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Unified Fractional Reaction-Diffusion Equations with Solutions

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Abstract

In the present paper we obtain the solution of a unified fractional reaction-diffusion equation associated with the Caputo fractional derivative as the time derivative and Riesz-Feller fractional derivative as the space-derivative. The results are derived by the applications of the Laplace and Fourier transforms in compact and elegant form in terms of Mittag-Leffler and H-functions, which are suitable for numerical computation. The results obtained here are of general nature and include the results investigated earlier by many authors. The main result is given in the form of a theorem. A number of interesting special cases of the theorem are also given as corollaries.

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1 Introduction

In recent years, it has turned out that many phenomena in engineering, physics, chemistry and other sciences can be described very successfully by models using mathematical tools from fractional calculus. For example, the nonlinear oscillation of earthquake can be modeled with fractional derivatives and the fluid-dynamic traffic model with fractional derivatives can eliminate the deficiency arising from the assumption of continuum traffic flow. Fractional derivatives are also used in modeling of many chemical processes, mathematical biology and many other problems in physics and engineering. These findings invoked the growing interest of studies of the fractional calculus in various fields such as physics, chemistry and engineering. Fractional differential equations have gained importance and popularity during the past three decades or so, mainly due to exact description of nonlinear phenomena, especially in fluid mechanics, e.g. nano-hydrodynamics, where continuum assumption does not well, and fractional model can be considered to be a best candidate. Hence, great attention has been given to finding solutions of fractional differential equations.

In recent years, fractional reaction-diffusion models are studied due to their usefulness and importance in many areas of science and engineering. The reaction-diffusion equations arise naturally as description models of many evolution problems in the real world, as in chemistry [38, 39], biology [29], etc. As is well known, complex behavior is peculiarity of systems modeled by reaction-diffusion equations and the Belousov-Zhabotinskii reaction [28, 40] provides a classic example.

The reaction-diffusion equations describes a population of diploid individuals (i.e., the ones that carry two genes) distributed in a two-dimensional habitat. Assuming that a gene occurs in two forms a and A, called *alleles*, one can divide the population into three genotypes aa, aA and AA. The reaction-diffusion equations are employed to describe the co-oxidation on Pt (110) [1], the study of temporal and spatial patterns of cytoplasmic Ca^{+2} dynamics under the effects of Ca^{+2}-release activated Ca^{+2} (CRAC) channels in T cells [6], problems in finance [14, 25, 32] and hydrology [2]. Burke et al. [4] obtained solutions for an enzyme-suicide substrate reaction with an instantaneous point source of substrate. In 1993, Grimson and Barker [17] introduced a continuum model for the spatio-temporal growth of bacterial colonies on the surface of a solid substrate which utilizes a reaction-diffusion equation for growth. Many cellular and sub-cellular biological processes [10] can be described in terms of diffusing and chemically reacting species (e.g. enzymes). A traditional approach to the mathematical modeling of such reaction-diffusion processes is to describe each biochemical species by its (spatially dependent) concentration.

In recent time, interest in fractional reaction-diffusion equations [15, 16, 18, 35, 36, 37] has increased because the equation exhibits self-organization phenomena and introduces a new parameter, the fractional index, into the equation. Additionally, the analysis of fractional reaction-diffusion equations is of great importance from the analytical and numerical point of view. In the present article, we investigate the solution of a unified fractional reaction-

diffusion equation associated with the Caputo derivative and the Riesz-Feller derivative.
The Riemann-Liouville fractional integral of order ν is defined by ([27, p.45] and [22]).

$$ {}_0D_t^{-\nu} N\,(x,t) \;=\; \frac{1}{\Gamma(\nu)} \int_0^t (t-u)^{\nu-1}\, N(x,u)\, du, \tag{1}$$

where $\mathrm{Re}(\nu) > 0$.
The fractional derivative of order $\alpha > 0$ is introduced by Caputo [7], see also Kilbas et al.[22] in the following form

$$ {}_0D_t^{\alpha} f(x,t) \;=\; \frac{1}{\Gamma(m-\alpha)} \int_0^t \frac{f^m(x,\tau)\, d\tau}{(t-\tau)^{\alpha+1-m}}\,, $$

$$ m\,-1 \;<\; \alpha \;\le\; m, \mathrm{Re}\,(\alpha) \;>\; 0,\; m \;\in\; N $$

$$ = \frac{\partial^m f(x,t)}{\partial\, t^m}\,,\ \text{if}\ \alpha \;=\; m, \tag{2}$$

where $\frac{\partial^m f(x,t)}{\partial\, t^m}$ is the m-th order partial derivative of f(x,t) with respect to t. The Laplace transform of the Caputo derivative is given by Caputo [7], see also Kilbas et al. [22] in the form

$$ L\,\{{}_0D_t^{\alpha} f(x,t)\,;\ s\} \;=\; s^{\alpha}\,F(x,s) \;-\; \sum_{r=0}^{m-1} s^{\alpha-r-1}\, f^{(r)}\,(x,0+)\,,\ (m-1 \;<\; \alpha \;\le\; m). \tag{3}$$

Following Feller [11-12], it is conventional to define the Riesz-Feller space-fractional derivative of order α and skewness θ in terms of its Fourier transform as

$$ F\,\{{}_xD_{\theta}^{\alpha} f(x)\,;\ k\} \;=\; -\,\psi_{\alpha}^{\theta}\,(k)\, f^{*}(k)\,, \tag{4}$$

where

$$ \psi_{\alpha}^{\theta}(k) \;=\; |\,k\,|^{\alpha} \exp\left[i\ (\mathrm{signk})\ \frac{\theta\pi}{2}\right],\ 0 \;<\; \alpha \;\le\; 2,\ |\,\theta\,| \;\le\; \min\,\{\alpha, 2-\alpha\}. $$

2 Unified fractional reaction-diffusion equation

In this section, we will investigate the solution of the reaction-diffusion equation (5). The result is given in the form of the following Theorem.
Theorem 1. Consider the unified fractional reaction-diffusion equation associated with the Caputo derivative and Riesz-Feller fractional derivative in the form

$$ {}_0D_t^{\alpha} N(x,t) \;+\; a\,{}_0D_t^{\beta}\, N(x,t) \;=\; \eta\,{}_xD_{\theta}^{\gamma}\, N(x,t) \;+\; \xi\,{}_xD_{\sigma}^{\lambda}\, N(x,t) \;+\; \phi(x,t), \tag{5}$$

where $a, \eta, \xi, t > 0$, $x \in R$; $\gamma, \lambda, \theta, \sigma, \alpha, \beta$ are real parameters with the conditions $0 < \gamma \le 2$, $0 < \lambda \le 2$, $|\theta| \le \min(\gamma, 2-\gamma)$, $|\sigma| \le \min(\lambda, 2-\lambda)$, $0 < \alpha \le 2$, $0 < \beta \le 2$ and the initial conditions

$$N(x,0) = f(x),\ N_t(x,0) = g(x);\ \text{for } x \in R, \lim_{|x| \to \pm\infty} N(x,t) = 0,\ t > 0. \tag{6}$$

Here $N_t(x,0)$ means the first partial derivative of N(x,t) with respect to t evaluated at $t = 0$, η and ξ are diffusion constants and $\phi(x,t)$ is a nonlinear function belonging to the area of reaction-diffusion. Then for the solution of (5), subject to the above constraints, there holds the formula

$$\begin{aligned} N(x,t) = {} & \frac{1}{2\pi}\int_{-\infty}^{\infty} f^*(k) \sum_{r=0}^{\infty}(-a)^r t^{(\alpha-\beta)r} E^{r+1}_{\alpha,(\alpha-\beta)r+1} \\ & [-\{\eta\,\psi^{\theta}_{\gamma}(k) + \xi\,\psi^{\sigma}_{\lambda}(k)\}t^{\alpha}]\exp(-ikx)dk \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} g^*(k)\, t \sum_{r=0}^{\infty}(-a)^r t^{(\alpha-\beta)r} E^{r+1}_{\alpha,(\alpha-\beta)r+2} \\ & [-\{\eta\,\psi^{\theta}_{\gamma}(k) + \xi\,\psi^{\sigma}_{\lambda}(k)\}t^{\alpha}]\exp(-ikx)dk \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} a\, f^*(k)\, t^{\alpha-\beta} \sum_{r=0}^{\infty}(-a)^r t^{(\alpha-\beta)r} E^{r+1}_{\alpha,(\alpha-\beta)r-\beta+1} \\ & [-\{\eta\,\psi^{\theta}_{\gamma}(k) + \xi\,\psi^{\sigma}_{\lambda}(k)\}t^{\alpha}]\exp(-ikx)dk \\ & + \frac{1}{2\pi}\int_{-\infty}^{\infty} ag^*(k)\, t^{\alpha-\beta+1} \sum_{r=0}^{\infty}(-a)^r t^{(\alpha-\beta+1)r} E^{r+1}_{\alpha,(\alpha-\beta+1)r-\beta+2} \\ & [-\{\eta\,\psi^{\theta}_{\gamma}(k) + \xi\,\psi^{\sigma}_{\lambda}(k)\}t^{\alpha}]\exp(-ikx)dk \\ & + \frac{1}{2\pi}\int_{0}^{t} u^{\alpha-1} \sum_{r=0}^{\infty}(-a)^r u^{(\alpha-\beta)r}\, du \int_{-\infty}^{\infty} \phi^*(k, t-u)\, E^{r+1}_{\alpha,(\alpha-\beta)r} \\ & [-\{\eta\,\psi^{\theta}_{\gamma}(k) + \xi\,\psi^{\sigma}_{\lambda}(k)\}u^{\alpha}]\exp(-ikx)dk. \end{aligned} \tag{7}$$

In equation (7) and the following, $E^{\gamma}_{\alpha,\beta}(z)$ denotes the generalized Mittag-Leffler function ([2], [8] and [33]).

Proof. Applying the Laplace transform with respect to the time variable t and using the initial conditions (6), we find that

$$\begin{aligned} & s^{\alpha}\bar{N}(x,s) - s^{\alpha-1} f(x) - s^{\alpha-2} g(x) + a\, s^{\beta}\bar{N}(x,s) - a\, s^{\beta-1} f(x) - a\, s^{\beta-2} g(x) \\ & \quad = \eta\, {}_xD^{\gamma}_{\theta}\,\bar{N}(x,s) + \xi\, {}_xD^{\lambda}_{\sigma}\,\bar{N}(x,s) + \bar{\phi}(x,s). \end{aligned} \tag{8}$$

If we apply the Fourier transform with respect to space variable x and use the formula (4), it yields

$$s^{\alpha}\bar{N}^*(k,s) - s^{\alpha-1}f^*(k) - s^{\alpha-2}g^*(k) + a\, s^{\beta}\bar{N}^*(k,s) - as^{\beta-1}f^*(k) - a\, s^{\beta-2}g^*(k)$$

$$= \eta\, \psi_\gamma^\theta(\mathrm{k})\, \bar{\mathrm{N}}^*(\mathrm{k,s}) \;+\; \xi\, \psi_\lambda^\sigma(\mathrm{k})\, \bar{\mathrm{N}}^*(\mathrm{k,s}) \;+\; \bar{\phi}^*(\mathrm{k,s}). \tag{9}$$

Solving for $\bar{\mathrm{N}}^*(\mathrm{k,s})$,it gives

$$\bar{\mathrm{N}}^*(\mathrm{k,s}) \;=\; \frac{\mathrm{f}^*(\mathrm{k})\, \mathrm{s}^{\alpha-1}}{\mathrm{s}^\alpha + \mathrm{a\, s}^\beta + \eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})} + \frac{\mathrm{g}^*(\mathrm{k})\, \mathrm{s}^{\alpha-2}}{\mathrm{s}^\alpha + \mathrm{a\, s}^\beta + \eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})}$$
$$+ \frac{\mathrm{a\, f}^*(\mathrm{k})\, \mathrm{s}^{\beta-1}}{\mathrm{s}^\alpha + \mathrm{a\, s}^\beta + \eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})} + \frac{\mathrm{a\, g}^*(\mathrm{k})\, \mathrm{s}^{\beta-2}}{\mathrm{s}^\alpha + \mathrm{as}^\beta + \eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})}$$
$$+ \frac{\bar{\phi}^*(\mathrm{k,s})}{\mathrm{s}^\alpha + \mathrm{a\, s}^\beta + \eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})}\,. \tag{10}$$

If we take the inverse Laplace transform of (10) and apply the formula [37], it is seen that

$$\mathrm{N}^*(\mathrm{k,t}) \;=\; \mathrm{f}^*(\mathrm{k}) \sum_{\mathrm{r}=0}^{\infty} (-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta)\mathrm{r}} \mathrm{E}^{\mathrm{r}+1}_{\alpha,(\alpha-\beta)\mathrm{r}+1}[-\{\eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})\}\; \mathrm{t}^\alpha]$$
$$+ \mathrm{g}^*(\mathrm{k})\, \mathrm{t} \sum_{\mathrm{r}=0}^{\infty} (-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta)\mathrm{r}}\, \mathrm{E}^{\mathrm{r}+1}_{\alpha,\alpha+(\alpha-\beta)\mathrm{r}-\alpha+2}[-\{\eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})\}\mathrm{t}^\alpha]$$
$$+ \mathrm{a\, f}^*(\mathrm{k})\, \mathrm{t}^{\alpha-\beta} \sum_{\mathrm{r}=0}^{\infty} (-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta)\mathrm{r}}\, \mathrm{E}^{\mathrm{r}+1}_{\alpha,\alpha+(\alpha-\beta)\mathrm{r}-\beta+1}[-\{\eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})\}\; \mathrm{t}^\alpha]$$
$$+ \mathrm{a\, g}^*(\mathrm{k})\, \mathrm{t}^{\alpha-\beta+1} \sum_{\mathrm{r}=0}^{\infty} (-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta+1)\mathrm{r}}\, \mathrm{E}^{\mathrm{r}+1}_{\alpha,\alpha+(\alpha-\beta+1)\mathrm{r}-\beta+2}\,[-\{\eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})\}\mathrm{t}^\alpha]$$
$$+ \int_0^{\mathrm{t}} \phi^*(\mathrm{k, t-u})\, \mathrm{u}^{\alpha-1} \sum_{\mathrm{r}=0}^{\infty} (-\mathrm{a})^{\mathrm{r}}\, \mathrm{u}^{(\alpha-\beta)\mathrm{r}}\, \mathrm{E}^{\mathrm{r}+1}_{\alpha,\alpha+(\alpha-\beta)\mathrm{r}}[-\{\eta\, \psi_\gamma^\theta(\mathrm{k}) + \xi\, \psi_\lambda^\sigma(\mathrm{k})\}\mathrm{u}^\alpha]\mathrm{du}. \tag{11}$$

Finally, the required solution (7) is obtained by taking inverse Fourier transform of (11).

3 Special Cases

If we set $\xi = 0$ in Theorem 1, then our result reduces in the following interesting result given in the form of corollary.

Corollary 1. Consider the fractional reaction-diffusion equation

$${}_0\mathrm{D}_{\mathrm{t}}^\alpha \mathrm{N}(\mathrm{x,t}) \;+\; \mathrm{a}\; {}_0\mathrm{D}_{\mathrm{t}}^\beta\, \mathrm{N}(\mathrm{x,t}) \;=\; \eta\; {}_{\mathrm{x}}\mathrm{D}_\theta^\gamma\, \mathrm{N}(\mathrm{x,t}) \;+\; \phi(\mathrm{x,t}), \tag{12}$$

where a,η, $\mathrm{t} > 0$, $\mathrm{x} \in \mathrm{R}$; $\gamma, \theta, \alpha, \beta$are real parameters with the conditions $0 < \gamma \le 2$, $|\,\theta\,| \le \min(\gamma, 2-\gamma)$, $0 < \alpha \le 2$, $0 < \beta \le 2$ and the initial conditions

$$\mathrm{N}\,(\mathrm{x},0) \;=\; \mathrm{f(x)}\,,\; \mathrm{N_t}(\mathrm{x},0) \;=\; \mathrm{g(x)}\,;\;\; \text{for } \mathrm{x} \in \mathrm{R}, \lim_{|\,\mathrm{x}\,| \to \pm\infty} \mathrm{N}\,(\mathrm{x,t}) \;=\; 0,\; \mathrm{t} \;>\; 0, \tag{13}$$

η is a diffusion constant and ϕ(x,t) is a non linear function belonging to the area of reaction-diffusion. Then for the solution of (12) subject to the above initial conditions, there holds the formula

$$\begin{aligned}
\mathrm{N}\,(\mathrm{x},\mathrm{t}) \;=\; & \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{f}^*(\mathrm{k}) \sum_{\mathrm{r}=0}^{\infty}(-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta)\mathrm{r}}\, \mathrm{E}^{\mathrm{r}+1}_{\alpha,(\alpha-\beta)\mathrm{r}+1}[-\eta\,\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{t}^{\alpha}]\exp(-\mathrm{ikx})\mathrm{dk} \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{g}^*(\mathrm{k})\, \mathrm{t}\sum_{\mathrm{r}=0}^{\infty}(-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta)\mathrm{r}}\, \mathrm{E}^{\mathrm{r}+1}_{\alpha,(\alpha-\beta)\mathrm{r}+2}[-\eta\,\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{t}^{\alpha}]\exp(-\mathrm{ikx})\mathrm{dk} \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{a}\,\mathrm{f}^*(\mathrm{k})\, \mathrm{t}^{\alpha-\beta}\sum_{\mathrm{r}=0}^{\infty}(-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta)\mathrm{r}}\mathrm{E}^{\mathrm{r}+1}_{\alpha,(\alpha-\beta)\mathrm{r}-\beta+1}\,[-\eta\,\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{t}^{\alpha}]\exp(-\mathrm{ikx})\mathrm{dk} \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{a}\mathrm{g}^*(\mathrm{k})\, \mathrm{t}^{\alpha-\beta+1}\sum_{\mathrm{r}=0}^{\infty}(-\mathrm{a})^{\mathrm{r}}\, \mathrm{t}^{(\alpha-\beta+1)\mathrm{r}}\, \mathrm{E}^{\mathrm{r}+1}_{\alpha,(\alpha-\beta+1)\mathrm{r}-\beta+2}[-\eta\,\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{t}^{\alpha}]\exp(-\mathrm{ikx})\mathrm{dk} \\
& + \frac{1}{2\pi}\int_{0}^{\mathrm{t}} \mathrm{u}^{\alpha-1}\sum_{\mathrm{r}=0}^{\infty}(-\mathrm{a})^{\mathrm{r}}\, u^{(\alpha-\beta)\mathrm{r}}\, du\int_{-\infty}^{\infty}\phi^*(\mathrm{k},\mathrm{t}-\mathrm{u})\mathrm{E}^{\mathrm{r}+1}_{\alpha,(\alpha-\beta)\mathrm{r}}[-\eta\,\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{u}^{\alpha}]\exp(-\mathrm{ikx})\mathrm{dk}.
\end{aligned} \tag{14}$$

If we take $\mathrm{a} = 0 = \xi$, then we obtain the following result which is the same result as given by Haubold et al. [21].

Corollary 2. Consider the fractional reaction-diffusion model

$${}_0\mathrm{D}^{\alpha}_{\mathrm{t}}\mathrm{N}(\mathrm{x},\mathrm{t}) \;=\; \eta\,{}_{\mathrm{x}}\mathrm{D}^{\gamma}_{\theta}\,\mathrm{N}(\mathrm{x},\mathrm{t}) \;+\; \phi(\mathrm{x},\mathrm{t}), \tag{15}$$

where η, $\mathrm{t} > 0$, $\mathrm{x} \in \mathrm{R}$; γ, θ, α, are real parameters with the constraints $0 < \gamma \le 2$, $|\,\theta\,| \le \min\,(\gamma, 2-\gamma)$, $0 < \alpha \le 2$, and the initial conditions

$$\mathrm{N}\,(\mathrm{x},0) \;=\; \mathrm{f}(\mathrm{x})\;,\;\; \mathrm{N}_{\mathrm{t}}(\mathrm{x},0) \;=\; \mathrm{g}(\mathrm{x})\;;\;\; \text{for } \mathrm{x} \in \mathrm{R},\; \lim_{|\,\mathrm{x}\,|\to\pm\infty}\mathrm{N}\,(\mathrm{x},\mathrm{t}) \;=\; 0,\; \mathrm{t} > 0, \tag{16}$$

η is a diffusion constant and ϕ(x,t) is a non linear function belonging to the area of reaction-diffusion. Then the solution of (15), subject to the above initial conditions, is given by

$$\begin{aligned}
\mathrm{N}\,(\mathrm{x},\mathrm{t}) \;=\; & \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{f}^*(\mathrm{k})\mathrm{E}_{\alpha,1}[-\eta\;\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{t}^{\alpha}]\,\exp(-\mathrm{ikx})\mathrm{dk} \\
& + \frac{1}{2\pi}\int_{-\infty}^{\infty}\mathrm{t}\;\mathrm{g}^*(\mathrm{k})\mathrm{E}_{\alpha,2}[-\eta\;\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{t}^{\alpha}]\;\exp(-\mathrm{ikx})\mathrm{dk} \\
& + \frac{1}{2\pi}\int_{0}^{\mathrm{t}}\mathrm{u}^{\alpha-1}\mathrm{du}\int_{-\infty}^{\infty}\phi^*(\mathrm{k},\mathrm{t}-\mathrm{u})\;\mathrm{E}_{\alpha,\alpha}[-\eta\;\psi^{\theta}_{\gamma}(\mathrm{k})\mathrm{u}^{\alpha}]\,\exp\,(-\mathrm{ikx})\;\mathrm{dk}.
\end{aligned} \tag{17}$$

Further, if we set $\mathrm{a} = 0 = \xi$ with $\mathrm{g}(\mathrm{x}) = 0$, then by the application of the convolution theorem of the Fourier transform to the solution (7) of the theorem, it readily yields

Corollary 3. Consider the fractional reaction-diffusion model

$$\frac{\partial^\alpha}{\partial t^\alpha} N(x,t) - \eta \frac{\partial^\gamma}{\partial x^\gamma} N(x,t) = \phi(x,t), \; x \in R, \; t > 0, \eta > 0, \tag{18}$$

with initial conditions

$$N(x,0) = f(x), \; N_t(x,0) = 0 \text{ for } x \in R, \; 1 < \alpha \leq 2, \lim_{x \to \pm\infty} N(x,t) = 0, \tag{19}$$

where η is a diffusion constant and ϕ(x,t) is a non linear function belonging to the area of reaction-diffusion. Then for the solution of (18) with initial conditions, there holds the formula

$$N(x,t) = \int_0^x G_1(x-\tau,t)\, f(\tau) d\tau + \int_0^t (t-u)^{\alpha-1} du \int_0^x G_2(x-\tau, t-u)\, \phi(\tau,u)\, d\tau, \tag{20}$$

where

$$\rho = \frac{\gamma - \theta}{2\gamma},$$

$$G_1(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)\, E_{\alpha,1}(-\eta\, t^\alpha\, \psi_\gamma^\theta(k))\, dk$$
$$= \frac{1}{\gamma|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\gamma}\, t^{\alpha/\gamma}}\Big|_{(1,1/\gamma),\,(1,1),\,(1,\rho)}^{(1,1/\gamma),\,(\alpha.\alpha/\gamma),\,(1,\rho)}\right], \; (\alpha > 0) \tag{21}$$

and

$$G_2(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp(-ikx)\, E_{\alpha,\alpha}(-\eta\, t^\alpha\, \psi_\gamma^\theta(k))\, dk$$
$$= \frac{1}{\gamma|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\gamma}\, t^{\alpha/\gamma}}\Big|_{(1,1/\gamma),\,(1,1),\,(1,\rho)}^{(1,1/\gamma),\,(\alpha.\alpha/\gamma),\,(1,\rho)}\right], \; (\alpha > 0). \tag{22}$$

In deriving the above results, we have used the inverse Fourier transform formula

$$F^{-1}\left[E_{\alpha,\beta}(-\eta\, t^\alpha\, \psi_\theta^\gamma(k))\,;\, x\right] = \frac{1}{\gamma|x|} H_{3,3}^{2,1}\left[\frac{|x|}{\eta^{1/\gamma}\, t^{\alpha/\gamma}}\Big|_{(1,1/\gamma),\,(1,1),\,(1,\rho)}^{(1,1/\gamma),\,(\beta,\alpha/\gamma),\,(1,\rho)}\right], \tag{23}$$

where $\mathrm{Re}(\alpha) > 0$, $\mathrm{Re}(\beta) > 0$, which can be established by following a procedure similar to that employed by Mainardi et al. [23]. Next, if we set $f(x) = \delta(x)$, $\phi(x,t) = 0$, $g(x) = 0$, $a = 0$, $\xi = 0$ where $\delta(x)$ is the Dirac delta-function, then we arrive at the following interesting result obtained by Mainardi et al. [24].

Corollary 4. The solution of the fractional reaction-diffusion equation

$$\frac{\partial^\alpha N(x,t)}{\partial t^\alpha} = \eta\, {}_xD_\theta^\gamma N(x,t), \; \eta > 0, \; x \in R, \; 0 < \alpha \leq 2, \tag{24}$$

with the initial conditions

$$\mathrm{N}\,(\mathrm{x},0) \;=\; \delta(\mathrm{x}),\; \mathrm{N}_{\mathrm{t}}(\mathrm{x},0) \;=\; 0,\; \lim_{\mathrm{x}\,\to\,\pm\infty} \mathrm{N}\,(\mathrm{x},\mathrm{t}) \;=\; 0, \tag{25}$$

where η is a diffusion constant and $\delta(\mathrm{x})$ is the Dirac delta function, is given by

$$\mathrm{N}\,(\mathrm{x},\mathrm{t}) \;=\; \frac{1}{\gamma\,|\,\mathrm{x}\,|}\,\mathrm{H}^{2,1}_{3,3}\left[\frac{|\,\mathrm{x}\,|}{(\eta\,\mathrm{t}^{\alpha})^{1/\gamma}}\;\Big|^{(1,1/\gamma),\,(1,\alpha/\gamma),\,(1,\rho)}_{(1,1/\gamma),\,(1,1),\,(1,\rho)}\right], \tag{26}$$

where $\rho \;=\; \frac{\gamma-\theta}{2\gamma}$.
Some interesting special cases of (24) are given below
(i) We note that for $\gamma = \alpha$, Mainardi et al. [24] have shown that the corresponding solution of (24), denoted by $\mathrm{N}^{\theta}_{\gamma}$, which we call as the neutral fractional diffusion, can be expressed in terms of elementary function and can be defined for $\mathrm{x} > 0$ as
Neutral fractional diffusion: $0 < \gamma = \alpha < 2$; $\theta \le \min\,\{\gamma,\, 2-\gamma\}$,

$$\mathrm{N}^{\theta}_{\gamma}(\mathrm{x}) \;=\; \frac{1}{\pi}\,\frac{\mathrm{x}^{\gamma-1}\sin\,[(\pi/2)(\gamma-\theta)]}{1+\,2\mathrm{x}^{\gamma}\cos\,[(\pi/2)(\gamma-\theta)]+\,\mathrm{x}^{2\gamma}}. \tag{27}$$

Next, we obtain some stable densities in terms of the H-functions as special cases of the solution of the equation (24).
(ii) When $\alpha \;=\; 1,\; 0 \;<\; \gamma \;<\; 2\,;\theta \;\le\; \min\,\{\gamma, 2-\gamma\}$, then equation (24) reduces to space fractional diffusion equation, which we denote by $\mathrm{L}^{\theta}_{\gamma}(\mathrm{x})$ is the fundamental solution of the following space-time fractional diffusion equation

$$\frac{\partial\,\mathrm{N}(\mathrm{x},\mathrm{t})}{\partial\,\mathrm{t}} \;=\; \eta\,{}_{\mathrm{x}}\mathrm{D}^{\gamma}_{\theta}\,\mathrm{N}(\mathrm{x},\mathrm{t}),\;\; \eta \;>\; 0,\;\; \mathrm{x} \;\in\; \mathrm{R}, \tag{28}$$

with the initial conditions $\mathrm{N}\,(\mathrm{x},0) \;=\; \delta(\mathrm{x})\,,\; \lim_{\mathrm{x}\,\to\,\pm\infty} \mathrm{N}(\mathrm{x},\mathrm{t}) \;=\; 0$, where $\eta$is a diffusion constant and $\delta(\mathrm{x})$ is the Dirac-delta function. Then the solution of equation (28) is given by

$$\mathrm{L}^{\theta}_{\gamma}(\mathrm{x}) \;=\; \frac{1}{\gamma(\eta\mathrm{t})^{1/\gamma}}\,\mathrm{H}^{1,1}_{2,2}\left[\frac{(\eta\mathrm{t})^{1/\gamma}}{|\,\mathrm{x}\,|}\;\Big|^{(1,1),(\rho,\rho)}_{(1/\gamma,1/\gamma),\,(\rho,\rho)}\right],\; 0 \;<\; \gamma \;<\; 1,\; |\,\theta\,| \;<\; \gamma, \tag{29}$$

where $\rho \;=\; \frac{\gamma-\theta}{2\gamma}$.
The density represented by the above expression is known as γ-stable Lévy density.
Finally, if we take $\alpha = 1/2$, $\mathrm{a} = 0$, $\xi = 0$ in equation (5), then we get the following result which is the same result as derived by Haubold et al. [21].
Corollary 5. Consider the following fractional reaction-diffusion model

$$\mathrm{D}^{1/2}_{\mathrm{t}}\,\mathrm{N}(\mathrm{x},\mathrm{t}) \;=\; \eta\,{}_{\mathrm{x}}\mathrm{D}^{\gamma}_{\theta}\,\mathrm{N}(\mathrm{x},\mathrm{t}) \;+\; \phi\,(\mathrm{x},\mathrm{t}), \tag{30}$$

where η, t > 0,x $\in$ R ; γ, θ are real parameters with the constraints $0 < \gamma \leq 2$, $|\theta| \leq \min(\gamma, 2-\gamma)$, and the initial conditions

$$\mathrm{N}(\mathrm{x}, 0) = \mathrm{f}(\mathrm{x}), \text{ for } \mathrm{x} \in \mathrm{R}, \lim_{\mathrm{x} \to \pm\infty} \mathrm{N}(\mathrm{x}, \mathrm{t}) = 0. \tag{31}$$

Here η is diffusion constant and ϕ(x,t) is a non linear function belonging to the area of reaction-diffusion. Then for the solution of (30), subject to the above initial conditions, there holds the formula

$$\mathrm{N}(\mathrm{x}, \mathrm{t}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \mathrm{f}^*(\mathrm{k})\, \mathrm{E}_{1/2,1}(-\eta\, \mathrm{t}^{\alpha}\, \psi_{\gamma}^{\theta}(\mathrm{k})) \exp(-\mathrm{ikx})\, \mathrm{dk}$$

$$+\frac{1}{2\eta} \int_0^{\mathrm{t}} \mathrm{u}^{-1/2} \mathrm{du} \int_{-\infty}^{\infty} \phi^*(\mathrm{k}, \mathrm{t}-\mathrm{u})\, \mathrm{E}_{\frac{1}{2},\frac{1}{2}}(-\eta\, \mathrm{t}^{1/2}\, \psi_{\gamma}^{\theta}(\mathrm{k})) \exp(-\mathrm{ikx})\, \mathrm{dk}. \tag{32}$$

If we take $\theta = 0$ in (32), then it reduces to the result obtained by Saxena et al. [35] for the fractional reaction-diffusion equation.

4 Conclusions and Discussions

Reaction-diffusion models have found numerous applications in pattern formation in biology, chemistry, and physics. These systems show that diffusion can produce the spontaneous formation of spatio-temporal patterns. In this paper, we have presented a solution of a unified fractional reaction-diffusion equation. The solution has been developed in terms of the generalized Mittag-Leffler and H-functions in a compact and elegant form with the help of Laplace and Fourier transform and their inverses. Most of the results obtained are in a form suitable for numerical computation. The importance of the derived results lies in the fact that numerous results on fractional reaction, fractional diffusion, anomalous diffusion problems, and fractional telegraph equations scattered in the literature can be derived, as special cases, of the results investigated in this article. Finally, we remark that the results derived are of a more general nature and than those investigated earlier by many authors, notably by Mainardi et al. [23-24], Saxena et al. [35-37], and Haubold et al. [21].

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