is rejected if $\mid t_0 \mid > t_{(n-k),\alpha/2}$. Under ANOVA, $\hat{\beta}_i$ are linear functions of observations say $\lambda' y$ and $SE(\hat{\beta}_i)$ is given by $\hat{\sigma}\ \sqrt{\sum \lambda_i^2}$ where $\hat{\sigma}$ is estimated by square root of the residual sum of squares. For LOG a parallel procedure is adopted with $\underline{Y}$ replaced by the log transformed variable $\underline{W}$. For GLM approach $\hat{\beta}_i$ is m.l.e. of $\beta_i$ obtained through an iterative procedure (it has closed form expression for the saturated model given in Section 2) and its standard error is the square root of $i^{th}$ diagonal element of the inverse information matrix, given by $((a(\phi))^2 (X' \Delta \overset{\wedge}{V} \Delta X)^{-1})$ which for exponential response under saturated model of Section 2 equals $\sqrt{\frac{1}{rk}}$.

The deviance test for GLM is based on the difference between the deviances for the two nested models; the full model and the reduced model that is obtained under $H_0$ by eliminating the $i^{th}$ column of the coefficient matrix $X$. The deviance of a fitted model is defined as

$$D = -2[lnL(fitted\ model) - lnL(saturated\ model)](a(\phi))$$

For testing $H_0 : \beta_i = 0$, the deviance statistic takes the form

$$F = \frac{D(\beta_i \mid \beta_{(i)})/(df_2 - df_1)}{D(\beta)/df_2}$$

For exponentially distributed response,

$$D(\beta) = -2r\Big(\sum_{i=1}^{k} ln\bar{Y}_{i.} + k\Big), \quad D(\beta_i \mid \beta_{(i)}) = -2\sum_{i=1}^{k}\sum_{j=1}^{r}\Big((y_{ij} - \hat{\mu}_i)/\hat{\mu}_i - ln(y_{ij}/\hat{\mu}_i)\Big)$$

and $\beta_{(i)}$ is the vector $\underline{\beta}$ with $i^{th}$ component removed and $\hat{\mu_i}$ is the m.l.e of $\mu_i$ for the reduced model. These four test procedures henceforth will be denoted by t-ANOVA, t-LOG, t-GLM and DEV respectively.

## 4.2 The Power functions

The power functions of the above four test procedures were simulated for a $2^2$ factorial experiment with r=2 and 3 replications for each of the three hypotheses $H_{01} : \beta_1 = 0, H_{02} : \beta_2 = 0, H_{03} : \beta_{12} = 0$, which respectively correspond to the two main effects and the interaction between them. Power functions of each of these three hypotheses were exactly same hence, we have presented only those for $H_0 : \beta_1 = 0$.

Noting that, the models under ANOVA and GLM are different and the coefficient $\beta_i$ represents respective effect on a linear scale for ANOVA and on logarithmic scale for GLM and LOG approaches, the power functions are simulated based on 5000 simulations $\underline{Y} \sim exponential(\underline{\mu})$ for $2^2$ experiments with the number of replications $r = 2$ and $r = 3$ varying $\beta_1$ between -4 to 4 with increment of 0.1, under the following situations:
(1) $\underline{\beta} = (\beta_0, \beta_2, \beta_{12})$ at (1, 1, 1) and (1, 1, 2) under the GLM model $\underline{\mu} = \exp(X'\underline{\beta})$. The corresponding Power functions are plotted in Figures 3(a) and 3(b) for two replications and in Figures 3(c) and 3(d) for three replications.
(2) Fixing $(\beta_0, \beta_2, \beta_{12})$ at (6.1, 1, 1) and (7.1, 2, 1) under ANOVA model $\underline{\mu} = X\underline{\beta}$. The corresponding power functions are plotted in Figures 4(a) and 4(b) for two replications and in Figures 4(c) and 4(d) for three replications. Here the component $\beta_0$ had to be adjusted so as to ensure $\underline{\mu} = X\underline{\beta}$ non-negative.
(3) For a few sets of $\underline{\mu}$ considered in the first column of Table 2, taking $\underline{\beta} = A^{-1}log(\underline{\mu})$ for GLM, $\beta_1$ varying systematically as above. The Power functions of these tests are plotted in Figures 5(a) and 5(b) respectively for $\underline{\mu} = [25.7903, 2.117, 5.7546, 0.1738]$ and $\underline{\mu} = [0.9375, 0.875, 1.0625, 1.125]$ for two replications and similarly in 5(c) and 5(d) for three replications.

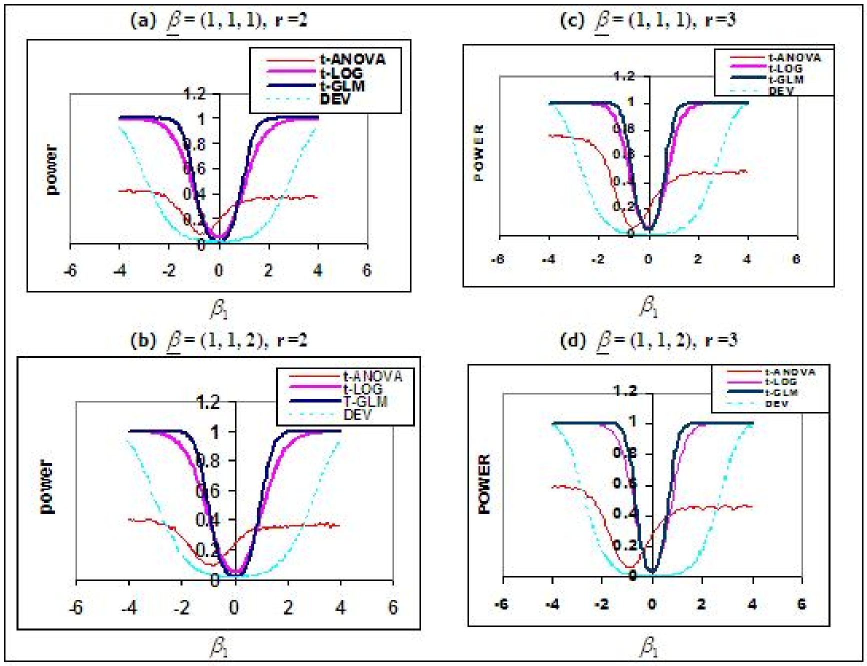

**Fig. 3: Graphs of power functions for $H_0 : \beta_1 = 0$ for various parameter combinations $\underline{\beta} = (\beta_0, \beta_1, \beta_2)$ when $\beta_1$ is systematically varied under GLM model.**

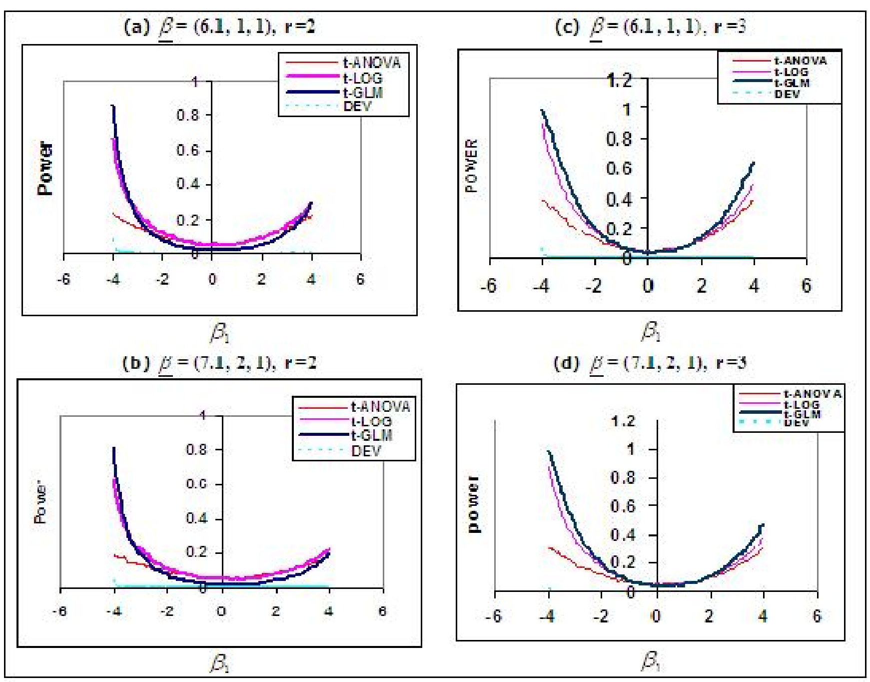

**Fig. 4: Graphs of power functions for $H_0 : \beta_1 = 0$ for various parameter combinations $\underline{\beta} = (\beta_0, \beta_1, \beta_2)$ when $\beta_1$ is systematically varied under ANOVA model.**

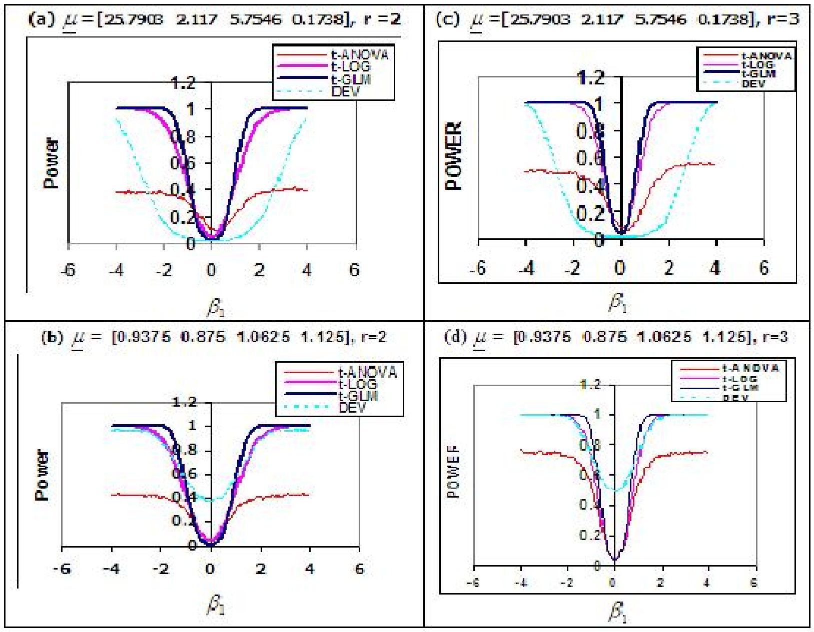


**Fig. 5: Graphs of power functions of the four tests for testing $H_0 : \beta_1 = 0$ for various parameter combinations $\underline{\beta} = (\beta_0, \beta_1, \beta_2)$ generated from $\underline{\mu}$ selected from Table 2, varying $\beta_1$ systematically.**

### 4.3 Observations and recommendations

From all the simulated Power functions it is clear that,
(1)t-LOG and t-GLM are having best power functions among the four tests and have small type-I error rates, GLM being a little more superior to LOG.
(2)The power functions of t-ANOVA are very small while the Type-I error rates are large.
(3)The deviance test is quite less powerful but has small type-I error rates.

These observations suggest that t-test under LOG is equally best as under GLM approach and hence both are recommended for testing significance of factorial effects. The ANOVA has worst performance with respect to expected coverage, length of CI and power in the entire parameter space and hence should be strictly avoided.

## 5 Concluding Remarks

For interval estimation of expected response with small replications, LOG approach has larger coverage probability than GLM but still little below the desired level and with larger expected lengths. The suggested modified GLM CI resolves this problem at the cost of a little increased expected lengths than GLM. For tests of significance for factorial effects, t-test based inference under

LOG and GLM are equally good and recommended. Classical ANOVA approach is unstable and misleading under both the inference problems and be strictly avoided.

Further study is demanded regarding the unsaturated models and is in the progress.

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