Applied Mathematical Sciences, Vol. 5, 2011, no. 60, 2975 - 2988

Weak Convergence Theorem for Finding Common Fixed Points of a Family of Firmly Nonexpansive Mappings and a Nonspreading Mapping in Hilbert Spaces

Somyot Plubtieng [1] **and Sukanya Chornphrom**

Department of Mathematics, Faculty of Science
Naresuan University, Phitsanulok 65000, Thailand

Abstract

In this paper, we introduce an iterative method and prove a weak convergence theorem for finding common fixed points of a family of firmly nonexpansive mappings and a nonspreading mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of generalized mixed equilibrium problem and a common fixed point set nonspreading mappings. Using the result, we improve and unify several results in fixed point problems and equilibrium problems.

Keywords: Equilibrium problem, Fixed point problem, Firmly nonexpansive mapping, Nonspreading mapping

1 Introduction

Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Then a mapping $T : C \longrightarrow C$ is said to be nonexpansive if $\|Tx-Ty\| \leq \|x - y\|$ for all $x, y \in C$. We denote by $F(T)$ the set of fixed points of T. A mapping F is said to be firmly nonexpansive if

$$\|Fx - Fy\|^2 \leq \langle x - y, Fx - Fy\rangle,$$

for all $x, y \in C$; see, for instance, [2, 4, 5, 13, 14]. On the other hand, a mapping $Q : C \longrightarrow C$ is said to be quasi-nonexpansive if $F(Q) \neq \emptyset$ and

$$\|Qx - y\| \leq \|x - y\|,$$

[1] Somyotp@nu.ac.th

for all $x \in C$ and $y \in F(Q)$, where $F(Q)$ is the set of fixed points of Q. If $T : C \longrightarrow C$ is nonexpansive and the set $F(T)$ of fixed points of T is nonempty, then T is quasi-nonexpansive.

Recently, Kohsaka and Takahashi [8] introduced the following nonlinear mapping: Let E be a a Hilbert space and let C be a nonempty closed convex subset of E. Then, a mapping $S : C \longrightarrow C$ is said to be nonspreading if

$$2\|Sx - Sy\|^2 \leq \|Sx - y\|^2 + \|x - Sy\|^2,$$

for all $x, y \in C$. We know in a Hilbert space that every firmly nonexpansive mapping is nonspreading and that if the set of fixed points of a nonspreading mapping is nonempty, the nonspreading mapping is quasi-nonexpansive; see [8]. Let $A : C \longrightarrow H$ be a mapping of C into H is called monotone if

$$\langle Au - Av, u - v\rangle \geq 0 \;\; \forall u, v \in C.$$

A mapping $A : C \longrightarrow H$ is called λ-inverse-strongly monotone if there exists a positive real number λ such that

$$\langle Au - Av, u - v\rangle \geq \lambda\|Ax - Ay\|^2 \;\; \forall u, v \in C.$$

Let $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction. The equilibrium problem for F is to determine its equilibrium points, i.e. the set

$$EP(F) = \{x \in C : F(x, y) \geq 0, \; \forall y \in C\}.$$

Many problems in physics, optimization, and economics require some elements of $EP(F)$, see [2, 3, 9, 15, 16, 17]. Several iterative methods have been proposed to solve the equilibrium problem, see for instance [3, 15, 16, 17]. In 2005, Combettes and Hirstoaga [3] introduced an iterative scheme for finding the best approximation to the initial data when $EP(F)$ is nonempty and proved a strong convergence theorem.

The variational inequality problem is to find $u \in C$ such that

$$\langle Au, v - u\rangle \geq 0$$

for all $v \in C$. The set of solutions of the variational inequality is denoted by $VI(C, A)$. The generalized equilibrium problem for F and A is to find $x \in C$ such that

$$F(x, y) + \langle Ax, y - x\rangle \geq 0 \;\; \text{for all } y \in C. \tag{1.1}$$

Problem (1.1) was introduce by Takahashi and Takahashi [16] and the set of solution of (1.1) is denoted by $GEP(F, A)$. The generalized mixed equilibrium problem for F, ψ and A is to find $x \in C$ such that

$$F(x, y) + \varphi(y) - \varphi(x) + \langle Ax, y - x\rangle \geq 0 \;\; \text{for all } y \in C. \tag{1.2}$$

The set of solution of (1.2) is denoted by $GMEP(F, \varphi, A)$.

On the other hand, Halpern [6] introduced the following iterative scheme for approximating a fixed point of T:

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)Tx_n \tag{1.3}$$

for all $n \in \mathbb{N}$, where $x_1 = x \in C$ and $\{\alpha_n\}$ is a sequence of $[0, 1]$. Recently, Aoyama et al. [1] introduced a Halpern type iterative sequence for finding a common fixed point of a countable family of nonexpansive mappings. Let $x_1 = x \in C$ and

$$x_{n+1} = \alpha_n x + (1 - \alpha_n)T_n x_n \tag{1.4}$$

for all $n \in \mathbb{N}$, where C is a nonempty closed convex subset of a Banach space, $\{\alpha_n\}$ is a sequence in $[0, 1]$ and $\{T_n\}$ is a sequence of nonexpansive mappings of C into itself which satisfies the AKTT-condition, that is,

$$\sum_{n=1}^{\infty} \sup\{\|T_{n+1}z - T_n z\| : z \in C\} < \infty. \tag{1.5}$$

They proved that the sequence $\{x_n\}$ defined by (1.4) converges strongly to a common fixed point of $\{T_n\}$.

In this paper, motivated by Plubtieng and Thammathiwat [12], Iemoto and Takahashi [7], we introduce a new iterative sequence and prove a weak convergence theorem for finding common fixed points of a families of nonexpansive mappings and a nonspreading mapping in Hilbert spaces. Moreover, we apply our result to finding common element of a solution set of generalized mixed equilibrium problem and a common fixed point set nonspreading mappings.

2 Preliminaries

This section collects some lemmas which will be used in the proofs for the main results in the next section. Let H be a real Hilbert space with inner product $\langle ., . \rangle$ and norm $\|.\|$. In a Hilbert space, it is known that

$$\|\alpha x + (1 - \alpha)y\|^2 = \alpha\|x\|^2 + (1 - \alpha)\|y\|^2 - \alpha(1 - \alpha)\|x - y\|^2.$$

By definition of the metric projection P_C we known that P_C is a nonexpansive mapping of H onto C and satisfies

$$\|P_C x - P_C y\|^2 \leq \langle P_C x - P_C y, x - y\rangle, \quad \forall x, y \in H.$$

Further, for any $x \in H$ and $y \in C$, $y = P_C x$ if and only if $\langle x - y, y - z\rangle \geq 0$, $\forall z \in C$.

A space X is said to satisfy Opial's condition [10] if for each sequence $\{x_n\}_{n=1}^{\infty}$ in X which converges weakly to point $x \in X$, we have

$$\liminf_{n\longrightarrow\infty} \|x_n - x\| < \liminf_{n\longrightarrow\infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x$$

and

$$\limsup_{n\longrightarrow\infty} \|x_n - x\| < \limsup_{n\longrightarrow\infty} \|x_n - y\|, \ \forall y \in X, \ y \neq x.$$

Lemma 2.1. [8] *Let H be a Hilbert space, C a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself. Then $F(S)$ is closed and convex.*

In order to prove the main result, we shall use the following lemmas in the sequel.

Lemma 2.2. [7] *Let H be a Hilbert space, C a closed convex subset of H, and $S : C \longrightarrow C$ a nonspreading mapping with $F(S) \neq \emptyset$. Then S is demiclosed, i.e., $x_n \rightharpoonup u$ and $x_n - Sx_n \rightarrow 0$ imply $u \in F(S)$.*

Lemma 2.3. [7] *Let H be a Hilbert space, C a nonempty closed convex subset of a real Hilbert space H and let S be a nonspreading mapping of C into itself and let $A = I - S$. Then*

$$\|Ax - Ay\|^2 \leq \langle x - y, Ax - Ay\rangle + \frac{1}{2}(\|Ax\|^2 + \|Ay\|^2).$$

Lemma 2.4. [11] *Let C be a nonempty bounded closed convex subset of Hilbert space E and $\{T_n\}$ a sequence of mappings of C into itself. Suppose that*

$$\lim_{k,l\longrightarrow\infty} \rho_l^k = 0 \tag{2.1}$$

where $\rho_l^k = \sup\{\|T_k z - T_l z\| : z \in C\} < \infty$, for all $k, l \in \mathbb{N}$. Then for each $x \in C$, $\{T_n x\}$ converges strongly to some point of C. Moreover, let T be a mapping from C in to itself defined by

$$Tx = \lim_{n\longrightarrow\infty} T_n x, \quad for\ all\ x \in C.$$

Then $\lim_{n\longrightarrow\infty} \sup\{\|Tz - T_n z\| : z \in C\} = 0.$

In fact, Aoyama et al. [1] proved Lemma 2.4 in case the sequence $\{T_n\}$ satisfies the AKTT-condition. We note that if a sequence $\{T_n\}$ satisfies the AKTT-condition then $\{T_n\}$ satisfies the condition (2.1).

3 Weak convergence theorem

In this section, we prove a weak convergence theorem for finding common fixed points of a family of nonexpansive mappings and a nonspreading mapping in Hilbert space.

Theorem 3.1. *Let H be a real Hilbert space and let C be a nonempty closed convex subset of H. Let S be a nonspreading mapping of C into itself and let $\{T_n\}$ be the sequences of firmly nonexpansive mappings of C into itself such that $F(S)\cap(\cap_{n=1}^{\infty}F(T_n))$ is nonempty. Let $\{\alpha_n\}\subset[a,b]$ for some $a,b\in(0,1)$. Let $\{x_n\}$ be a sequence defined by $x_0=x\in C$ and*

$$x_{n+1}=\alpha_n x_n+(1-\alpha_n)ST_nx_n, \quad n\geq 0. \tag{3.1}$$

Suppose that $\{T_n\}$ satisfy the AKTT-condition and T be the mappings of C into itself defined by $Ty=\lim_{n\longrightarrow\infty}T_ny$ for all $y\in C$ and suppose that $F(T)=\cap_{n=1}^{\infty}F(T_n)$. Then $\{x_n\}$ converges weakly to $\hat{z}\in F(S)\cap(\cap_{n=1}^{\infty}F(T_n))$.

Proof. Take a point $v\in F(S)\cap(\cap_{n=1}^{\infty}F(T_n))$ and put $y_n=T_nx_n$. We shall show that the sequences $\{x_n\}$ is bounded. First, we note that

$$\begin{aligned}\|Sy_n-v\| &\leq \|y_n-v\|\\ &= \|T_nx_n-v\|\\ &\leq \|x_n-v\|,\end{aligned}$$

we obtain,

$$\begin{aligned}\|x_{n+1}-v\|^2 &= \|\alpha_nx_n+(1-\alpha_n)Sy_n-v\|^2\\ &= \|\alpha_n(x_n-v)+(1-\alpha_n)(Sy_n-v)\|^2\\ &= \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|Sy_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\ &\leq \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|y_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\ &\leq \alpha_n\|x_n-v\|^2+(1-\alpha_n)\|x_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\ &= \|x_n-v\|^2-\alpha_n(1-\alpha_n)\|Sy_n-x_n\|^2\\ &\leq \|x_n-v\|^2.\end{aligned}$$

Hence $\{\|x_{n+1}-v\|\}$ is a decreasing sequence and therefore $\lim_{n\longrightarrow\infty}\|x_n-v\|$ exists. This implies that $\{x_n\},\{y_n\}$ and $\{Sy_n\}$ are bounded. Since $\{T_n\}$ is firmly nonexpansive, it follows that

$$\begin{aligned}\|T_nx_n-v\|^2 &= \|T_nx_n-T_nv\|^2\\ &\leq \langle T_nx_n-v,x_n-v\rangle\\ &= \frac{1}{2}\big(\|T_nx_n-v\|^2+\|x_n-v\|^2-\|x_n-T_nx_n\|^2\big),\end{aligned}$$

for all $v \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$ and hence

$$\|T_n x_n - v\|^2 \le \|x_n - v\|^2 - \|x_n - T_n x_n\|^2.$$

Thus, we have

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|\alpha_n x_n + (1-\alpha_n) S y_n - v\|^2 \\
&= \|\alpha_n (x_n - v) + (1-\alpha_n)(S y_n - v)\|^2 \\
&\le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|S y_n - v\|^2 \\
&\le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|y_n - v\|^2 \\
&= \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|T_n x_n - v\|^2 \\
&\le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\big(\|x_n - v\|^2 - \|x_n - T_n x_n\|^2\big).
\end{aligned}$$

we obtain

$$\begin{aligned}
(1-\alpha_n)\|x_n - T_n x_n\|^2 &\le \alpha_n \|x_n - v\|^2 + (1-\alpha_n)\|x_n - v\|^2 - \|x_{n+1} - v\|^2 \\
&= \|x_n - v\|^2 - \|x_{n+1} - v\|^2.
\end{aligned}$$

Since, $0 < a \le \alpha_n \le b < 1$ and $\lim_{n\longrightarrow\infty} \|x_n - v\|^2 = \lim_{n\longrightarrow\infty} \|x_{n+1} - v\|^2$, we obtain

$$\|x_n - T_n x_n\| = \|x_n - y_n\| \longrightarrow 0.$$

Put $A_n = I - ST_n$. From $A_n v = 0$, it follows by Lemma 2.3 that

$$\begin{aligned}
\|x_{n+1} - v\|^2 &= \|\alpha_n x_n + (1-\alpha_n) S T_n x_n - v\|^2 \\
&= \|\alpha_n x_n + (1-\alpha_n) x_n - (1-\alpha_n) x_n + (1-\alpha_n) S T_n x_n - v\|^2 \\
&= \|x_n - v - (1-\alpha_n)(x_n - S T_n x_n)\|^2 \\
&= \|x_n - v - (1-\alpha_n) A_n x_n\|^2 \\
&= \|x_n - v\|^2 - 2(1-\alpha_n)\langle x_n - v, A_n x_n\rangle + (1-\alpha_n)^2 \|A_n x_n\|^2 \\
&= \|x_n - v\|^2 - 2(1-\alpha_n)\langle x_n - v, A_n x_n - A_n v\rangle + (1-\alpha_n)^2 \|A_n x_n\|^2 \\
&\le \|x_n - v\|^2 - 2(1-\alpha_n)\big\{\|A_n x_n - A_n v\|^2 - \frac{1}{2}(\|A_n x_n\|^2 + \|A_n v\|^2) \\
&\quad +(1-\alpha_n)^2 \|A_n x_n\|^2 \\
&= \|x_n - v\|^2 - (1-\alpha_n)\|A_n x_n\|^2 + (1-\alpha_n)^2 \|A_n x_n\|^2 \\
&= \|x_n - v\|^2 - \alpha_n (1-\alpha_n)\|A_n x_n\|^2
\end{aligned}$$

and hence

$$\alpha_n (1-\alpha_n)\|A_n x_n\|^2 \le \|x_n - v\|^2 - \|x_{n+1} - v\|^2.$$

Since $\liminf_{n\longrightarrow\infty} \alpha_n (1-\alpha_n) > 0$, we get

$$\lim_{n\longrightarrow\infty} \|A_n x_n\| = \lim_{n\longrightarrow\infty} \|x_n - S T_n x_n\| = 0.$$

So, we have

$$\begin{aligned}\|y_n - Sy_n\| &= \|T_nx_n - ST_nx_n\| \\ &= \|T_nx_n - x_n + x_n - ST_nx_n\| \\ &\leq \|T_nx_n - x_n\| + \|x_n - ST_nx_n\| \longrightarrow 0 \text{ as } n \longrightarrow \infty.\end{aligned}$$

Since $\{y_n\}$ is bounded, there exists a subsequence $\{y_{n_j}\}$ of $\{y_{n_i}\}$ which converges weakly to $\hat{z}$. Without loss of generality, we can assume that $y_{n_i} \rightharpoonup \hat{z}$. By Lemma 2.2, we have $\hat{z} \in F(S)$. From $\lim_{n\longrightarrow\infty} \|x_n - y_n\| \longrightarrow 0$ and $y_{n_i} \rightharpoonup \hat{z}$, we get $x_{n_i} \rightharpoonup \hat{z}$. We shall show that $\hat{z} \in F(T)$. From $\|T_nx_n - x_n\| \longrightarrow 0$ and AKTT-condition, we have $\|Tx_n - x_n\| \leq \|Tx_n - T_nx_n\| + \|T_nx_n - x_n\| \longrightarrow 0$. We next show that $\hat{z} \in F(T)$. Assume $\hat{z} \notin F(T)$. Since $x_{n_i} \rightharpoonup \hat{z}$ and $\hat{z} \neq T\hat{z}$. By the Opial's condition, we have

$$\begin{aligned}\liminf_{n\longrightarrow\infty} \|x_{n_i} - \hat{z}\| &< \liminf_{n\longrightarrow\infty} \|x_{n_i} - T\hat{z}\| \\ &\leq \liminf_{n\longrightarrow\infty} \{\|x_{n_i} - Tx_{n_i}\| + \|Tx_{n_i} - T\hat{z}\|\} \\ &\leq \liminf_{n\longrightarrow\infty} \|x_{n_i} - \hat{z}\|.\end{aligned}$$

This is a contradiction. So, we get $\hat{z} \in F(T)$. Hence $\hat{z} \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$. Let $\{x_{n_k}\}$ be another subsequence of $\{x_n\}$ such that $\{x_{n_k}\}$ converges weakly to $\bar{z}$. We may show that $\hat{z} = \tilde{z}$, suppose not. Since $\lim_{n\longrightarrow\infty} \|x_n - v\|$ exists for all $v \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$, it follows by the Opial's condition that

$$\begin{aligned}\lim_{n\longrightarrow\infty} \|x_n - \hat{z}\| &= \liminf_{i\longrightarrow\infty} \|x_{n_i} - \hat{z}\| < \liminf_{i\longrightarrow\infty} \|x_{n_i} - \bar{z}\| = \lim_{n\longrightarrow\infty} \|x_n - \bar{z}\| \\ &= \liminf_{k\longrightarrow\infty} \|x_{n_k} - \bar{z}\| < \liminf_{k\longrightarrow\infty} \|x_{n_k} - \hat{z}\| = \lim_{n\longrightarrow\infty} \|x_n - \hat{z}\|.\end{aligned}$$

This is a contradiction. Thus, we have $\hat{z} = \tilde{z}$. This implies that $\{x_n\}$ converges weakly to $\hat{z} \in F(S) \cap (\cap_{n=1}^{\infty} F(T_n))$. This completes the proof. □

4 Applications

In this section, using Theorem 3.1, we prove weak convergence theorem for finding a common element of the set of solutions of generalized mixed equilibrium problem and the fixed point set of a nonspreading mapping in Hilbert space. Before, proving our theorems, we need the following lemmas. For solving the equilibrium problem for a bifunction $F : C \times C \longrightarrow \mathbb{R}$, let us assume that F satisfies following conditions:

(A1) $F(x, x) = 0$ for all $x \in C$.

(A2) F is monotone, that is, $F(x, y) + F(y, x) \leq 0$ for all $x, y \in C$.

(A3) for each $x, y, z \in C$. $\lim_{t\downarrow 0} F(tz + (1-t)x, y) \leq F(x, y)$.

(A4) for each $x \in C, y \mapsto F(x, y)$ is convex and lower semicontinuous.

The following lemma appears implicitly in [2].

Lemma 4.1. [2] *Let C be a nonempty closed convex subset of H and let F be a bifunction of $C \times C$ into $\mathbb{R}$ satisfying (A1)-(A4). Let $r > 0$ and $x \in H$. Then, there exists $z \in C$ such that*

$$F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0 \text{ for all } y \in C.$$

The following lemma was also given in [3].

Lemma 4.2. [3] *Assume that $F : C \times C \longrightarrow \mathbb{R}$ satisfies (A1)-(A4). For $r > 0$ and $x \in H$, define a mapping $T_r : H \longrightarrow C$ as follows:*

$$T_r(x) = \{z \in C : F(z, y) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\}$$

for all $z \in H$. Then, the following hold:

(1) T_r is single- valued;

(2) T_r is firmly nonexpansive, i.e., for any $x, y \in H$, $\|T_r x - T_r y\|^2 \leq \langle T_r x - T_r y, x - y\rangle$;

(3) $F(T_r) = EP(F)$;

(4) $EP(F)$ is closed and convex.

Lemma 4.3. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping, $\psi : C \longrightarrow \mathbb{R}$ a lower semi-continuous and convex function and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfies (A1)-(A4). Let $r > 0$ and $x \in H$. Then*

(I) there exists $z \in C$ such that

$$F(z, y) + \langle Az, y - z\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \quad \forall y \in C. \quad (4.1)$$

(II) If we define a mapping $K_r : C \longrightarrow C$ as follows:

$$K_r(x) = \{z \in C : F(z, y) + \langle Az, y - z\rangle + \psi(y) - \psi(z) + \frac{1}{r}\langle y - z, z - x\rangle \geq 0, \forall y \in C\} \quad (4.2)$$

for all $z \in H$. Then, the following hold:

(1) K_r *is single- valued;*

(2) K_r *is firmly nonexpansive, i.e., for any* $x, y \in H$,
$$|K_r x - K_r y\|^2 \leq \langle K_r x - K_r y, x - y\rangle;$$

(3) $F(K_r) = GMEP(F, A, \psi)$;

(4) $GMEP(F, A, \psi)$ *is closed and convex.*

Proof. Define a bifunction $\Theta : C \times C \longrightarrow \mathbb{R}$ as follows:

$$\Theta(x, y) = F(x, y) + \langle Ax, y - x\rangle + \psi(y) - \psi(x), \forall x, y \in C.$$

Next we prove that F satisfies the conditions (A1) - (A4).

(i) In fact, since

$$\Theta(x, x) = F(x, x) + \langle Ax, x - x\rangle + \psi(x) - \psi(x) = 0, \forall x \in C,$$

the condition (A1) is satisfied.

(ii) Since F satisfies the condition (A2), ψ is lower semi-continuous and convex function and $A : C \longrightarrow H$ is a continuous monotone mapping, for any $x, y \in C$, we have

$$\begin{aligned}
\Theta(x, y) + \Theta(y, x) &= F(x, y) + F(y, x) + \langle Ax, y - x\rangle + \langle Ay, x - y\rangle + \psi(y) \\
&\quad -\psi(x) + \psi(x) - \psi(y) \\
&\leq 0 + \langle Ay - Ax, x - y\rangle + 0 \leq 0.
\end{aligned}$$

The condition (A2) is proved.

(iii) Since A is continuous and monotone, ψ is convex and lower semi-continuous and F satisfies the condition (A3), we have

$$\begin{aligned}
\limsup_{t\downarrow 0} \Theta(x + t(u - x), y) &= \limsup_{t\downarrow 0} \big\{F(x + t(u - x), y) + \langle A(x + t(u - x)), \\
&\quad y - (x + t(u - x))\rangle + \psi(y) - \psi(x + t(u - x))\big\} \\
&\leq \limsup_{t\downarrow 0} F(x + t(u - x), y) + \limsup_{t\downarrow 0} \big\{\langle A(x + t(u - x)), \\
&\quad y - (x + t(u - x))\rangle\big\} + \psi(y) - \liminf_{t\downarrow 0} \psi(x + t(u - x)) \\
&\leq F(x, y) + \lim_{t\downarrow 0} \big\{\langle A(x + t(u - x)), y - (x + t(u - x))\rangle\big\} \\
&\quad +\psi(y) - \psi(x) \\
&= F(x, y) + \langle A(x), y - x\rangle + \psi(y) - \psi(x) = \Theta(x, y).
\end{aligned}$$

The condition (A3) is proved.

(iv) By the assumption that the function $y \mapsto F(x, y)$ and ψ both are convex and lower semi-continuous. Again since the function $y \mapsto \langle A(x), y - x\rangle$

is convex and continuous, thus the function $y \mapsto \Theta(x, y)$ is convex and lower semi-continuous, i.e., Θ satisfies the condition (A4).

Hence the conclusions (I) and (II) of Lemma 4.3 can be obtained from Lemma 4.2 immediately. □

Theorem 4.4. *Let C be a nonempty closed convex subset of a real Hilbert space H. Let $A : C \longrightarrow H$ be a continuous monotone mapping, $\psi : C \longrightarrow \mathbb{R}$ a lower semi-continuous and convex function and $F : C \times C \longrightarrow \mathbb{R}$ be a bifunction satisfies (A1)-(A4). Let S be a nonspreading mapping of C into itself such that $F(S) \cap GMEP(F, A, \psi) \neq \emptyset$. Suppose $x_0 = x \in C$ and define the sequence $\{x_n\}$ and $\{y_n\}$ by*

$$\begin{cases} F(y_n, y) + \langle Ay_n, y - y_n\rangle + \psi(y) - \psi(y_n) + \frac{1}{r_n}\langle y - y_n, y_n - x_n\rangle \geq 0, \ \forall y \in C, \\ x_{n+1} = \alpha_n x_n + (1 - \alpha_n)Sy_n, \end{cases}$$

for all $n \in \mathbb{N}$, where $\{r_n\} \in (0, \infty)$ with $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$ and $\{\alpha_n\} \subset [a, b]$ for some $a, b \in (0, 1)$. Then $\{x_n\}$ converges weakly to $\hat{z} \in F(S) \cap GMEP(F, A, \psi)$.

Proof. Setting $T_n \equiv T_{r_n}$ in Theorem 3.1 then we have $y_n = T_{r_n}x_n$. Let $v \in F(S) \cap GMEP(F, A, \psi)$. For $n \in \mathbb{N}$, let $z_n = T_{r_n}z$. we first prove that

$$\sum_{n=1}^{\infty} \sup\left\{\|T_{r_{n+1}}z - T_{r_n}z\| : z \in C\right\} < \infty \tag{4.3}$$

We note that

$$F(z_n, y) + \langle Az_n, y - z_n\rangle + \psi(y) - \psi(z_n) + \frac{1}{r_n}\langle y - z_n, z_n - z\rangle \geq 0 \tag{4.4}$$

for all $y \in C$ and

$$F(z_{n+1}, y) + \langle Az_{n+1}, y - z_{n+1}\rangle + \psi(y) - \psi(z_{n+1}) + \frac{1}{r_{n+1}}\langle y - z_{n+1}, z_{n+1} - z\rangle \geq 0 \tag{4.5}$$

for all $y \in C$. Setting $y = z_{n+1}$ in (4.4) and $y = z_n$ in (4.5), we have

$$F(z_n, z_{n+1}) + \langle Az_n, z_{n+1} - z_n\rangle + \psi(z_{n+1}) - \psi(z_n) + \frac{1}{r_n}\langle z_{n+1} - z_n, z_n - z\rangle \geq 0$$

and

$$F(z_{n+1}, z_n) + \langle Az_{n+1}, z_n - z_{n+1}\rangle + \psi(z_n) - \psi(z_{n+1}) + \frac{1}{r_{n+1}}\langle z_n - z_{n+1}, z_{n+1} - z\rangle \geq 0.$$

Adding the two inequalities and by (A2), we have

$$\langle Az_n - Az_{n+1}, z_{n+1} - z_n \rangle + \left\langle z_{n+1} - z_n, \frac{z_n - z}{r_n} - \frac{z_{n+1} - z}{r_{n+1}} \right\rangle \geq 0.$$

Thus, we have

$$\left\langle z_{n+1} - z_n, \frac{z_n - z}{r_n} - \frac{z_{n+1} - z}{r_{n+1}} \right\rangle \geq \langle Az_{n+1} - Az_n, z_{n+1} - z_n \rangle$$

and hence

$$\langle z_{n+1} - z_n, z_n - z_{n+1} \rangle + \left\langle z_{n+1} - z_n, (1 - \frac{r_n}{r_{n+1}})(z_{n+1} - z) \right\rangle \quad \geq \quad \langle Az_{n+1} - Az_n, z_{n+1} - z_n \rangle$$

Since A is continuous monotone mapping, we have

$$-\|z_{n+1} - z_n\|^2 + \left\langle z_{n+1} - z_n, \left(1 - \frac{r_n}{r_{n+1}}\right)(z_{n+1} - z) \right\rangle \geq 0.$$

So, we get

$$\begin{aligned} \|z_{n+1} - z_n\|^2 &\leq \left\langle z_{n+1} - z_n, \left(1 - \frac{r_n}{r_{n+1}}\right)(z_{n+1} - z) \right\rangle \\ &\leq \|z_{n+1} - z_n\| \left|1 - \frac{r_n}{r_{n+1}}\right| \|z_{n+1} - z\|. \end{aligned}$$

Without loss of generality, let us assume that there exists a real number b such that $r_n > b > 0$ for all $n \in \mathbb{N}$. Then

$$\begin{aligned} \|T_{r_{n+1}} z - T_{r_n} z\| &= \|z_{n+1} - z_n\| \\ &\leq \frac{1}{r_{n+1}} |r_{n+1} - r_n| \|T_{r_{n+1}} z - z\| \\ &\leq \frac{1}{b} |r_{n+1} - r_n| \|T_{r_{n+1}} z - z\|. \end{aligned} \tag{4.6}$$

Let $u \in GMEP(F, A, \psi)$ and $M = \sup \left\{ \|z - u\| : z \in C \right\}$. Then

$$\begin{aligned} \|T_{r_{n+1}} z - z\| &\leq \|T_{r_{n+1}} z - u\| + \|u - z\| \\ &= \|T_{r_{n+1}} z - T_{r_{n+1}} u\| + \|u - z\| \\ &\leq 2\|z - u\|. \end{aligned}$$

This together with (4.6), we have

$$\sup \|T_{r_{n+1}} z - T_{r_n} z\| : z \in C \leq \frac{2M}{b} |r_{n+1} - r_n|.$$

Since $\sum_{n=1}^{\infty} |r_{n+1} - r_n| < \infty$, we obtain $\sum_{n=1}^{\infty} \sup \{\|T_{r_{n+1}}z - T_{r_n}z\| : z \in C\} < \infty$. By Lemma 2.4, we define a mapping T by $Tx = \lim_{n \longrightarrow \infty} T_{r_n}x$ for all $x \in C$.

Next, we prove that $F(T) = \cap_{n=1}^{\infty} F(T_{r_n})$. It easy to see that $\cap_{n=1}^{\infty} F(T_{r_n}) \subset F(T)$. Let $w \in F(T)$. For $n \in \mathbb{N}$, let $w_n = T_{r_n}w$. Then

$$F(w_n, y) + \langle Aw_n, y - w_n\rangle + \psi(y) - \psi(w_n) + \frac{1}{r_n}\langle y - w_n, w_n - w\rangle \geq 0$$

for all $y \in C$. By (A2), we obtain $\frac{1}{r_n}\langle y - w_n, w_n - w\rangle \geq F(y, w_n) + \langle Aw_n, w_n - y\rangle - \psi(y) + \psi(w_n)$ for all $y \in C$. Since $w_n \longrightarrow w$, A is continuous monotone mapping, ψ is lower semi-continuous mapping and from (A4), we have $0 \geq F(y, w) + \langle Aw, w - y\rangle - \psi(y) + \psi(w)$ for all $y \in C$. Put $u_t = ty + (1 - t)w$ for all $t \in (0, 1]$ and $y \in C$. Thus, we note that

$$\begin{aligned}
0 &= F(u_t, u_t) + \langle Aw, u_t - u_t\rangle + \psi(u_t) - \psi(u_t)\\
&= F(ty + (1-t)w, ty + (1-t)w) + \big\langle Aw, (ty + (1-t)w) - u_t\big\rangle\\
&\leq tF(ty + (1-t)w, y) + (1-t)F(ty + (1-t)w, w) + t\big\langle Aw, y - u_t)\big\rangle\\
&\quad +(1-t)\big\langle Aw, w - u_t)\big\rangle\\
&= t\big[F(ty + (1-t)w, y) + \big\langle Aw, y - u_t)\big\rangle\big]\\
&\quad +(1-t)\big[F(ty + (1-t)w, w) + \big\langle Aw, w - u_t)\big\rangle\big]\\
&\leq t\big[F(ty + (1-t)w, y) + \big\langle Aw, y - u_t)\big\rangle\big].
\end{aligned}$$

So, $F(ty + (1 - t)w, y) + \big\langle Aw, y - u_t)\big\rangle \geq 0$ for all $y \in C$. Letting $t \longrightarrow 0^+$ and using (A3), A is continuous monotone mapping. So, we obtain $F(w, y) + \big\langle Aw, y - w)\big\rangle \geq 0$ for all $y \in C$. Thus $w \in GMEP(F, A, \psi)$. It follows that $w \in \cap_{n=1}^{\infty} F(T_{r_n})$. Hence $\{T_{r_n}\}$ satisfy condition in Theorem 3.1. So, we obtain the desired result by using Theorem 3.1. □

Acknowledgement. The authors would like to thank the referees for the insightful comments and suggestions. Moreover, the authors gratefully acknowledge the Thailand Research Fund Master Research Grants (TRF-MAG, MRG-WII515S029) for funding this paper.

References

[1] K.Aoyama, Y. Kimura, W. Takahashi and M. Toyoda, Approximation of common fixed points of a countable family of nonexpansive mappings in a Banach space. Nonlinear Anal. 67 (2007) 2350-2360.

[2] E. Blum and W. Oettli, From optimization and variationnal inequalities to equilibrium problems, Math. Student 63 (1994) 123-145.

[3] P. L. Combettes and S. A. Hirstoaga, Equilibrium programming in Hilbert spaces, J. Nonlinear Convex Anal. 6 (2005) 117-136.

[4] K. Goebel and W.A. Kirk, Topics in Metric Fixed Point Theory, Cambridge University Press, Cambridge, 1990.

[5] K. Geobel and S. Reich, Uniform Convexity, Hyperbolic Geometry and Nonexpansive Mappings, Marcel Dekker inc. New York, 1984.

[6] B. Halpern, Fixed point of nonexpanding maps, Bull. amer. Math. Soc. 73 (1967) 957-961.

[7] S. Iemoto and W. Takahashi, Approximating common fixed points of nonexpasive mappings and nonspreading mappings in a Hilbert space, Nonlinear Analysis (2009), doi:10.1016/j.na.2009.03.064.

[8] F. Kohsaka and W. Takahashi, Fixed point theorems for a class of nonlinear mappings related to maximal monotone operators in Banach spaces, Arch. Math. 91(2008) 166-177.

[9] A. Moudafi and M. Thera, Proximal and Dynamical Approaches to Equilibrium Problems, in: Lecture Notes in Economics and Mathematical Systems, 477 (1999) 187-201.

[10] Z. Opial, Weak convergence of the sequence of successive approximations for nonexpansive mappings, Bull. Amer. Math. Soc. 73(1967) 595-597.

[11] S. Plubtieng and T. Thammathiwat, A viscosity approximation method for finding a common fixed point of nonexpansive and firmly nonexpansive mappings in hilbert spaces. Thai. J. Math. 6 (2008) 377390.

[12] S. Plubtieng and T. Thammathiwat, A viscosity approximation method for finding a common solution of fixed point and equilibrium problems in Hilbert spaces. J. Glob. Optium, (2010) doi:10.1007/s10898-010-9583-z.

[13] S. Reich and D. Shoikhet, Nonlinear Semigroups, Fixed Points, and Geometry of Domains in Banach Spaces, Imperial College Press, London, 2005.

[14] W. Takahashi, Nonlinear Functional Analysis-Fixed Point Theory and its Applications, Yokohama Publishers, Yokohama, 2000.

[15] A. Tada and W. Takahashi, Strong convergence theorem for an equilibrium problem and a nonexpansive mapping, J. Optim. Theory Appl. 133 (2007) 359-370.

[16] S. Takahashi and W. Takahashi, Viscosity approximation methods for equilibrium problems and fixed point problems in Hilbert spaces, J. Math. Anal. Appl. 331 (2007) 506-515.

[17] S. Takahashi and W. Takahashi, Strong convergence theorem for a generalized equilibrium problem and a nonexpansive mapping in a Hilbert space, Nonlinear Anal. 69 (2008) 1025-1033.

Received: April, 2011