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Regularization for a Common Solution of a System of Ill-Posed Equations Involving Linear Bounded Mappings[1]

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Abstract. The purpose of this paper is to give a regularization method for solving a system of ill-posed equations involving linear bounded mappings in real Hilbert spaces and an example of finding a common solution of two systems of linear algebraic equations with singular matrices.

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1. INTRODUCTION

Let X and Y_j be Hilbert spaces with scalar product and norm of X denoted by the symbols $\langle .,. \rangle_X$ and $\|.\|_X$, respectively. Let $A_j, j = 1, ..., N$, be N linear bounded mappings from X into Y_j.

Consider the following problem: find an element $\tilde{x} \in X$ such that

$$A_j \tilde{x} = f_j, \quad \forall j = 1, ..., N, \tag{1.1}$$

where f_j is given in Y_j a priori. Set

$$S_j = \{\overline{x} \in X : A_j \overline{x} = f_j\}, j = 1, ..., N, \text{ and } \quad S = \cap_{j=1}^N S_j.$$

Here, we assume that $S \neq \emptyset$. From the properties of A_j it is easy to see that each set S_j is a closed convex set in X. Therefore, S is also a closed convex subset in X.

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We are specially interested in the situation where the data f_j is not exactly known, i.e., we have only an approximation $f_j^{\delta_j}$ of the data f_j satisfying

$$\|f_j - f_j^{\delta_j}\|_{Y_j} \leq \delta_j, \quad \delta_j \to 0,\ j = 1, ..., N. \tag{1.2}$$

With the above conditions on A_j, each j-equation in (1.1) is ill-posed. By this we mean that the solution set S_j does not depend continuously on the data f_j. Therefore, to find a solution of each j-equation in (1.1) one has to use stable methods. One of those methods is the variational variant of Tikhonov's regularization that consists of minimizing the functional

$$\|A_j x - f_j^{\delta_j}\|_{Y_j}^2 + \alpha\|x - x_*\|_X^2, \tag{1.3}$$

where x_* is some element in $X \setminus S_j$, and chosing a value of the regularization parameter $\alpha = \alpha(\delta_1, \cdots \delta_N) > 0$. It proved in [1] that each j-minimization problem of (1.3) has unique solution $x_j^{\alpha\delta_j}$, and if $\delta_j^2/\alpha, \alpha \to 0$ then $\{x_j^{\alpha\delta_j}\}$ converges to a solution $\tilde{x}_j$ satisfying

$$\|\tilde{x}_j - x_*\|_X = \min_{x \in S_j} \|x - x_*\|_X, \quad j = 1, ..., N.$$

Our problem: find $x_\alpha^{\delta_j}$ such that $x_\alpha^{\delta_j} \to \tilde{x}$ as $\delta_j, \alpha \to 0$, a relation $\alpha = \alpha(\delta_1, \cdots \delta_N)$ such that $x_{\alpha(\delta_1,\cdots\delta_N)}^{\delta_j} \to \tilde{x}$ as $\delta_j \to 0$, and finally estimate the value $\|x_{\alpha(\delta_1,\cdots\delta_N)}^{\delta_j} - \tilde{x}\|$ where $\tilde{x}$ is a x_*-minimal norm element in S (x_*-MNS).

Burger and Kaltenbacher [2] used the Newton-Kaczmarz method cyclically for regularizing each separate equation in (1.1) under a *source condition* on each mapping A_j. The Steepest-Descent-Kaczmarz method is used cyclically by Haltmeier, Kowar, Leitão, and Scherzerfor [3] for regularizing each separate equation in (1.1) under a *local tangential cone condition* on each mapping A_j.

Note that the system of equations (1.1) can be written in the form

$$\mathcal{A}x = f, \tag{1.4}$$

where $\mathcal{A} : X \to Y := Y_1 \times ... \times Y_N$ by $\mathcal{A}x := (A_1x, ..., A_Nx)$, and $f := (f_1, ..., f_N)$. Very recently, Halmeier, Leitão and Scherzer [4] considered the Landweber-Kaczmarz method to solve (1.4) under a *local tangential cone condition* also on each mapping A_j. Equation (1.4) can be seen as a special case of (1.1) with $N = 1$. Howerver, one potential advantage of (1.1) over (1.4) can be that it might better reflect the structure of the underlying information $(f_1, ..., f_N)$ leading to the couplet system, than a plain concantenation into one single data element f could.

In [5], for finding a common zero for a finite family of potential hemicontinuous monotone mappings from a reflexive Banach space E into E^*, the adjoint of E, the first author proposed a regularization method by solving the following regularized equation

$$\begin{aligned}&\sum_{j=0}^{N}\alpha^{\mu_j}A_j(x)+\alpha U(x)=\theta,\\&\mu_0=0<\mu_j<\mu_{j+1}<1, j=1,2,...,N-1,\end{aligned}\tag{1.5}$$

where U is a normalized duality mapping of E and estimated convergence rate of the regularized solution under a smooth condition only for one A_1, i.e., $A_1'(\tilde{x})^*z = U(x_0)$, for some element $z \in E$.

In this paper, to solve (1.1), we consider a new regularization method based on the following unconstrained optimization problem:

$$\min_{x\in X}\sum_{j=1}^{N}\|A_jx-f_j^{\delta_j}\|_{Y_j}^2+\alpha\|x-x_*\|_X^2. \tag{1.6}$$

We shall show convergence rate of regularized solution in (1.6) under a source condition only on A_1 in the next section. In section 3, we give an example showing that problem (1.4), perhaps, is well-posed, although each equation in (1.1) is ill-posed. So, the cyclical regularization for each separate equation in (1.1) as in [2] and [3] is dispensable. Moreover, it does not exploir the well-posed property of the given system of equations. Meantime, our method (1.6) still uses the property.

Above and below, the symbols $\rightharpoonup$ and $\to$ denote the weak convergence and convergence in norm, respectively, and $a \sim b$ is meant $a = O(b)$ and $b = O(a)$.

2. MAIN RESULTS

Under the assumptions on A_j it can be easy to show that problem (1.6) admits a unique solution. We shall first address two questions. Is the problem (1.6) stable in the sense of continuous dependence of the solution on the data $f_j^{\delta_j}$? Secondly, do the solutions of (1.6) converge toward a solution of (1.1) as $\alpha, \delta_j \to 0$. In [6], stability has been proved for the case $N = 1$. For the convenience of the reader, we provide the whole argument in the case of arbitrary $N \geq 1$.

Theorem 2.1. *Let $\alpha > 0$, $f_j^{\delta_{jk}} \to f_j^{\delta_j}$ with $\delta_j \geq 0$, as $k \to \infty$, and x_k be a minimizer of (1.6) with $f_j^{\delta_j}$ replaced by $f_j^{\delta_{jk}}$. Then the sequence of $\{x_k\}$ converges to a minimizer of (1.6).*

Proof. Obviously, x_α^δ is a solution of (1.6) if and only if it is a solution of the following equation

$$\mathcal{B}x + \alpha(x - x_*) = \tilde{f}_\delta, \tag{2.1}$$

where

$$\mathcal{B} = \sum_{i=1}^{N} A_j^* A_j \quad \text{and} \quad \tilde{f}_\delta = \sum_{i=1}^{N} A_j^* f_j^{\delta_j},$$

where A_j^* denotes the adjoint mapping of A_j. Obviously, if $f_j^{\delta_{jk}} \to f_j^{\delta_{jk}}$ then

$$\tilde{f}_\delta^k = \sum_{i=1}^N A_j^* f_j^{\delta_j k} \to \tilde{f}_\delta = \sum_{i=1}^N A_j^* f_j^{\delta_j}$$

as $k \to \infty$. Denote by $x_\alpha^{\delta k}$ the solution of the following equation

$$\mathcal{B}x + \alpha(x - x_*) = \tilde{f}_\delta^k, \quad \tilde{f}_\delta^k = \sum_{i=1}^N A_j^* f_j^{\delta_j k}. \tag{2.2}$$

From (2.1), (2.2) and the monotone property of the mapping $\mathcal{B}$ it implies that

$$\|x_\alpha^{\delta k} - x_\alpha^\delta\|_X \le \|\tilde{f}_\delta^k - \tilde{f}_\delta\|_X/\alpha,$$

for each $\alpha > 0$. This together with $\tilde{f}_\delta^k \to \tilde{f}_\delta$ follows that $x_\alpha^{\delta k} \to x_\alpha^\delta$ as $k \to \infty$. Theorem is proved. □

Further, without loss of generality, assume that $\delta_j = \delta,\ \delta \to 0$.

Theorem 2.2. *Let $\alpha(\delta)$ be such that $\alpha(\delta) \to 0, \delta/\alpha(\delta) \to 0$ as $\delta \to 0$. Then the sequence $\{x_\alpha^\delta\}$, where $\delta \to 0, \alpha = \alpha(\delta)$ and x_α^δ is a solution of* (1.6), *converges to an x_*-MNS $\tilde{x}$ of* (1.1).

Proof. Clearly, we have, for each $y \in S$, that

$$\mathcal{B}y = \tilde{f}, \quad \tilde{f} = \sum_{i=1}^N A_j^* f_j. \tag{2.3}$$

So, from (2.1) and (2.3), we obtain that

$$\langle \mathcal{B}x_\alpha^\delta - \mathcal{B}y, x_\alpha^\delta - y\rangle + \alpha\langle x_\alpha^\delta - x_*, x_\alpha^\delta - y\rangle = \langle \tilde{f}_\delta - \tilde{f}, x_\alpha^\delta - y\rangle.$$

This together with the nonegative property of $\mathcal{B}$ implies that

$$\|x_\alpha^\delta - y\|_X \le \|x_* - y\|_X + \sum_{j=1}^N \|A_j^*\|_{(Y_j^* \to X)} \frac{\delta}{\alpha} \quad \forall y \in S. \tag{2.4}$$

Since A_j are the bounded linear mappings and $\delta/\alpha(\delta) \to 0$, the sequence $\{x_{\alpha(\delta)}^\delta\}$ is bounded. Then, there exists a subsequence $\{x_k := x_{\alpha(\delta_k)}^{\delta_k}\}$ of the sequence $\{x_{\alpha(\delta)}^\delta\}$ converging weakly to some element $\tilde{x} \in H$ as $k \to \infty$. Now, we shall prove that $\tilde{x} \in S$. Indeed, from (1.6) we can obtain the following inequalities

$$\begin{aligned}\|A_l x_k - f_l^{\delta_k}\|_{Y_l}^2 &\le \sum_{j=1}^N \|A_j x_k - f_j^{\delta_k}\|_{Y_j}^2 \\ &\le \sum_{j=1}^N \|A_j y - f_j^{\delta_k}\|_{Y_j} + \alpha(\delta_k)\|y - x_*\|_X^2 \\ &\le N\delta_k^2 + \alpha(\delta_k)\|y - x_*\|_X^2,\end{aligned}$$

for any $y \in S$ and $l = 1, ..., N$. Since each functional $\|A_l x - f_l^{\delta_k}\|^2$ is weakly lower semicontinuous in x [7], $x_k \rightharpoonup \tilde{x}$ and $\delta_k, \alpha(\delta_k) \to 0$ as $k \to \infty$, we obtain from the last inequality that $A_l\tilde{x} = f_l, l = 1, ..., N$. So, $\tilde{x} \in S$. Now, from (2.4), $\delta/\alpha(\delta) \to 0$, as $\delta \to 0$, the weakly lower semicontinuity of norm and that any closed convex subset in H has only one x_*-MNS, it follows that $x^\delta_{\alpha(\delta)} \to \tilde{x}$ as $\delta \to 0$. This completes the proof. □

Theorem 2.3. *Assume that there exists* $\omega \in Y_1$ *such that* $\tilde{x} - x_* = A_1^*\omega$. *Then for the choice* $\alpha \sim \delta^p, 0 < p < 1$, *we obtain*

$$\|x^\delta_{\alpha(\delta)} - \tilde{x}\|_X = O(\delta^{p/2}).$$

Proof. Using (1.6) with $x = \tilde{x}$ and $\delta_j = \delta$ for all $j = 1, ..., N$, we obtain

$$\sum_{j=1}^{N} \|A_j x^\delta_{\alpha(\delta)} - f_j^\delta\|^2_{Y_j} + \alpha(\delta)\|x^\delta_{\alpha(\delta)} - x_*\|^2_X \leq \sum_{j=1}^{N} \|A_j\tilde{x} - f_j^\delta\|^2_{Y_j} + \alpha(\delta)\|\tilde{x} - x_*\|^2_X.$$

So, we have that

$$\sum_{j=1}^{N} \|A_j x^\delta_{\alpha(\delta)}) - f_j^\delta\|^2_{Y_j} + \alpha(\delta)\|x^\delta_{\alpha(\delta)} - \tilde{x}\|^2_X \leq N\delta^2 + \alpha(\delta)(\|\tilde{x} - x_*\|^2_X - \|x^\delta_{\alpha(\delta)} - x_*\|^2_X + \|x^\delta_{\alpha(\delta)} - \tilde{x}\|^2_X).$$

Since

$$\|\tilde{x} - x_*\|^2_X - \|x^\delta_{\alpha(\delta)} - x_*\|^2_X + \|x^\delta_{\alpha(\delta)} - \tilde{x}\|^2_X = 2\langle \tilde{x} - x_*, \tilde{x} - x^\delta_{\alpha(\delta)}\rangle$$

we obtain that

$$\begin{aligned} \|A_1 x^\delta_{\alpha(\delta)} - f_1^\delta\|^2_{Y_1} + \alpha(\delta)\|x^\delta_{\alpha(\delta)} - \tilde{x}\|^2_X &\leq N\delta^2 \\ &\quad + 2\alpha(\delta)\langle \omega, A_1(\tilde{x} - x^\delta_{\alpha(\delta)})\rangle \\ &\leq N\delta^2 + 2\alpha(\delta)\langle \omega, f_1 - f_1^\delta + f_1^\delta - A_1 x^\delta_{\alpha(\delta)}\rangle \\ &\leq N\delta^2 + 2\alpha(\delta)\|\omega\|_{Y_1}(\delta + \|A_1 x^\delta_{\alpha(\delta)} - f_1^\delta\|_{Y_1}). \end{aligned} \tag{2.5}$$

Therefore,

$$\begin{aligned} \|A_1 x^\delta_{\alpha(\delta)} - f_1^\delta\|^2_{Y_1} + \alpha(\delta)\|x^\delta_{\alpha(\delta)} - \tilde{x}\|^2_X &\leq N\delta^2 \\ &\quad + 2\alpha(\delta)\|\omega\|_{Y_1}\delta + 2\alpha(\delta)\|\omega\|_{Y_1}\|A_1 x^\delta_{\alpha(\delta)} - f_1^\delta\|_{Y_1}. \end{aligned} \tag{2.6}$$

This together with the implication

$$(a, b, c \geq 0, a^2 \leq ab + c^2) \Rightarrow a \leq b + c$$

implies

$$\|A_1 x^\delta_{\alpha(\delta)} - f^\delta_1\|_{Y_1} \leq \Big[N\delta^2 + 2\|\omega\|_{Y_1}\alpha(\delta)\delta\Big]^{1/2} + 2\|\omega\|_{Y_1}\alpha(\delta)$$

and hence

$$\|x^\delta_{\alpha(\delta)} - \tilde{x}\|_X = O(\delta^{p/2}),$$

if $\alpha(\delta) \sim \delta^p, 0 < p < 1$. □

3. NUMERICAL EXAMPLES

For illustration, we consider the following problem of finding a common solution of two systems of linear algebraic equations

$$A_j x = f_j,\ \ j = 1, 2, \tag{3.1}$$

where

$$A_1 = \begin{bmatrix} 1 & 2 & -1 \\ 2 & 0 & 1 \\ 3 & 2 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 1 & -2 & -1 \\ -2 & 1 & 0 \\ -1 & -1 & -1 \end{bmatrix}, f_1 = \begin{bmatrix} 2 \\ 3 \\ 5 \end{bmatrix}, f_2 = \begin{bmatrix} -2 \\ -1 \\ -3 \end{bmatrix}.$$

It is easy to verify that system (3.1) possesses a unique common solution $\tilde{x} = (1; 1; 1)$. Since det A_1 = det $A_2 = 0$, each equation in (3.1) is ill-posed. Based on the results in section 2, the common solution of (3.1) can be found by solving the following optimization problem

$$\min_{x \in R^3} \ \|A_1 x - f_1\|^2 + \|A_2 x - f_2\|^2 + \alpha\|x - x_*\|^2, \tag{3.2}$$

where $\|x\| = \sqrt{x_i^2 + x_2^2 + x_3^2}$ for every $x = (x_1; x_2; x_3) \in R^3$. It is not difficult to verify that (3.2) is equivalent to the following equation

$$\mathcal{B}x + \alpha(x - x_*) = \tilde{f}, \tag{3.3}$$

where

$$\mathcal{B} = A_1^* A_1 + A_2^* A_2 = \begin{bmatrix} 20 & 5 & 1 \\ 5 & 14 & 1 \\ 1 & 1 & 4 \end{bmatrix}, \tilde{f} = A_1^* f_1 + A_2^* f_2 = \begin{bmatrix} 26 \\ 20 \\ 6 \end{bmatrix},$$

and x_* is any vector in R^3. Since det $\mathcal{B} = 996$ and (3.3) can be considered as a regularization equation for the well-posed equation $\mathcal{B}x = \tilde{f}$, we can use Jacoby or Gauss-Seidel iteration methods for finding a unique solution x_α of (3.3). The following table 1 shows the calculation results for the approximation solution $x_\alpha = (x_1^\alpha; x_2^\alpha; x_3^\alpha)$ at 15^{th} iteration with the started point $x_0 = (2, 2, 2)$.

- Table 1. 15^{th} approximation solution with started point $x_0 = (2; 2; 2)$.

α	x_1^α	x_2^α	x_3^α	$\|\tilde{x} - x_\alpha\|$
0.1000	0.9973	0.9955	0.9974	0.0232
0.0100	0.9997	0.9995	0.9977	0.0024
0.0010	1.000	1.0000	0.9998	0.0002
0.0001	1.0000	1.0000	1.0000	0.0000

Now, we give another example with det $\mathcal{B} = 0$. We consider the case that

$$A_1 = \begin{bmatrix} 0.1 & -0.2 & 0.1 & -0.1 \\ 0.2 & -0.1 & 0.0 & 0.2 \\ 0.3 & -0.3 & 0.1 & 0.1 \\ 0.1 & 0.1 & -0.1 & 0.3 \end{bmatrix}, \; A_2 = \begin{bmatrix} 0.1 & 0.2 & -0.1 & 0.1 \\ 0.2 & -0.1 & 0.0 & 0.2 \\ 0.0 & 0.3 & -0.2 & 0.4 \\ -0.1 & 0.5 & -0.3 & 0.5 \end{bmatrix},$$

and

$$f_1 = \begin{bmatrix} -0.1 \\ 0.3 \\ 0.2 \\ 0.4 \end{bmatrix}, \; f_2 = \begin{bmatrix} 0.3 \\ 0.3 \\ 0.5 \\ 0.6 \end{bmatrix}.$$

Then,

$$\mathcal{B} = \begin{bmatrix} 0.21 & -0.17 & 0.05 & 0.09 \\ -0.17 & 0.54 & -0.29 & 0.37 \\ 0.05 & -0.29 & 0.17 & -0.27 \\ 0.09 & 0.37 & -0.27 & 0.61 \end{bmatrix}, \; \tilde{f} = \begin{bmatrix} 0.18 \\ 0.45 \\ -0.34 \\ 0.80 \end{bmatrix}.$$

Since the rank of the matrix $\mathcal{B} = 3$, it is not difficult to verify that the set of common solutions of (3.1) is a line passed through two points $\tilde{x} = (1; 1; 1; 1)$ and $x' = (1; 3; 6; 2)$. So, the solution of minimal norm is the vector $\tilde{x} = (1; 7/15; -1/3; 11/15) \approx (1; 0.466667; -0.333333; 0.733333)$.

Table 2 shows our calculation result with $\delta = 10^{-n}$. To solve regularized equation (3.3) with $\tilde{f}$ replaced by $\tilde{f}_\delta$ we use the iterative regularization [8]

$$x^{(k+1)} = x^{(k)} - \alpha_k(\mathcal{B}x^{(k)} + \varepsilon_k(x^{(k)} - x^{(0)}) - \tilde{f}_\delta), x^{(0)} \in R^4 \text{ any vector},$$

with the stopping rule in [9]:

$$\|\mathcal{B}x^{(K)} - \tilde{f}_\delta\|^2 \le \tau\delta < \|\mathcal{B}x^{(k)} - \tilde{f}_\delta\|^2, \; \tau > 1,$$

for all $k = 1, ..., K-1$ and every fixed δ. In our example, we have that $L = \|\mathcal{B}\| = 1.1847$. We chose $\varepsilon_0 = 0.1$ and $\varepsilon_{k+1} = \varepsilon_k/(1 + \varepsilon_k^3)$. Then, the sequence $\{\varepsilon_k\}$ satisfies the conditions: $\varepsilon_k > 0, \varepsilon_k \searrow 0$ and $(\varepsilon_k - \varepsilon_{k+1})/(\varepsilon_k^3\varepsilon_{k+1}) = 1$ and $\lambda = (\varepsilon_0 + L^2)/2 = 0.7068$. By taking $\alpha_k = c\varepsilon_k$ with $c = 1/(2\lambda) = 0.7075$ we have that $(1 - c\lambda)c\varepsilon_0^2 = 0.0035 < 1$. Clearly, condition (2.6) in [8] is equivalent to

$$\tau \ge \left(\sqrt{\frac{\|\tilde{x} - x^{(0)}\|[(1 + \varepsilon_0^3)(1 + \varepsilon_0^2) + 2\varepsilon_0]}{(1 - c\lambda - 2(\varepsilon_0/c))\varepsilon_0}} + 1 \right)^2.$$

Therefore, with $x^{(0)} = (0; 0; 0; 0)$, we obtain that $\|\tilde{x} - x^{(0)}\| = \|\tilde{x}\| = 1.3663$, and hence $\tau \ge 94.5933$.

• Table 2. Approximation solutions with started point $x^{(0)} = (0;0;0;0)$, $\tau = 100$ and $\delta = 10^{-n}, n = 1, 2, \cdots$

n	K	$x_1^{(K)}$	$x_2^{(K)}$	$x_3^{(K)}$	$x_4^{(K)}$	$\|\mathcal{B} - \tilde{f}_\delta\|$	$\tau\delta$
1	0	0	0	0	0	1.1857	10
2	1	0.01337	0.03212	-0.02441	0.05780	0.94057	1
3	19	0.02337	0.25683	-0.22184	0.59552	0.31553	0.1
4	123	0.47880	0.21716	-0.25959	0.86361	0.09222	0.01
5	17452	0.70121	0.29095	-0.28953	0.86577	0.03162	0.001
6	694405	0.85935	0.37766	-0.31246	0.80696	0.00999	0.0001
7	24868554	0.94756	0.43269	-0.32551	0.76217	0.00316	0.00001

References

[1] A.N. Tikhonov and V.Y. Arsenin, Solutions of ill-posed problems, Wiley, N.Y. 1977.

[2] M. Burger and B. Kaltenbacher, Regularization Newton-Kaczmarz methods for nonlinear ill-posed problems, SIAM J. Numer. Analysis, **44** (2006) 153-182.

[3] M. Haltmeier, R. Kowar, A. Leitao, and O. Scherzer, Kaczmarz methods for nonlinear ill-posed equations I: convergence analysis, Inverse problem and Imaging, **1** (2) (2007) 289-298, II: Applications, **1** (3) (2007) 507-523.

[4] A.D. Cezaro, M. Haltmeier, A. Leitao, and O. Scherzer, On steepest-descent-Kaczmarz method for regularizing systems of nonlinear ill-posed equations, Applied Mathematics and Computations, **202** (2) (2008) 596-607.

[5] Ng. Buong, Regularization for unconstrained vector optimization of convex functionals in Banach spaces, Zh. Vychisl. Mat. i Mat. Fiziki, **46**(3) (2006) 372-378.

[6] H.W. Engl, K. Kunisch, and A. Neubauer, Convergence rates for Tikhonov regularization of non-linear ill-posed problems, Inverse Problems **5** (1989) 523-540.

[7] M.M. Vainberg, Variational method and method of monotone mappings, Moscow, Nauka 1972 (in Russian).

[8] A.B. Bakushinsky, A. Goncharsky, Ill-posed problems: Theory and Applications, Kluwer Academic 1994.

[9] A.B. Bakushinsky and A. Smirnova, A posteriori stopping rule for regularized fixed point iterations, *Nonl. Anal.* **64** (2006) 1255-1261.

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