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The Generalized Broer-Kaup-Kupershmidt System and its Hamiltonian Extension

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Abstract

The generalized Broer-Kaup-Kupershmidt (generalized BKK) isospectral problem, including the x-derivative of potential, is considered based on Lie algebra A_1. The variational trace identity is extended to construct Hamiltonian structure of generalized BKK system. The Lie algebra A_1 is extended to the non-semi-simple Lie algebra of 4×4 matrix form, from which a hierarchy of soliton equations related to generalized BKK system are given. The Hamiltonian structure of the resulting system is established, by the generalized trace identity.

Keywords: Lie algebra, Generalized Broer-Kaup-Kupershmidt hierarchy, Veriational trace identity, Hamiltonian structure

1 Introduction

The theory of integrable Hamiltonian systems of infinite dimensions has undergone a rapid development since the late 1960's. The representation of a nonlinear system as the compatibility condition of linear equations, i.e., the zero-curvature representation, is central to our understanding of the word "integrability". Meanwhile, the role of Lie algebra has attracted much attention [1-3], among of which the Lie algebra A_1 has served as the ground in which the principal elements of Lax and zero-curvature equations grow.

Let G be a finite dimensional Lie algebra over $\mathbb{C}$, and $\widetilde{G}$ be the corresponding loop algebra $\widetilde{G} = G \otimes \mathrm{c}[\lambda, \lambda^{-1}]$. We assume that a pair of matrix spectral problems

$$\begin{cases} \varphi_x = U\varphi = U(u, \lambda)\varphi, \\ \varphi_t = V\varphi = V(u, u_x, u_{xx}, \cdots, \dfrac{\partial^{m_0} u}{\partial x^{m_0}}; \lambda)\varphi, \end{cases} \tag{1.1}$$

where $U, V \in \widetilde{G}, u = (u_1, u_2, \cdots, u_l)^T$ are potential functions, λ is a spectral parameter with $\lambda_t = 0$, m_0 is a natural number indicating the differential order, determines a Lax integrable equation [4-7]

$$u_t = K(u), \tag{1.2}$$

through the zero-curvature equation

$$U_t - V_x + [U, V] = 0. \tag{1.3}$$

The system (1.2) may be casted in the Hamiltonian form

$$u_t = J\frac{\delta \widetilde{H}_n}{\delta u}, \tag{1.4}$$

where J is a symplectic operator and $\{\widetilde{H}_n\}$ are a sequence of scalar functions, and $\frac{\delta}{\delta u}$ stands for the variational derivatives [3] defined by

$$\frac{\delta}{\delta u} = \sum_{n\geq 0}(-\partial)^n \frac{\partial}{\partial u^{(n)}}, \left(\partial = \frac{d}{dx}, u^{(n)} = \partial^n u\right). \tag{1.5}$$

The variational trace identity,

$$\frac{\delta}{\delta u}\langle V, \frac{\partial U}{\partial \lambda}\rangle = \left(\lambda^{-\varepsilon}\frac{\partial}{\partial \lambda}\lambda^{\varepsilon}\right)\langle V, \frac{\partial U}{\partial u}\rangle, \tag{1.6}$$

that produces Hamiltonian structures of infinite dimensional integrable systems, has been established by Tu [3]. To some extent, the establishment of trace identity has shed light on certain exchangeability of operations between $\delta/\delta u$ and $\partial/\partial\lambda$ [3,8,9], i.e.,

$$\frac{\delta}{\delta u}\langle \bar{V}, \frac{\partial U}{\partial \lambda}\rangle = \frac{\partial}{\partial \lambda}\langle \bar{V}, \frac{\partial U}{\partial u}\rangle,$$

where γ is a constant such that $\bar{V} = \lambda^{\gamma} V$ is again a solution of $V_x = [U, V]$. By using of trace identity, quite a number of infinite-dimensional Liouville integrable Hamiltonian systems are discussed. [4-7] Recently, a good deal of original work on developing trace identities have been given to construct Hamiltonian structures of integrable multi-component systems and integrable couplings in cases of semi-simple and non-semi-simple Lie algebras, as well as super integrable systems [10-14].

Generally, to the best of our knowledge, very few of the matrix eigenvalue problems are involved with the x-derivatives of potentials, e.g.,

$$\begin{cases} \varphi_x = U\varphi = U(u, u_x, u_{xx}, \cdots, \dfrac{\partial^{m_1} u}{\partial x^{m_1}}; \lambda)\varphi, \\ \varphi_t = V\varphi = V(u, u_x, u_{xx}, \cdots, \dfrac{\partial^{m_2} u}{\partial x^{m_2}}; \lambda)\varphi, \end{cases} \tag{1.7}$$

m_1, m_2 are natural numbers indicating the differential order. In such a case, the variational trace identity (1.6) should be changed to [8,9]

$$\frac{\delta}{\delta u}\langle V, \frac{\partial U}{\partial \lambda}\rangle = \left(\lambda^{-\varepsilon}\frac{\partial}{\partial \lambda}\lambda^{\varepsilon}\right)\langle V, \frac{\delta U}{\delta u}\rangle. \tag{1.8}$$

For example, let $U = U(u, u_x, \lambda), V = V(u, u_x, u_{xx}, \cdots, \frac{\partial^{m_0} u}{\partial x^{m_0}}; \lambda)$ in (1.7), then, from (1.5), the variational trace identity is casted into

$$\frac{\delta}{\delta u}\langle V, \frac{\partial U}{\partial \lambda}\rangle = \left(\lambda^{-\varepsilon}\frac{\partial}{\partial \lambda}\lambda^{\varepsilon}\right)\left(\langle V, \frac{\partial U}{\partial u}\rangle - \frac{\partial}{\partial x}\langle V, \frac{\partial U}{\partial u_x}\rangle\right). \tag{1.9}$$

This paper, based on the Lie algebra A_1, is devoted to revisiting the generalized Broer-Kaup-Kupershmidt matrix eigenvalue problem [15]

$$\varphi_x = U\varphi = \begin{pmatrix} \lambda + u & v + \alpha u_x \\ 1 & -\lambda - u \end{pmatrix}\varphi, \varphi = \begin{pmatrix} \varphi_1 \\ \varphi_2 \end{pmatrix}, \tag{1.10}$$

where the spectral matrix U depends not only on potentials u, v but also on the x-derivative of potential u, α is a arbitrary constant. By constructing a proper time evolution equation, a hierarchy of soliton equations are furnished. One representative hierarchy of the resulting Lax integrable equations is presented and shown to possess Hamiltonian structure based on the variational trace identity (1.9). By using of semidirect sum of Lie algebras, a 4×4 matrix Lie algebra is constructed, based on which a hierarchy of Lax integrable equations are derived by zero-curvature representation. The Hamiltonian structure of the enlarged system is constructed by using of generalized variational trace identity through a non-degenerate symmetric bilinear form. Then, we construct infinitely many common commuting conserved functionals for the resulting hierarchy.

2 The generalized BKK hierarchy and its Hamiltonian structure

The Lie algebra A_1 is presented as [1-6]

$$A_1 = span\{\bar{w}_1, \bar{w}_2, \bar{w}_3, \}, \tag{2.1}$$

with

$$\bar{w}_1 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \bar{w}_2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \bar{w}_3 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix},$$

equipped with the commutative relations

$$[\bar{w}_1, \bar{w}_2] = 2\bar{w}_2, [\bar{w}_1, \bar{w}_3] = -2\bar{w}_3, [\bar{w}_2, \bar{w}_3] = \bar{w}_1. \tag{2.2}$$

Then, the associated loop Lie algebra may be defined by

$$\widetilde{A}_1 = \{P | P \in \mathbb{R}[\lambda] \otimes A_1\},$$

where $\mathbb{R}[\lambda] \otimes A_1$ means $span\{\lambda^n Q | n \in \mathbb{Z}, Q \in A_1\}$.

Based on loop Lie algebra $\widetilde{A}_1$, the generalized Broer-Kaup-Kupershmidt spectral problem (1.10) is given by

$$\varphi_x = U\varphi, U = \bar{w}_1(1) + u\bar{w}_1(0) + (v + \alpha u_x)\bar{w}_2(0) + \bar{w}_3(0), \varphi = (\varphi_1, \varphi_2)^T, \quad (2.3)$$

where $U = U(u, v, u_x; \lambda)$, $\mathcal{U} = (u, v)^T$ is potential function, λ is a spectral parameter with $\lambda_t = 0$, and α is an arbitrary constant.

Consider the auxiliary spectral problem associated with (2.3)

$$\varphi_{t_m} = V_m \varphi, m \geq 0, \quad (2.4)$$

with

$$V_m = \sum_{j=0}^{m} \begin{pmatrix} a_j & b_j \\ c_j & -a_j \end{pmatrix} \lambda^{m-j} + \begin{pmatrix} -c_{m+1} & 0 \\ 0 & c_{m+1} \end{pmatrix}.$$

Then the Eqs. (2.4) yield

$$V_{m_x} - [U, V_m] = -c_{m+1_x}\bar{w}_1(0) + 2[b_{m+1} - (v + \alpha u_x)c_{m+1}]\bar{w}_2(0),$$

which is consistent with U_{t_m}. Then, the zero-curvature equations

$$U_{t_m} - V_{m_x} + [U, V_m] = 0, m \geq 0,$$

give rise to the following hierarchy of soliton equations

$$\mathrm{U}_{t_m} = K_m(\mathrm{U}) = \begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = \begin{pmatrix} -c_{m+1_x} \\ -2a_{m+1_x} + \alpha c_{m+1_{xx}} \end{pmatrix}, m \geq 0. \quad (2.5)$$

When $m = 2$, the soliton hierarchy (2.5) is reduced to the generalized BKK equation

$$\begin{cases} u_t = -\frac{1-\alpha}{2}u_{xx} + \frac{1}{2}v_x - 2uu_x, \\ v_t = \frac{\alpha(2-\alpha)}{2}u_{xxx} + \frac{1-\alpha}{2}v_{xx} - 2(uv)_x, \\ \frac{d}{dt} = \frac{d}{dt_2}. \end{cases} \quad (2.6)$$

In what follows, the trace identity (1.8), i.e., (1.9), [8,9] is used to construct the Hamiltonian structure for the system (2.5). It is easy to see that

$$\langle V, \frac{\partial U}{\partial \lambda} \rangle = 2a, \langle V, \frac{\delta U}{\delta v} \rangle = \langle V, \frac{\partial U}{\partial v} \rangle = c,$$

$$\langle V, \frac{\delta U}{\delta u} \rangle = \langle V, \frac{\partial U}{\partial u} \rangle - \frac{\partial}{\partial x}\langle V, \frac{\partial U}{\partial u_x} \rangle = 2a - \alpha c_x.$$

Substituting above expressions with

$$a=\sum_{m=0}^{\infty}a_m\lambda^{-m}, b=\sum_{m=0}^{\infty}b_m\lambda^{-m}, c=\sum_{m=0}^{\infty}c_m\lambda^{-m},$$

into trace identity (1.8), and equating the coefficients of λ^{-m-1} on both sides of (1.8) yield

$$\frac{\delta}{\delta \mathrm{U}}\int(2a_{m+1})dx=(\gamma-m)\left(\begin{array}{c}2a_m-\alpha c_{m_x}\\ \\ c_m\end{array}\right).$$

To fix the constant γ, we simply set $m=0$, then, we have $\gamma=0$. Therefor, we obtain

$$\frac{\delta}{\delta \mathrm{U}}\int\left(\frac{2a_{m+1}}{m}\right)dx=\left(\begin{array}{c}-2a_m+\alpha c_{m_x}\\ \\ -c_m\end{array}\right).$$

It is easy to verify that the system (2.5) is Liouville integrable and possess the bi-Hamiltonian structure

$$\mathrm{U}_{t_m}=\left(\begin{array}{c}u\\ v\end{array}\right)_{t_m}=J\frac{\delta\widetilde{H}_m}{\delta \mathrm{U}}=M\frac{\delta\widetilde{H}_{m-1}}{\delta \mathrm{U}}, m\geq 1. \tag{2.7}$$

The Hamiltonian operators J, M and the Hamiltonian functionals $\widetilde{H}_m$ are given by

$$J=\left(\begin{array}{cc}0&\partial\\ \partial&0\end{array}\right), M=\left(\begin{array}{cc}\frac{1}{2}\partial & \frac{\alpha-1}{2}\partial^2-\partial u\\ \frac{1-\alpha}{2}\partial^2-u\partial & M_{22}\end{array}\right), \tag{2.8}$$

with

$$M_{22}=(\alpha-\frac{\alpha^2}{2})\partial^3+\alpha(\partial^2u-u\partial^2)-(v+\alpha u_x)\partial-\partial(v+\alpha u_x),$$

and

$$\widetilde{H}_m=\int H_m dx=\int\frac{2a_{m+1}}{m}dx, m\geq 1, \tag{2.9}$$

where [3,8,9]

$$\frac{\delta}{\delta \mathrm{U}}=\left(\frac{\delta}{\delta \mathrm{U}_1},\frac{\delta}{\delta \mathrm{U}_2},\cdots,\frac{\delta}{\delta \mathrm{U}_l}\right)^T, \frac{\delta\widetilde{H}_m}{\delta \mathrm{U}_i}=\sum_{m\geq 0}(-\partial)^m\frac{\delta H_m}{\delta \mathrm{U}_i^m},$$

$$\partial=\frac{d}{dx}, \mathrm{U}_i^m=\partial^m \mathrm{U}_i, i=1,2,\cdots,l.$$

If we denote

$$\frac{\delta \widetilde{H}_m}{\delta \mathrm{U}} = \Phi \frac{\delta \widetilde{H}_{m-1}}{\delta \mathrm{U}},$$

then, asking for the help of the recursion relation (2.7), we get the recursion operator

$$\Phi = \begin{pmatrix} \frac{1}{2}\partial - \partial^{-1}u\partial - \frac{1}{2}\alpha\partial & \alpha(1-\frac{\alpha}{2} - \partial^{-1}u)\partial^2 + \alpha\partial u - \partial^{-1}(v+\alpha u_x)\partial - (v+\alpha u_x) \\ \frac{1}{2} & \frac{\alpha}{2}\partial - \frac{1}{2}\partial - u \end{pmatrix}. \tag{2.10}$$

Remark 1. As $\alpha = 0$, the spectral problem (2.3) reduces to the Broer-Kaup-Kupershmidt spectral problem [17]

$$\varphi_x = U_{BKK}(\mathrm{U}, \lambda)\varphi,$$

$$U_{BKK}(\mathrm{U}, \lambda) = \bar{w}_1(1) + u\bar{w}_1(0) + v\bar{w}_2(0) + \bar{w}_3(0),$$

and the soliton hierarchy (2.5) gives the well-known Broer-Kaup-Kupershmidt soliton hierarchy

$$\mathrm{U}_{tm} = \begin{pmatrix} u \\ v \end{pmatrix}_{t_m} = J_{BKK}\frac{\delta \widetilde{H}_m}{\delta \mathrm{U}} = M_{BKK}\frac{\delta \widetilde{H}_{m-1}}{\delta \mathrm{U}}, m \geq 1,$$

in which

$$J_{BKK} = \begin{pmatrix} 0 & \partial \\ \partial & 0 \end{pmatrix}, M_{BKK} = \begin{pmatrix} \frac{1}{2}\partial & -\frac{1}{2}\partial^2 - \partial u \\ \frac{1}{2}\partial^2 - u\partial & -v\partial - \partial v \end{pmatrix},$$

and $H_m, m \geq 1$ satisfy (2.9) as $\alpha = 0$.

Remark 2. When $\alpha = 1$ the spectral problem (2.3) reduces to the Boussinesq-Burgers spectral problem. [15] The first nonlinear system of Boussinesq-Burgers hierarchy is as follows

$$\begin{cases} u_t = \frac{1}{2}v_x - 2uu_x, \\ v_t = \frac{1}{2}u_{xxx} - 2(uv)_x, \end{cases}$$

whose multi-soliton solutions are discussed by Darboux transformation. [16]

3 Hamiltonian extension of the generalized BKK soliton hierarchy (2.9)

With the development of the soliton theory, integrable coupling [18] has become a new and important topic in the study of integrable systems. The concept of integrable couplings and related theories were brought forward in recent years (see e.g., Refs. [10,11,19-27] and references therein). The corresponding results show various mathematical structures that integrable equations possess, such as Lax representations, infinitely many symmetries, conserved quantities and bi-Hamiltonian structures, etc.

In Refs. [10,11,22,23], by considering semi-direct sum of Lie algebras, a technologically-practicable approach to derive integrable and nonlinear integrable couplings is proposed. In order to construct the Hamiltonian structures of the corresponding integrable coupling systems, in the case of non-semi-simple Lie algebras, a generalized trace identity is established [10,11,22,23], which undoes the constraint on the standard trace identity [3,8,9].

Let us first consider the extension of the Lie algebra A_1 into Lie algebra of 4×4 matrix by semidirect sum of Lie algebras. Note

$$F = span\{w_1, w_2, w_3, w_4, w_5, w_6\}, F_0 = span\{w_1, w_2, w_3\}, F_c = span\{w_4, w_5, w_6\},$$

with

$$w_1 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, w_2 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}, w_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

$$w_4 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, w_5 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, w_6 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verify that F, F_0, F_c construct three Lie algebras with the communication operation

$$[w_i, w_j] = w_i w_j - w_j w_i, (i, j = 1, 2, ..., 6),$$

and

$$F = F_0 \uplus F_c, [F, F_c] = \{AB - BA | A \in F, B \in F_c\} \subseteq F_c.$$

Note

$$\widetilde{F} = \{A | A \in R[\lambda] \otimes F\}, \widetilde{F}_0 = \{A | A \in R[\lambda] \otimes F_0\}, \widetilde{F}_c = \{A | A \in R[\lambda] \otimes F_c\},$$

where $R[\lambda] \otimes F$ means the loop algebra defined by $span\{\lambda^n A | n \geq 0, A \in F\}$. Obviously, $\widetilde{F}_c$ is an Abelian ideal of the loop Lie algebra $\widetilde{F}$, and $\widetilde{F}_0$ and $\widetilde{F}_c$ is closed under the multiplication of matrix. Thus, $\widetilde{F}$ forms a semi-direct sum of $\widetilde{F}_0$ and $\widetilde{F}_c$.

In terms of $\widetilde{F}$, the spectral matrix $\mathcal{W}$ is of the form

$$\mathcal{W} = \begin{pmatrix} \mathcal{W}_0 & \mathcal{W}_c \\ 0 & \mathcal{W}_0 \end{pmatrix} \in \widetilde{F},$$

where $\mathcal{W}_0, \mathcal{W}_c$ are 2×2 matrices, 0 stands for a 2×2 zero matrix. For generalized BKK hierarchy (2.5), we consider an isospectral problem

$$\bar{\varphi}_x = \bar{U}(\bar{\mathrm{U}}, \lambda)\bar{\varphi} = \begin{pmatrix} U & U_c \\ 0 & U \end{pmatrix} \bar{\varphi}, \bar{\varphi} = \begin{pmatrix} \bar{\varphi}_1 \\ \bar{\varphi}_2 \\ \bar{\varphi}_3 \\ \bar{\varphi}_4 \end{pmatrix}, \tag{3.1}$$

where U is defied by (2.3) and $U_c = \begin{pmatrix} r & s + \alpha r_x \\ 0 & -r \end{pmatrix}$, i.e.,

$$\bar{U}(\bar{\mathrm{U}}, \lambda) = w_1(1) + uw_1(0) + (v + \alpha u_x)w_2(0) + w_3(0) + rw_4(0) + (s + \alpha r_x)w_5(0).$$

The stationary zero-curvature equation

$$\bar{V}_x - [\bar{U}, \bar{V}] = 0, \tag{3.2}$$

with

$$\begin{aligned} \bar{V} &= \begin{pmatrix} V & V_c \\ 0 & V \end{pmatrix} \\ &= aw_1(0) + bw_2(0) + cw_3(0) + ew_4(0) + fw_5(0) + gw_6(0) \in \widetilde{F}, \end{aligned}$$

leads to

$$\begin{cases} a_x = (v + \alpha u_x)c - b, b_x = 2(\lambda + u)b - 2(v + \alpha u_x)a, \\ c_x = -2(\lambda + u)c + 2a, e_x = (v + \alpha u_x)g + (s + \alpha r_x)c - f, \\ f_x = 2(\lambda + u)f - 2(v + \alpha u_x)e + 2rb - 2(s + \alpha r_x)a, g_x = -2(\lambda + u)g + 2e - 2rc. \end{cases} \tag{3.3}$$

Upon setting

$$\begin{aligned} \bar{V} &= \sum_{j \geq 0} \bar{V}_j \lambda^{-j} = \sum_{j \geq 0} \begin{pmatrix} V_j & V_{c_j} \\ 0 & V_j \end{pmatrix} \lambda^{-j} \\ &= \sum_{j \geq 0} (a_j w_1(j) + b_j w_2(j) + c_j w_3(j) + e_j w_4(j) + f_j w_5(j) + g_j w_6(j)) \in \widetilde{F}, \end{aligned}$$

and choosing the initial data

$$a_0 = e_0 = 1, b_0 = c_0 = f_0 = g_0 = 0,$$

then, Eqs (3.3) give rise to the following recursion relations

$$\begin{aligned}
a_{j+1} &= \partial^{-1}[-\tfrac{1}{2}(v+\alpha u_x)c_{j_x} - \tfrac{1}{2}b_{j_x} - ua_{j_x}], j \geq 0,\\
b_{j+1} &= \tfrac{1}{2}b_{j_x} - ub_j + (v+\alpha u_x)a_j, j \geq 0,\\
c_{j+1} &= -\tfrac{1}{2}c_{j_x} - uc_j + a_j, j \geq 0,\\
e_{j+1} &= \partial^{-1}[-\tfrac{1}{2}(v+\alpha u_x)g_{j_x} - \tfrac{1}{2}(s+\alpha r_x)c_{j_x} - \tfrac{1}{2}f_{j_x} - ra_{j_x} - ue_{j_x}], j \geq 0,\\
f_{j+1} &= \tfrac{1}{2}f_{j_x} - uf_j + (v+\alpha u_x)e_j - rb_j + (s+\alpha r_x)a_j, j \geq 0,\\
g_{j+1} &= -\tfrac{1}{2}g_{j_x} - ug_j + e_j - rc_j, j \geq 0.
\end{aligned} \tag{3.4}$$

Assume that the constants of integration are selected to be zero. Then the recursion relations (3.4) uniquely determine a series of sets of differential polynomial functions in $\bar{\mathrm{U}}$ with respect to x. The first few are listed as follows

$$a_1 = e_1 = 1, b_1 = v+\alpha u_x, c_1 = 1, f_1 = v+\alpha u_x + s + \alpha r_x, g_1 = 1,$$

$$a_2 = -\tfrac{1}{2}(v+\alpha u_x), b_2 = \tfrac{1}{2}(v+\alpha u_x)_x - u(v+\alpha u_x),$$

$$c_2 = -u, e_2 = -\tfrac{1}{2}(v+\alpha u_x + s+\alpha r_x), g_2 = -u-r,$$

$$f_2 = \tfrac{1}{2}(v+\alpha u_x+s+\alpha r_x)_x - u(v+\alpha u_x+s+\alpha r_x) - r(v+\alpha u_x), \cdots$$

Let the time evolution of the eigenfunction of the spectral problem (3.1) obey the differential equations

$$\bar{\varphi}_{t_m} = \bar{V}_m \bar{\varphi}, m \geq 0, \tag{3.5}$$

where

$$\begin{aligned}
\bar{V}_m &= \sum_{j=0}^{m} \begin{pmatrix} V_j & V_{c_j} \\ 0 & V_j \end{pmatrix} \lambda^{m-j} + \begin{pmatrix} \Delta_m & \Delta_{c_m} \\ 0 & \Delta_m \end{pmatrix}\\
&= \sum_{j=0}^{m} [a_j w_1(m-j) + b_j w_2(m-j) + c_j w_3(m-j) + e_j w_4(m-j) + f_j w_5(m-j)\\
&\quad + g_j w_6(m-j)] - c_{m+1} w_1(0) - g_{m+1} w_4(0).
\end{aligned}$$

A direct computation gives

$$\begin{aligned}
\bar{V}_{m_x} - [\bar{U}, \bar{V}_m] &= -c_{m+1_x} w_1(0) + 2[b_{m+1} - (v+\alpha u_x)c_{m+1}]w_2(0)\\
&\quad - g_{m+1_x} w_4(0) + 2[f_{m+1} - (v+\alpha u_x)g_{m+1} - (s+\alpha r_x)c_{m+1}]w_5(0),
\end{aligned}$$

which is consistent with $\bar{U}_{t_m}$. Then the compatibility condition of (3.1) and (3.5), i.e., the enlarged zero-curvature equations [3,10,11,21]

$$\bar{U}_{t_m} = (\bar{V}_m)_x - [\bar{U}, \bar{V}_m], m \geq 0,$$

give rise to the following equations

$$\bar{\mathrm{U}}_{t_m} = K_m(\bar{\mathrm{U}}) = \begin{pmatrix} u \\ v \\ r \\ s \end{pmatrix}_{t_m} = \begin{pmatrix} -c_{m+1_x} \\ -2a_{m+1_x} + \alpha(c_{m+1})_{xx} \\ -g_{m+1_x} \\ -2e_{m+1_x} + \alpha(g_{m+1})_{xx} \end{pmatrix}, m \geq 0. \quad (3.6)$$

The first nontrivial equation, when $m = 2$, gives

$$\bar{\mathrm{U}}_{t_2} = K_2(\bar{\mathrm{U}}) = \begin{pmatrix} \frac{\alpha-1}{2}u_{xx} + \frac{1}{2}v_x - 2uu_x \\ \frac{\alpha(2-\alpha)}{2}u_{xxx} + \frac{1-\alpha}{2}v_{xx} - 2(uv)_x \\ \frac{\alpha-1}{2}(u+r)_{xx} + \frac{1}{2}(v+s)_x - 2uu_x - 2(ur)_x \\ \frac{\alpha(2-\alpha)}{2}(u+r)_{xxx} + \frac{1-\alpha}{2}(v+s)_{xx} - 2(uv+us+vr)_x \end{pmatrix}.$$

The first two members in above system are same as those in (2.6), hence, it is a kind of integrable coupling system of equation (2.6), and, the system (3.6) is the integrable coupling system of generalized Broer-Kaup-Kupershmidts hierarchy (2.5). Further, the integrable couplings of Broer-Kaup-Kupershmidts [27] and Boussinesq-Burgers equations are given by

$$\begin{cases} u_t = -\frac{1}{2}u_{xx} + \frac{1}{2}v_x - 2uu_x, \\ v_t = \frac{1}{2}v_{xx} - 2(uv)_x, \\ r_t = -\frac{1}{2}(u+r)_{xx} + \frac{1}{2}(v+s)_x - 2uu_x - 2(ur)_x, \\ s_t = \frac{1}{2}(v+s)_{xx} - 2(uv+us+vr)_x, \end{cases}$$

when $\alpha = 0$ and

$$\begin{cases} u_t = \frac{1}{2}v_x - 2uu_x, \\ v_t = \frac{1}{2}u_{xxx} - 2(uv)_x, \\ r_t = \frac{1}{2}(v+s)_x - 2uu_x - 2(ur)_x, \\ s_t = \frac{1}{2}(u+r)_{xxx} - 2(uv+us+vr)_x, \end{cases}$$

when $\alpha = 1$, respectively.

In what follows, we are going to construct the Hamiltonian structure of the system (3.6). In order to do so, we should introduce a non-degenerate symmetric bilinear form. Let us consider the following map [10,11,21,27]

$$\Omega : \widetilde{F} \to R^6, A \mapsto a = (a_1, a_2, a_3, a_4, a_5, a_6)^T, A = \begin{pmatrix} a_1 & a_2 & a_4 & a_5 \\ a_3 & -a_1 & a_6 & -a_4 \\ 0 & 0 & a_1 & a_2 \\ 0 & 0 & a_3 & -a_1 \end{pmatrix} \in \widetilde{F}, \quad (3.7)$$

which induces a Lie algebraic structure on R^6, and, is isomorphic to the matrix loop algebra $\widetilde{F}$. The commutator $[.,.]_{R^6}$ on R^6 is derived by the commutator $[.,.]_{\widetilde{F}}$ on $\widetilde{F}$,

$$[a,b]_{R^6}^T = \Omega([A,B]_{\widetilde{F}}^T) = a^T R(b),$$

where $a, b \in R^6, A, B \in \widetilde{F}, R(b)$ is a square matrix

$$R(b) = \begin{pmatrix} 0 & 2b_2 & -2b_3 & 0 & 2b_5 & -2b_6 \\ b_3 & -2b_1 & 0 & b_6 & -2b_4 & 0 \\ -b_2 & 0 & 2b_1 & -b_5 & 0 & 2b_4 \\ 0 & 0 & 0 & 0 & 2b_2 & -2b_3 \\ 0 & 0 & 0 & b_3 & -2b_1 & 0 \\ 0 & 0 & 0 & -b_2 & 0 & 2b_1 \end{pmatrix},$$

which is in fact defined by communication operation in $\widetilde{F}$ [10,11]. According to [10,11], we introduce the matrix

$$H = \begin{pmatrix} 2 & 0 & 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 \\ 2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

It is easy to verified that

$$H^T = H, h(R(b))^T = -R(b)H, \quad b \in R^6.$$

Therefor, we can define a non-degenerate symmetric bilinear form on R^6

$$\langle a, b\rangle = a^T H b. \tag{3.8}$$

Then, from (3.7) and (3.8), a non-degenerate bilinear form on $\widetilde{F}$ is given by

$$\begin{aligned}\langle A, B\rangle_{\widetilde{F}} &= \langle \Omega(A), \Omega(B)\rangle_{R^6} = (a_1, a_2, a_3, a_4, a_5, a_6)H(b_1, b_2, b_3, b_4, b_5, b_6)^T \\ &= 2a_1b_1 + 2a_1b_4 + a_2b_3 + a_2b_6 + a_3b_2 + a_3b_5 + 2a_4b_1 + a_5b_3 + a_6b_2,\end{aligned} \tag{3.9}$$

which is symmetric and invariant associated with the Lie product, i.e.,

$$\langle A, B\rangle_{\widetilde{F}} = \langle B, A\rangle_{\widetilde{F}}, \langle A, [B, C]\rangle_{\widetilde{F}} = \langle [A, B], C\rangle_{\widetilde{F}}, A, B, C \in \widetilde{F}.$$

Through a direct computation, by utilizing (3.9) and (1.5), we have

$$\begin{gathered}\langle \bar{V}, \frac{\partial \bar{U}}{\partial \lambda}\rangle_{\widetilde{F}} = 2a + 2e, \langle \bar{V}, \frac{\delta \bar{U}}{\delta v}\rangle_{\widetilde{F}} = \langle \bar{V}, \frac{\partial \bar{U}}{\partial v}\rangle_{\widetilde{F}} = c + g, \\ \langle \bar{V}, \frac{\delta \bar{U}}{\delta u}\rangle_{\widetilde{F}} = \langle \bar{V}, \frac{\partial \bar{U}}{\partial u}\rangle - \frac{\partial}{\partial x}\langle \bar{V}, \frac{\partial \bar{U}}{\partial u_x}\rangle_{\widetilde{F}} = 2(a+e) - \alpha(c+g)_x, \\ \langle \bar{V}, \frac{\delta \bar{U}}{\delta r}\rangle_{\widetilde{F}} = \langle \bar{V}, \frac{\partial \bar{U}}{\partial r}\rangle - \frac{\partial}{\partial x}\langle \bar{V}, \frac{\partial \bar{U}}{\partial r_x}\rangle_{\widetilde{F}} = 2a - \alpha c_x, \\ \langle \bar{V}, \frac{\delta \bar{U}}{\delta s}\rangle_{\widetilde{F}} = \langle \bar{V}, \frac{\partial \bar{U}}{\partial s}\rangle_{\widetilde{F}} = c.\end{gathered} \tag{3.10}$$

The substitution of (3.10) with

$$a=\sum_{j=0}^{\infty}a_j\lambda^{-j}, b=\sum_{j=0}^{\infty}b_j\lambda^{-j}, c=\sum_{j=0}^{\infty}c_j\lambda^{-j},$$

$$e=\sum_{j=0}^{\infty}e_j\lambda^{-j}, f=\sum_{j=0}^{\infty}f_j\lambda^{-j}, g=\sum_{j=0}^{\infty}g_j\lambda^{-j},$$

into the generalized trace identity [8,9,10,11]

$$\frac{\delta}{\delta\bar{\mathrm{U}}}\int\langle\bar{V},\frac{\partial\bar{U}}{\partial\lambda}\rangle_{\widetilde{F}}dx=\left(\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\right)\langle\bar{V},\frac{\delta\bar{U}}{\delta\bar{\mathrm{U}}}\rangle_{\widetilde{F}}, \tag{3.11}$$

and equating the coefficients of λ^{-m-1} on both sides of (3.11) yield

$$\frac{\delta}{\delta\bar{\mathrm{U}}}\int 2(a_{m+1}+e_{m+1})dx=(\gamma-m)\begin{pmatrix}2(a_m+e_m)-\alpha(c_m+g_m)_x\\ c_m+g_m\\ 2a_m-\alpha c_{m_x}\\ c_m\end{pmatrix}.$$

To fix the constant γ, we simply set $m=0$, then, we have $\gamma=0$. Hence, we could construct the Hamiltonian structure of the system (3.6) by

$$\bar{\mathrm{U}}_{t_m}=K_m(\bar{\mathrm{U}})=\begin{pmatrix}u\\ v\\ r\\ s\end{pmatrix}_{t_m}=\bar{J}\frac{\delta\tilde{\bar{H}}}{\delta\bar{\mathrm{U}}}, m\geq 1, \tag{3.12}$$

in which the Hamiltonian operators $\bar{J}$ and the Hamiltonian functionals $\tilde{\bar{H}}_m$ are given by

$$\bar{J}=\begin{pmatrix}0&0&0&\partial\\ 0&0&\partial&0\\ 0&\partial&0&-\partial\\ \partial&0&-\partial&0\end{pmatrix}=\begin{pmatrix}0&J\\ J&-J\end{pmatrix},$$

$$\tilde{\bar{H}}_m=\int\frac{2(a_{m+1}+e_{m+1})}{m+1}dx, m\geq 0,$$

and, J is defined by (2.12). Now, if we set

$$\frac{\delta\tilde{\bar{H}}_{m+1}}{\delta\bar{\mathrm{U}}}=\bar{\Phi}\frac{\delta\tilde{\bar{H}}_m}{\delta\bar{\mathrm{U}}},$$

by utilizing the recursion relations (3.4), we get

$$\bar{\Phi}=\begin{pmatrix}\Phi&\Phi_c\\ 0&\Phi\end{pmatrix},$$

where Φ is defined by Eq. (2.14), 0 stands for 2×2 zero matrix, and

$$\Phi_c = \begin{pmatrix} -\partial^{-1}r\partial & \alpha\partial r - \alpha\partial^{-1}r\partial^2 - \partial^{-1}(s+\alpha r_x)\partial - (s+\alpha r_x) \\ 0 & -r \end{pmatrix}.$$

It is easy to verify that

$$\bar{M} = \bar{J}\bar{\Phi} = \begin{pmatrix} 0 & M \\ M & M_c \end{pmatrix},$$

is a skew-symmetric operator, in which M is given by (2.12), and

$$M_c = \begin{pmatrix} M_c^{11} & M_c^{12} \\ M_c^{21} & M_c^{22} \end{pmatrix},$$

with

$$\begin{aligned} M_c^{11} &= -\frac{1}{2}\partial, M_c^{12} = -\partial r + \frac{1-\alpha}{2}\partial^2 + \partial u, M_c^{21} = -r\partial + \frac{\alpha-1}{2}\partial^2 + u\partial, \\ M_c^{22} &= \alpha(\partial^2 r - r\partial^2) - [(s+r_x)\partial + \partial(s+r_x)] + (\frac{\alpha^2}{2} - \alpha)\partial^3 + \alpha(u\partial^2 - \partial^2 u) \\ &\quad +(v+u_x)\partial + \partial(v+u_x). \end{aligned}$$

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