Applied Mathematical Sciences, Vol. 5, 2011, no. 58, 2891 - 2898

A Note on Fixed Points

K. M. Sruthi and K. T. Ravindran

Department of Mathematics Payyanur College, Payyanur, India
sruthishonima@gmail.com
drravindran@gmail.com

Abstract

In this note we will discuss the existence of fixedpoints and Approximate fixedpoints of maps such as continuous and compact maps, condensing maps and 1-set contractive maps also we will discuss about the existence of common fixed point of both single valued maps and multivalued maps.

Mathematical Subject Classification: 47H09, 47H10

Keywords: Nonexpansive maps, demiclosed maps, condensing maps, 1-setcontractive maps, strongly continuous map, interior condition, demicompact map, uniformlyconvex Banach Space, fixed points, approximate fixedpoint, reflexive Banach space, multimap, Hausdorff metric

1 Introduction

Here we deal extensively with the existence of fixed points of sum of two maps,which satisfies Certain conditions.We are mostly concerned there with the existence of approximate fixed points of continuous maps.We emphasis the main role of interior condition,demicompactness and demiclosedness of maps in fixed point theory.Finally we will discuss about the existence of Common fixed point of both single valued maps and multi valued maps.Inorder to discuss the above first we recall some definitions and results as follows.

2 Preliminaries

Definition 2.1 [3] *Let Q be a closed subset of a Banach space E. A mapping $F : Q \to E$is said to be demiclosed if for any sequence x_n in Q we have $x_n \rightharpoonup x$ and $F(x_n) \to y \Longrightarrow F(x) = y$*

Definition 2.2 [7] *Let E be a Banach space. Q, bounded , open subset of E. A mapping $G : \overline{Q} \to E$ satisfies the interior condition ,if there exists $\delta > 0$ such that $G(x) \neq x$ for $x \in Q^*$ $\lambda > 1$ and $G(x) \notin \overline{Q}$. Where $Q^* = \{x \in Q : dist(x, \delta(Q)) < \delta\}$*

Result 2.3 [3] *It is known that for every single valued nonexpansive mapping $F : Q \longrightarrow E$,$I - F$is demiclosed if the underlying Banach space E is uniformly convex.*

Definition 2.4 [5] *Let Q be a non empty subset of a Banach space E .G maps Q into E,G is condensing if G is bounded and continuous and $\alpha(G(U)) < \alpha(U)$ for all bounded subsets of Q with $\alpha(U) > 0$ where α is the measure of noncompactness.*

Definition 2.5 [8] *Let E be a Banach space and Ω_E be the bounded subsets of E. The Kuratowskii measure of noncompactness is the map $\alpha : \Omega_E \to [0, 1)$ defined by $\alpha(Q)$=inf $\{\epsilon > 0 : Q \subseteq \bigcup_{i=1}^{n}(Q_i)$ and diam $Q_i \leqslant \epsilon$ for i=1,2,3...$\}$*
Let $A, B \in \Omega_E$
a) $\alpha(A) = 0$ if and only if $\overline{A}$ is compact
b) $\alpha(\lambda(A)) = |\lambda|\alpha(A)$
c) $\alpha(A + B) \leqslant \alpha(A) + \alpha(B)$

Definition 2.6 [8] *Let Q be a non empty subset of a Banach space E and $G : Q \to E$. G is k-set contractive $k \geqslant 0$ if $\alpha(G(U)) \leqslant k\alpha(U)$ for all bounded subsets U of Q.*

Theorem 2.7 [7] *Every compact map is 0-set contractive, and every condensing map is 1-set contractive.*

Theorem 2.8 [8] *Let E be a Banach space.Q closed, convex subset of E, U an open subset of Q and $p \in U$. Suppose $F : \overline{U} \to Q$ is a continuous , k-set contractive map $0 \leqslant k < 1$, with $F(\overline{U})$ is bounded set in Q. Then either F has a fixedpoint in $\overline{U}$ or there exits $u \in \delta(u)$ and $\lambda \in (0, 1)$ with $u = \lambda F(U)+(1-\lambda)p$.*

Theorem 2.9 [8] *(Schauder's theorem) : Let Q be a closed, convex subset of a normed linear space E.Then every compact continuous map $F : Q \to Q$ has at least one fixed point.*

Definition 2.10 [4] *Let E be a normed linear space and let $Q \subseteq E$. A map $H : Q \to E$ is said to be demicompact if for any bounded sequence (x_n) in Q, $x_n - Hx_n \to z$ as n tends to ∞. Then there exist a subsequence x_{ni} and a point $p \in Q$ such that $x_{ni} \to p$ as $i \to \infty$ and $(I - H)p = z$.*

3 Main Results

Theorem 3.1 *Let E be a Uniformly convex Banach space and Q bounded, open, convex subset of E with $0 \in Q$. Let $F : \overline{Q} \to E$ is strongly continuous and $G : \overline{Q} \to E$ is nonexpansive. If $H = F + G : \overline{Q} \to \overline{Q}$ Satisfies the interior condition then H has a fixedpoint in $\overline{Q}$.*

Proof: Since $F : \overline{Q} \to E$ is strongly continuous , it is continuous and compact, and $G : \overline{Q} \to E$ is nonexpansive. Clearly $H = F+G$ is a 1-set contractive map [1 corollary 1(b)] satisfies the interior condition. Then there exist a sequence $\{x_{nj}\}$ in $\overline{Q}$ such that $x_{nj} - Hx_{nj} \to 0$ [1corollary1(b)].Since $\overline{Q}$ is a bounded, closed subset of a reflexive space E, every bounded sequence x_{nj} has a weakly convergent subsequence x_{nj_p} and $x_{nj_p} \rightharpoonup x$ in $\overline{Q}$. By strong continuity of F, $F(x_{nj_p}) \to F(x)$. $x_{nj_p} - H(x_{nj_p})$ is a subsequence of $x_{nj} - H(x_{nj})$, hence we get $x_{nj_p} - H(x_{nj_p}) \to 0$. Therefore $x_{nj_p} - H(x_{nj_p}) + F(x_{nj_p}) \to F(x)$ it follows that $x_{nj_p} - G(x_{nj_p}) \to F(x)$.But I-G is demiclosed [2.3] . Hence $(I - G)x = F(x) \Longrightarrow H(x) = x$.

Theorem 3.2 *Let E be a reflexive Banach space, Q closed subset of E and U an open subset of E with $p \in U$. Suppose $F : \overline{U} \to Q$ is strongly continuous, $G : \overline{U} \to Q$ condensing,$G(\overline{U})$ is bounded set in Q and I-G is demiclosed. Then either $F + G$ has a fixedpoint in $\overline{U}$ or There exists $u \in \delta(U)$ and $\lambda_i \in (0, 1)$ $i = 1, 2$ with $u = \lambda_1 F(u) + \lambda_2 G(u) + [1 - (\lambda_1 + \lambda_2)]p$.*

Proof: Suppose $u \neq \lambda_1 F(u) + \lambda_2 G(u) + [1 - (\lambda_1 + \lambda_2)]p$. For n = 2 ,define $F_n^1 : \overline{U} \to Q$ as $F_n^1 = (1 - \frac{1}{n})F$, $F_n^{11} = (1 - \frac{1}{n})G$ Since F is continuous and compact , F_n^1 is continuous and compact and F_n^{11} is condensing.
Consider

$$
\begin{aligned}
[\alpha(F_n^1+F_n^{11})(\overline{U})] &= \alpha[(1-\frac{1}{n})F(\overline{U})+(1-\frac{1}{n})G(\overline{U})] \\
&= \alpha[(1-\frac{1}{n})(F(\overline{U}))+G(\overline{U})] \\
&= |(1-\frac{1}{n})|\ \alpha\{F(\overline{U})+G(\overline{U})\} \\
&\leq |(1-\frac{1}{n})|\{\alpha(F(\overline{U}))+\alpha(G(\overline{U}))\} \\
&= |(1-\frac{1}{n}|\alpha(G(\overline{U})) \\
&\leqslant (1-\frac{1}{n})\alpha(\overline{U})
\end{aligned}
$$

It follows that ${F_n}^1+{F_n}^{11}$ is a continuous k-set contractive map with $k=1-\dfrac{1}{n}$
Then by [2.7] there exists $u\in\overline{U}$ and $\lambda\in(0,1)$ with
$u{=}\lambda\{{F_n}^1{+}{F_n}^{11}\}(u){+}(1{-}\lambda)p$ A contradiction ,hence ${F_n}^1+{F_n}^{11}$ has a fixedpoint in $\overline{U}$ say u_n. Clearly $\{u_n\}$ is a bounded sequence in E then $\{u_n\}$ has a weakly convergent subsequence $\{u_{nk}\}$ and $u_{nk}\rightharpoonup u$. By strong continuity of F , $F(u_{nk})\rightharpoonup F(u)$ Consider

$$
\begin{aligned}
\|(I-G(u_n)-F(u)\| &= \|u_n-G(u_n)-F(u)\| \\
&= \|({F_n}^1+{F_n}^{11})(u_n)-G(u_n)-F(u)\| \\
&= \|(1-\frac{1}{n})\{F+G\}(u_n)-G(u_n)-F(u)\| \\
&\leqslant \|F(u_n)-F(u)\|+\frac{1}{n}\{\|F(u_n)\|+\|G(u_n)\|\}
\end{aligned}
$$

converges to 0 as $n\to\infty$. Hence $(I-G)(u_n)\to F(u)$.But I-G is demiclosed which implies $F(u)+G(u)=u$.

corollary 3.3 *In the above theorem take $G=0$ and by Remark of 2.6 we get , Let E be a reflexive Banach space , Q closed, convex subset of E.U an open subset of E with $p\in U$.Suppose $F:\overline{U}\to Q$ is a continuous k-set contractive map, $k=o$. Then either a) F has a fixedpoint in $\overline{U}$ or b) There exists $u\in\delta(U)$and $\lambda\in(0,1)$ with $u=\lambda F(u)+(1-\lambda)p$.*

Remark

By the above , Theorem (3.2) is a generalization of Theorem (2.7).

Definition 3.4 [9]*Let K be a nonempty, compact, convex subset of R^n and **A** denote the set of all continuous maps $F:K\to K$.For $F,G\in$ **A***
$d(F,G)=Sup\{\|F(x)-G(x)\|:x\in K\}$.

Theorem 3.5 *Let K be a non empty, compact, convex subset of R^n and $F \in$ $\boldsymbol{A}$.If F has an ϵ - approximate fixed point x^* in K then there exists a continuous map G in the ϵ - neighbourhood of F such that it has an approximate fixed point in some ϵ-neighbourhood of $F(x^*)$.*

Proof: Since F is continuous , given $\epsilon > 0$ there exists $\delta > 0$ such that $\|F(z) - F(x^*\| \leqslant \frac{\epsilon}{4}$ whenever $\|z - x^*\| \leqslant \delta$ (A)
By Urysohn's theorem , there exists a continuous map $\lambda : X \rightarrow [0, 1]$Such that $\lambda(z) = 1$ for each $z \in R^n$ satisfying $\|z - F(x^*) \leqslant \delta\|$ (B) And for $\lambda(z) = 0$ for each $z \in R^n$ satisfying $\|z - F(x^*)\| \geqslant 2\delta$ (C)
Define $G(z) = \lambda(z)F(x^*) + (1 - \lambda(z))F(z)$ for all $z \in K$.
G is continuous and $G(x^*) = F(x^*)$ (D)
For each $z \in K$ satisfying $\|z - F(x^{*)}\| \leqslant \delta$ $G(z) = F(x^*)$ [by B,D] For each $z \in K$ satisfying $\|z - F(x^*)\| \geqslant 2\delta$ we have $G(z) = F(z)$ [by C,D] and $\|G(z) - z\| = \|G(z) - F(x^*) + F(x^*) - z\| \leqslant \|G(z) - F(x^*)\| + \|F(x^*) - z\| \leqslant \delta$
Consider,$\|G(z) - F(z)\| = \|\lambda(z)F(x^*) + (1 - \lambda(z))F(z) - F(z)\|$
$= \|F(x^*) - F(z)\| \leqslant \epsilon$ whenever $\|z - F(x^*)\| \leqslant \delta$. Hence we get, $d(F, G) \leqslant \epsilon$.

Theorem 3.6 [10] *Let $K \subseteq R^n$ be a nonempty, bounded, convex subset of R^n and let $F : K \rightarrow K$ be a continuous function. Then F has an approximate fixedpoint property.*

Theorem 3.7 *Let $F \in \boldsymbol{A}$, $\epsilon > 0$ and $x^* \in K$ be an approximate fixed point of F. where K is a nonempty, compact, convex subset of R^n and let $G \in A$, $\delta > 0$ as guaranteed by Theorem 3.6 .Then for each continuous map H in the δ- neighbourhood of G has an approximate fixedpoint in the neighbourhood of $F(x^*)$.*

Proof: By 3.6 $d(F, G) \leqslant \epsilon$ and $G(z) = F(x^*)$, for each $Z \in K$ satisfying $\|z - F(x^*\| \leqslant \delta$. suppose $d(H, G) \leqslant \delta$
Set $\Omega = \{z \in K : \|z - F(x^*)\| \leqslant d(F, G)\}$
Clearly Ω is non empty, compact, convex subset of K. Consider
$\|H(z) - F(x^*\| = \|H(z) - G(z) + G(z) - F(x^*\| \leqslant \|H(z) - G(z)\| + \|G(z) - F(x^*\| \leqslant \delta$. Hence $H(z) \in \Omega$ for all $z \in \Omega$

Therefore $H(\Omega) \subseteq \Omega$ By Theorem 3.7 H has an approximate fixedpoint in Ω let it be y.

That is $\|H(y) - y\| \leqslant \epsilon$ clearly $\|y - F(x^*\| \leqslant d(F, G)$.

Theorem 3.8 *Let E be a Reflexive Banach space. Q non empty, bounded, closed, convex subset of E.*$F : Q \to Q$ *strongly continuous and* $G : Q \to E$ *is condensing satisfying the interior condition. Then F and G have common fixedpoint in Q if G satisfies the following condition. a) For any sequence* $\{x_n\}$, $x_n - G(x_n) \to 0$ *as* $n \to \infty$ *and (I-G) is demiclosed.*

Proof: $\{x_n\}$ be any sequence in Q and by assumption $x_n - G(x_n) \to 0$ as $n \to \infty$. since every condensing map is demicompact [2] $\{x_n\}$ has a convergent subsequence $\{x_{nk}\}$ and $x_{nk} \to x$ in Q. Define $F_{nk}^1 : Q \to Q$ as ${F_{nk}}^1 = (1 - \frac{1}{n})F$clearly ${F_{nk}}^1$ is continuous and compact. By [2.8] ${F_{nk}}^1$ has a fixedpoint in Q say x_n . That is $({F_{nk}}^1)(x_{nk}) = x_{nk}$. By strong continuity of F, $F(x_{nk}) \to F(x)$.
Consider $\|x_{nk} - F(x)\| = \|{F_{nk}}^1(x_{nk}) - F(x)\|$

$$=\|(1 - \frac{1}{n}F(x_{nk}) - F(x)\|$$
$$= \|F(x_{nk}) - F(x) - \frac{1}{n}F(x_{nk})\|$$
$$\leqslant \|F(x_{nk}) - F(x)\| + \frac{1}{n}\|F(x_{nk})\|$$

Converges to 0 as $n \to \infty$.Hence we get F(x)=x, and I-G is demiclosed.
Therefore (I-G)(x)=0 which implies G(x) = x.

Definition 3.9 [11] *Let E be Banach space, Q bounded ,closed,convex subset of E.C(Q) denote the collection of all non empty ,compact subsets of E.For any* $A, B \in C(Q)$ *the Hausdorff metric is defined as*
$H(A, B) = max\{supd(x, B)_{x\in A}, supd(A, y)_{y\in B}\}$

Theorem 3.10 [11] *Let E be a uniformlyconvex Banach space, Q be a bounded , closed, convex subset of E and let* $F : Q \to C(Q)$ *nonexpansive relative to the Hausdorff metric on C(Q).Then there exists a point* $x \in Q$ *such that* $x \in T(x)$.

Theorem 3.11 *Let E be a uniformlyconvex, compact Banach space, Q a closed, bounded, convex subset of E.*$F : Q \to C(Q)$ *be a Continuous multimap and* $G : Q \to C(Q)$ *nononexpansive multimap with (I-G) is demiclosed. Then F and G have common fixed point in Q if F satisfies the following*
** If for any sequence* $\{x_n\}$ *in Q* $d(x_n, F(x_n))$ *converges to 0 as* $n \to \infty$.

Proof: $\{x_n\}$ be any sequence in $Q \subseteq E$.Since E is compact $\{x_n\}$ has a convergent subsequence $\{x_{nk}\}$ and $x_{nk} \to x$. Since $F : Q \to C(Q)$ is continuous $H(F(x_{nk}), x_{nk} \to 0$ [11]

Consider $d(x, F(x)) \leqslant d(x, x_{nk}) + d(x_{nk}, F(x_n)_k) + d(F(x_{nk}), F(x)) \rightarrow 0$as $n \rightarrow \infty$.Which implies $x \in F(x)$. Define $G : Q \rightarrow C(Q)$ by $G_n = \lambda_n G$ where $\lambda_n \in (0, 1)$ and $\lambda_n \rightarrow 1$. Clearly G_n is nonexpansive. By (3.10) G_n has a fixedpoint in Q say x_{nk}.
That is $x_{nk} \in \lambda_n G_n$. Let $y_{nk} \in G(x_{nk})$.Then $x_{nk} = \lambda_n y_{nk}$.
Consider $\|x_{nk} - y_{nk}\| = \|x_{nk} - \frac{1}{\lambda_n} x_{nk}\|$
$=(1 - \frac{1}{\lambda_n})\|x_{nk}\| \rightarrow 0$. But $(I - G)$ is demi closed which implies $x \in G(x)$.

Acknowledgement: The authors thanks University Grant Commission (UGC) for giving financial support and this work was partially supported by UGC major research project.

REFERENCE

[1] A. J. Melado and C.H.Morales, Fixed point theorems under the interior-condition, Proc. Amer. Math. Soc. 134, (No,2) 501-504 (2005).

[2] Anoop.s.k.,K.T.Ravindran On Approximate Fixed point property and Fixed Points. International Mathematical Forum, Vol.6,2011 ,no.6,281-284.

[3] B.Sims, H. K. XU And X. Z.Yuan,The Homotopic Invariance for Fixed Points of Set-valued nonexpansive mappings.Josai Mathematical monographs1:55-65(1999)

[4] Chika Moore, Iterative Approximation of fixed points of demicontractive Mappings MIRAMARE-TRIESTE,Nov.1998.

[5] E.Zeidler,Nonlinear Functional Analysis and its Applications. Vol 1:.Fixed point Theorems,Springer,New York,1986,p.499

[6]Madalina Barinde, Approximate Fixed point Theorems , Studia Univ.?Babes-Bolvaii?,Mathematica,Vol LI,No.1(2006).

[7] Neeta singh,Shaini.p, Fixed point Theorems for Mappings satisfying Interior Condition.Int. journal of Math.Analysis, Vol.2, 2008, no.1,45-54

[8] Ravi P. Agarval, maria Meehan,Donal O'Regan Fixed point Theory and Applications-2001, Cambridge University Press

[9] Simon Reich and Alexander j. zaslavski:A Stability Result in Fixed pointTheory, Fixed point theory, Volume 6,No.1,2005,113 -118.

[10] Stef Tijs , Anna Torre. Rodica Branzei:Aproximate fixed point theorems

[11] William A Kirk and Brailey Sims. Handbook of Metric Fixed point Theory.

Received: March, 2011