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Application of the Improved $(\frac{G'}{G})$-Expansion Method for the Variant Boussinesq Equations

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Abstract

In this paper, by using the improved $(\frac{G'}{G})$-expansion method, we have successfully obtained some travelling wave solutions of the variant Boussinesq Equations. These exact solutions include the hyperbolic function solutions, trigonometric function solutions and rational function solutions.

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1 Introduction

In recent years, with the development of symbolic computation packages like Maple and Mathematica, which enable us to perform the tedious and complex computation on computer, much work has been focused on the direct methods [1-5] to construct exact solutions of the nonlinear evolution equations (NLEEs).

Recently, Wang et al. [6] introduced a new direct method called the $(\frac{G'}{G})$-expansion method to look for travelling wave solutions of NLEEs. Later, the further developed methods named the generalized $(\frac{G'}{G})$-expansion method, the modified $(\frac{G'}{G})$-expansion method, the extended $(\frac{G'}{G})$-expansion method and the improved $(\frac{G'}{G})$-expansion method have been proposed in Refs. [7-10].

In this paper, we will apply improved $(\frac{G'}{G})$-expansion method [10] to study the variant Boussinesq equations[11-16]

$$H_t + (Hu)_x + u_{xxx} = 0, \tag{1}$$

$$u_t + H_x + uu_x = 0. \tag{2}$$

As a model for water waves, u is the velocity and H the total depth, and the subscripts denote partial derivatives. Wang [11] has obtained two types of solitary wave solutions of the variant Boussinesq equations by homogeneous balance method. In [12], L has obtained abundantJacobi elliptic function solutions of the variant Boussinesq equations. In [13], several types of explicit and exact travelling wave solutions to a system of variant Boussinesq equations have been obtained by using an improved Sine-cosine method and Wu eliminationmethod. In Refs. [14,15], a series of new travelling wave solutions, which include soliton solutions, rational solutions, triangularperiodic solutions, Jacobi and Weierstrass doubly periodic wave solutions, have been constructed by the algebraicmethod. Wu and He [16] have obtained a generalized solitary solution with free parameters by Exp-function method.

2 The methods

Guo has summarized for using improved $(\frac{G'}{G})$-expansion method.

step 1. A PDE

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, \ldots\ldots) = 0, \tag{3}$$

can be converted to on ODE

$$p(u, u', u'', u''', \ldots\ldots) = 0, \tag{4}$$

upon using a wave variable $u(x,t) = u(\xi), \xi = x + kt$.

setp 2. We suppose that the NLODE (4) has the following solution:

$$u(\xi) = \sum_{i=n}^{-1} \frac{a_i \left(\frac{G'}{G}\right)^i}{\left(1+\sigma(\frac{G'}{G})\right)^i} + a_0 + \sum_{i=1}^{n} \frac{a_i \left(\frac{G'}{G}\right)^i}{\left(1+\sigma(\frac{G'}{G})\right)^i}. \tag{5}$$

where $\sigma, a_i, (i = -n, -n+1, \ldots, n-1, n)$ are constants to be determined later, n is a positive integer, and $G = G(\xi)$ satisfies the following second order linear ordinary differential equation:

$$G'' + \mu G = 0 \tag{6}$$

where μ is a real constant. The general solutions of (6) can be listed as follows.

when $\mu > 0$, we obtain the trigonometric function solution of Eq.(6)

$$G(\xi) = C_1 \sin(\sqrt{\mu}\xi) + C_2 \cos(\sqrt{\mu}\xi), \tag{7}$$

when $\mu < 0$, we obtain the hyperbolic function solution of Eq.(6)

$$G(\xi) = C_1 \cosh(\sqrt{-\mu}\xi) + C_2 \sinh(\sqrt{-\mu}\xi), \tag{8}$$

when $\mu = 0$, we obtain the rational function solution of Eq.(6)

$$G(\xi) = C_1 + C_2\xi. \tag{9}$$

where C_1 and C_2 are arbitrary constants.

step 3. Determine the positive interger n by balancing the highest order derivatives and nonlinear terms in Eq.(4).

step 4. substituting (5) along with (6) into Eq.(4) and then setting all the coefficients of $\left(\frac{G'}{G}\right)^i$ $(k = 1, 2 \ldots)$ of the resulting system's numerator to zero, yields a set of over-determined nonlinear algebraic equations for c, σ and $a_i (i = -n, -n+1, \ldots, n-1, n)$.

step 5. Assuming that the constants $c, \sigma, a_i (i = -n, -n+1, \ldots, n-1, n)$ can be obtained by solving the algebraic equations in Step 4, then substituting these constants and the known general solutions of Eq.(6) into (5), we can obtain the explicit solutions of Eq.(3) immediately.

3 New exact travelling wave solutions of the variant Boussinesq equations

In this section, we will apply the improved $(\frac{G'}{G})$-expansion method to study the variant Boussinesq equations, suppose that

$$H(x,t) = H(\xi), \quad u(x,t) = u(\xi), \quad \xi = x + kt \tag{10}$$

by using (10), Eqs.(1)-(2) are converted into the ODEs for $H = H(\xi)$ and $u = u(\xi)$ as follows:

$$kH' + (Hu)' + u''' = 0, \tag{11}$$

$$ku' + H' + uu' = 0. \tag{12}$$

Integrating the ODEs above with respect to ξ once yields

$$A_1 + kH + Hu + u'' = 0, \tag{13}$$

$$A_2 + ku + H + \frac{1}{2}u^2 = 0. \tag{14}$$

where A_1 and A_2 are integration constants that are to be determined later.

Considering the homogeneous balance between u'' and Hu in Eq.(13) and that between H and u^2 in Eq.(14) $(m_1 + m_2 = m_2 + 2, 2m_2 = m_1 \Rightarrow m_1 = 2, m_2 = 1)$, we suppose that

$$u(\xi) = a_0 + \frac{a_1(\frac{G'}{G})}{1+\sigma(\frac{G'}{G})} + \frac{b_1(1+\sigma(\frac{G'}{G}))}{(\frac{G'}{G})}, \tag{15}$$

$$H(\xi) = c_0 + \frac{c_1(\frac{G'}{G})}{1+\sigma(\frac{G'}{G})} + \frac{d_1(1+\sigma(\frac{G'}{G}))}{(\frac{G'}{G})} + \frac{c_2(\frac{G'}{G})^2}{(1+\sigma(\frac{G'}{G}))^2} + \frac{d_2(1+\sigma(\frac{G'}{G}))^2}{(\frac{G'}{G})^2}. \tag{16}$$

where $G = G(\xi)$ satisfies Eq.(6), $\sigma, a_0, a_1, b_1, c_0, c_1, c_2, d_1$ and d_2 are constants to be determined latter.

Substituting (15-16) along with Eq.(6) into Eq.(13-14) and then setting all the coefficients of $(\frac{G'}{G})^k (k = 1, 2\ldots)$ of the resulting system's numerator to zero, yields a set of over-determined nonlinear algebraic equations about $\sigma, a_0, a_1, b_1,\ c_0, c_1, c_2, d_1, d_2, k, A_1$ and A_2. Solving the over-determined algebraic equations by Maple, we can obtain the following results:

Cass 1.

$$A_1 = 0, A_2 = \frac{1}{2}k^2 + 8\mu, a_0 = -k, a_1 = \pm 2, b_1 = \mp 2\mu, c_0 = -4\mu, c_1 = 0,$$

$$c_2 = -2, d_1 = 0, d_2 = -2\mu^2, \sigma = 0. \tag{17}$$

Cass 2.

$$A_1 = 0, A_2 = \frac{1}{2}k^2 + 2\mu, a_0 = \mp 2\mu\sigma - k, a_1 = \pm 2 \pm 2\mu\sigma^2, b_1 = 0,$$

$$c_0 = -2\mu - 2\mu^2\sigma^2, c_1 = 4\mu\sigma + 4\mu^2\sigma^3, c_2 = -2(1+\mu\sigma^2)^2, d_1 = d_2 = 0. \tag{18}$$

Using Case 1, When $\mu > 0$, we obtain the trigonometric function solutions of the variant Boussinesq equations.

$$u = -k \pm \frac{2\sqrt{\mu}(C_1\cos(\sqrt{\mu}\xi) - C_2\sin(\sqrt{\mu}\xi))}{C_1\sin(\sqrt{\mu}\xi) + C_2\cos(\sqrt{\mu}\xi)} \mp \frac{2\sqrt{\mu}(C_1\sin(\sqrt{\mu}\xi) + C_2\cos(\sqrt{\mu}\xi)}{C_1\cos(\sqrt{\mu}\xi) - C_2\sin(\sqrt{\mu}\xi)},$$

$$H = -4\mu - \frac{2\mu(C_1\cos(\sqrt{\mu}\xi) - C_2\sin(\sqrt{\mu}\xi))^2}{(C_1\sin(\sqrt{\mu}\xi) + C_2\cos(\sqrt{\mu}\xi))^2} - \frac{2\mu(C_1\sin(\sqrt{\mu}\xi) + C_2\cos(\sqrt{\mu}\xi))^2}{(C_1\cos(\sqrt{\mu}\xi) - C_2\sin(\sqrt{\mu}\xi))^2}. \tag{19}$$

where $\xi = x + kx$, k are arbitrary constants, or

$$u = -k \pm \frac{2\sqrt{\mu}(C_1 - C_2\tan(\sqrt{\mu}\xi))}{C_1\tan(\sqrt{\mu}\xi) + C_2} \mp \frac{2\sqrt{\mu}(C_1\tan(\sqrt{\mu}\xi) + C_2)}{C_1 - C_2\tan(\sqrt{\mu}\xi)},$$

$$H = -4\mu - \frac{2\mu(C_1 - C_2\tan(\sqrt{\mu}\xi))^2}{(C_1\tan(\sqrt{\mu}\xi) + C_2)^2} - \frac{2\mu(C_1\tan(\sqrt{\mu}\xi) + C_2)^2}{(C_1 - C_2\tan(\sqrt{\mu}\xi))^2}. \tag{20}$$

In particular, setting $C_1 = 0$, the following solitary wave solutions of the variant Boussinesq equations are discovered.

$$u = -k \mp 2\sqrt{\mu}(\tan(\sqrt{\mu}\xi) - \cot(\sqrt{\mu}\xi))$$

$$H = -4\mu - 2\mu(\tan^2(\sqrt{\mu}\xi) + \cot^2(\sqrt{\mu}\xi)). \tag{21}$$

When $\mu < 0$, we obtain the hyperbolic function solutions of the variant Boussinesq equations.

$$u = -k \pm \frac{2\sqrt{-\mu}(C_1\sinh(\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\mu}\xi))}{C_1\cosh(\sqrt{-\mu}\xi) + C_2\sinh(\sqrt{-\mu}\xi)} \pm$$

$$\frac{2\sqrt{-\mu}(C_1\cosh(\sqrt{-\mu}\xi) + C_2\sinh(\sqrt{-\mu}\xi)}{C_1\sinh(\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\mu}\xi)},$$

$$H = -4\mu - \frac{2(\sqrt{-\mu}(C_1\sinh(\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\mu}\xi))^2}{(C_1\cosh(\sqrt{-\mu}\xi) + C_2\sinh(\sqrt{-\mu}\xi))^2} -$$

$$\frac{2(\sqrt{-\mu}(C_1\cosh(\sqrt{-\mu}\xi) + C_2\sinh(\sqrt{-\mu}\xi))^2}{(C_1\sinh(\sqrt{-\mu}\xi) + C_2\cosh(\sqrt{-\mu}\xi))^2}. \tag{22}$$

where $\xi = x + kx$,k are arbitrary constants, or

$$u = -k \pm \frac{2\sqrt{-\mu}(C_1\tanh(\sqrt{-\mu}\xi) + C_2)}{C_1 + C_2\tanh(\sqrt{-\mu}\xi)} \pm \frac{2\sqrt{-\mu}(C_1 + C_2\tanh(\sqrt{-\mu}\xi))}{C_1\tanh(\sqrt{-\mu}\xi) + C_2},$$

$$H = -4\mu - \frac{2(\sqrt{-\mu}(C_1\tanh(\sqrt{-\mu}\xi) + C_2)^2}{(C_1 + C_2\tanh(\sqrt{-\mu}\xi))^2} - \frac{2(\sqrt{-\mu}(C_1 + C_2\tanh(\sqrt{-\mu}\xi))^2}{(C_1\tanh(\sqrt{-\mu}\xi) + C_2)^2}. \tag{23}$$

In particular, setting $C_1 = 0$, the following solitary wave solutions of the variant Boussinesq Equations are discovered.

$$u = -k \pm 2\sqrt{-\mu}(\coth(\sqrt{\mu}\xi) + \tanh(\sqrt{-\mu}\xi))$$

$$H = -4\mu + 2\mu(\coth^2(\sqrt{-\mu}\xi) + \tanh^2(\sqrt{-\mu}\xi)). \tag{24}$$

When $\mu = 0$, we get the rational function solutions of the variant Boussinesq Equations.

$$u = -k \pm \frac{2C_2}{C_1 + C_2\xi}, \tag{25}$$

$$H = -\frac{2C_2^2}{(C_1 + C_2\xi)^2}. \tag{26}$$

Using Case 2, (15-16) and the general solutions of Eq.(6), we could obtain more exact solutions of the variant Boussinesq Equations, and here we do not list all of them.

4 Conclusions

In this paper, by using the improved $(\frac{G'}{G})$-expansion method, we have successfully obtained some travelling wave solutions of the variant Boussinesq Equations. These exact solutions include the hyperbolic function solutions, trigonometric function solutions and rational function solutions. When the parameters are taken as special values, the solitary wave solutions are derived from the hyperbolic function solutions. To the best of our knowledge, the solutions obtained in this paper have not been reported in previous literature. The work shows that the improved $(\frac{G'}{G})$-expansion method is direct, concise and effective, and can be applied to other NLEEs in mathematical physics.

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