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Existence of Fixed Point for a Class of Mappings in Partially Ordered Complete Metric Spaces

Vahid Parvaneh

Department of Mathematics, Gilan Gharb Branch
Islamic Azad University, Gilan Gharb, Iran
vahid.parvaneh@kiau.ac.ir

Abstract

The purpose of this paper is to present a fixed point result for a class of contractions in partially ordered complete metric spaces.

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1 Introduction

The Banach contraction principle [2] is an important tool in the theory of metric spaces. It guarantees the existence and uniqueness of fixed points for certain self-maps on metric spaces, and provides a constructive method to find fixed points. The theorem was named after S. Banach (1892–1945), and was first stated by him in 1922.

Up to now, many authors have generalized this theorem([3], [4], [9] and $\cdots$) in various ways. In recent years, the Banach contraction principle, also, has been extended to partially ordered complete metric spaces. We can see some of this generalizations in [1], [5], [6], [7], [8], $\cdots$ and references mentioned therein.

In this paper, we establish fixed point theorems for a class of mappings in a partially ordered complete metric space.

2 Preliminary Notes

Definition 2.1 *A mapping $T : X \to X$, where (X, d) is a metric space is said to be a C-contraction if there exists $\alpha \in (0, \frac{1}{2})$ such that for all $x, y \in X$ the following inequality holds:*

$$d(Tx, Ty) \leq \alpha(d(x, Ty) + d(y, Tx)).$$

The concept of C-contraction was defined by S. K. Chatterjea [3] in 1972 and he has proved that if (X,d) is a complete metric space, then every C-contraction on X admits a unique fixed point.

In 2009, Choudhury introduced a generalization of C-contraction given by the following definition.

Definition 2.2 *([4], Definition 1.3) A mapping $T : X \to X$, where (X,d) is a metric space is said to be weakly C-contractive (or a weak C-contraction) if for all $x, y \in X$,*

$$d(Tx,Ty) \leq \frac{1}{2}(d(x,Ty) + d(y,Tx)) - \varphi((d(x,Ty), d(y,Tx))),$$

where $\varphi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only if $x = y = 0$.

In ([4]), Binayak S. Choudhury has proved that if (X,d) is a complete metric space, then every weak C-contraction on X has a unique fixed point.

Harjani et al have presented the following theorem which is a version of the main result in [4] in the context of ordered metric spaces.

Theorem 2.3 *([5], Theorem 2.1) Let $(X,\preceq)$ be a partially ordered set and suppose that there exists a metric d in X such that (X,d) is a complete metric space. Let $T : X \longrightarrow X$ be a continuous and nondecreasing mapping such that*

$$d(Tx,Ty) \leq \frac{1}{2}(d(x,Ty) + d(y,Tx)) - \varphi(d(x,Ty), d(y,Tx)),\ for\ x \succeq y$$

where $\varphi : [0,\infty)^2 \to [0,\infty)$ is a continuous function such that $\varphi(x,y) = 0$ if and only if $x = y = 0$. If there exists $x_0 \in X$ with $x_0 \preceq Tx_0$, then T has a fixed point.

3 Main Results

In what follows, we use the following definitions.

Definition 3.1 *([1]) Let $(X,\preceq)$ be a partially ordered set. A mapping f is called dominating, if $x \preceq fx$ for each $x \in X$.*

Definition 3.2 *A mapping $T : X \to X$, where (X, d) is a metric space is said to be a P-contraction if there exists $\alpha \in (0,\frac{1}{4})$ such that for all $x, y \in X$,*

$$d(Tx,Ty) \ \ \leq \alpha(d(x,Tx) + d(x,Ty) + d(y,Tx) + d(y,Ty)). \qquad (1)$$

Definition 3.3 *A mapping $T : X \to X$, where (X, d) is a metric space is said to be weakly P-contractive (or a weak P-contraction) if for all $x, y \in X$,*

$$\begin{array}{ll} d(Tx,Ty) & \leq \frac{1}{4}(d(x,Tx)+d(x,Ty)+d(y,Tx)+d(y,Ty)) \\ & -\varphi(d(x,Tx),d(x,Ty),d(y,Tx),d(y,Ty)), \end{array} \quad (2)$$

where $\varphi : [0,\infty)^4 \to [0,\infty)$ is a lower semicontinuous function such that $\varphi(x, y, z, t) = 0$ if and only if $x = y = z = t = 0$.

Our first result is the following.

Theorem 3.4 *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a continuous P-contraction, that is, for all comparable $x, y \in X$, 2 be holds. If, T be dominating, then T has a fixed point.*

Proof 3.5 *Let $x_0 \in X$ be arbitrary. We know that $x_0 \preceq Tx_0$. If $Tx_0 = x_0$, then the proof is finished. Suppose that $x_0 \prec Tx_0$. We obtain by induction*

$$x_0 \prec Tx_0 \preceq T^2x_0 \preceq T^3x_0 \prec \cdots \preceq T^nx_0 \preceq T^{n+1}x_0 \preceq \cdots.$$

We define $x_{n+1} = Tx_n$. First, we show that $\{x_n\}$ is a Cauchy sequence in X.

We consider two cases.

(1) If for some n, $x_n = x_{n+1}$, then $Tx_{n-1} = Tx_n = T(Tx_{n-1})$ and the proof is completed.

(2) If $x_n \neq x_{n+1}$, for every positive integer n, we obtain that

$$\begin{array}{ll} d(x_{n+1},x_{n+2}) & = d(Tx_n,Tx_{n+1}) \\ & \leq \frac{1}{4}(d(x_n,Tx_n)+d(x_n,Tx_{n+1})+d(x_{n+1},Tx_n)+d(x_{n+1},Tx_{n+1})) \\ & -\varphi(d(x_n,Tx_n),d(x_n,Tx_{n+1}),d(x_{n+1},Tx_n),d(x_{n+1},Tx_{n+1})) \\ & = \frac{1}{4}(d(x_n,x_{n+1})+d(x_n,x_{n+2})+d(x_{n+1},x_{n+1})+d(x_{n+1},x_{n+2})) \\ & -\varphi(d(x_n,x_{n+1}),d(x_n,x_{n+2}),d(x_{n+1},x_{n+1}),d(x_{n+1},x_{n+2})) \\ & \leq \frac{1}{4}(d(x_n,x_{n+1})+d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})+d(x_{n+1},x_{n+2})) \\ & = \frac{1}{2}(d(x_n,x_{n+1})+d(x_{n+1},x_{n+2})). \end{array} \quad (3)$$

Thus, $d(x_{n+1}, x_n)$ is a decreasing sequence and so it is convergent.

Assume that, $\lim_{n\to\infty} d(x_{n+1}, x_n) = r$.

From the above argument we have

$$\begin{array}{ll} d(x_{n+1},x_n) & \leq \frac{1}{4}(d(x_{n-1},x_n)+d(x_{n-1},x_{n+1})+d(x_n,x_{n+1})) \\ & \leq \frac{1}{2}(d(x_{n-1},x_n)+d(x_n,x_{n+1})). \end{array} \quad (4)$$

If $n \to \infty$, we have

$$r \leq \lim_{n\to\infty} \frac{1}{4}[2r + d(x_{n-1}, x_{n+1})] \leq r.$$

Therefore

$$\lim_{n\to\infty} d(x_{n-1}, x_{n+1}) = 2r.$$

We have proved in 3

$$\begin{aligned} d(x_{n+1}, x_{n+2}) &\leq \tfrac{1}{4}(d(x_n, x_{n+1}) + d(x_n, x_{n+2}) + 0 + d(x_{n+1}, x_{n+2})) \\ &-\varphi(d(x_n, x_{n+1}), d(x_n, x_{n+2}), 0, d(x_{n+1}, x_{n+2})) \\ &\leq \tfrac{1}{2}(d(x_n, x_{n+1}) + d(x_{n+1}, x_{n+2})). \end{aligned} \tag{5}$$

Now, if $n \to \infty$ *and using the lower semicontinuity of* φ*, we obtain*

$$\begin{aligned} r \leq \tfrac{1}{4}4r &\; - \liminf_{n\to\infty} \varphi(d(x_n, x_{n+1}), d(x_n, x_{n+2}), 0, d(x_{n+1}, x_{n+2})) \\ &\leq \tfrac{1}{4}4r - \varphi(r, 2r, 0, r). \end{aligned} \tag{6}$$

Consequently, $\varphi(r, 2r, 0, r) = 0$*. This yields that*

$$r = \lim_{n\to\infty} d(x_n, x_{n+1}) = 0, \tag{7}$$

by our assumption about φ*.*

We show that $\{x_n\}$ *is a Cauchy sequence in* X*.*

Suppose that $\{x_n\}$ *is not a Cauchy sequence. Then there exists* $\varepsilon > 0$ *for which we can find subsequences* $\{x_{m(k)}\}$ *and* $\{x_{n(k)}\}$ *of* $\{x_n\}$ *such that* $n(k)$ *is the smallest index for which* $n(k) > m(k) > k$ *and* $d(x_{m(k)}, x_{n(k)}) \geq \varepsilon$*.*

This means that

$$d(x_{m(k)}, x_{n(k)-1}) < \varepsilon. \tag{8}$$

From 8 and triangle inequality

$$\begin{aligned} \varepsilon \leq d(x_{m(k)}, x_{n(k)}) &\leq d(x_{m(k)}, x_{n(k)-1}) + d(x_{n(k)-1}, x_{n(k)}) \\ &\leq \varepsilon + d(x_{n(k)-1}, x_{n(k)}). \end{aligned} \tag{9}$$

Letting $k \to \infty$ *and using 7 we can conclude that*

$$\lim_{k\to\infty} d(x_{m(k)}, x_{n(k)}) = \varepsilon. \tag{10}$$

Moreover,

$$|d(x_{n(k)}, x_{m(k)+1}) - d(x_{n(k)}, x_{m(k)})| \leq d(x_{m(k)}, x_{m(k)+1}) \tag{11}$$

and

$$|d(x_{n(k)-1}, x_{m(k)+1}) - d(x_{n(k)}, x_{m(k)+1})| \leq d(x_{n(k)}, x_{n(k)-1}). \tag{12}$$

and

$$|d(x_{n(k)-1}, x_{m(k)}) - d(x_{n(k)}, x_{m(k)})| \leq d(x_{n(k)}, x_{n(k)-1}). \tag{13}$$

Using 7, 10, 11, 12 and 13 we get

$$\lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)}) = \lim_{k\to\infty} d(x_{m(k)+1}, x_{n(k)}) = \lim_{k\to\infty} d(x_{n(k)-1}, x_{m(k)+1}) = \varepsilon. \tag{14}$$

As $x_{2m(k)-1}$ and $x_{2n(k)}$ are comparable, using 2 we have

$$\begin{aligned} d(x_{m(k)+1}, x_{n(k)}) \quad &= d(Tx_{m(k)}, Tx_{n(k)-1}) \\ &\le \tfrac{1}{4}(d(x_{m(k)}, Tx_{m(k)}) + d(x_{m(k)}, Tx_{n(k)-1}) \\ &+ d(x_{n(k)-1}, Tx_{m(k)}) + d(x_{n(k)-1}, Tx_{n(k)-1})) \\ &- \varphi(d(x_{m(k)}, Tx_{m(k)}), d(x_{m(k)}, Tx_{n(k)-1}) \\ &, d(x_{n(k)-1}, Tx_{m(k)}), d(x_{n(k)-1}, Tx_{n(k)-1})) \\ &\le \tfrac{1}{4}(d(x_{m(k)}, x_{m(k)+1}) + d(x_{m(k)}, x_{n(k)}) \\ &+ d(x_{n(k)-1}, x_{m(k)+1}) + d(x_{n(k)-1}, x_{n(k)})) \\ &- \varphi(d(x_{m(k)}, x_{m(k)+1}), d(x_{m(k)}, x_{n(k)}) \\ &, d(x_{n(k)-1}, x_{m(k)+1}), d(x_{n(k)-1}, x_{n(k)})). \end{aligned} \tag{15}$$

Making $k \to \infty$ and taking into accounts 10 and 14 and the lower semicontinuity of φ, we have

$$\varepsilon \le \frac{1}{4}(0 + \varepsilon + \varepsilon + 0) - \varphi(0, \varepsilon, \varepsilon, 0)$$

and from the last inequality $\varphi(0, \varepsilon, \varepsilon, 0) \le -\frac{1}{2}\varepsilon$. Therefore, $\varphi(0, \varepsilon, \varepsilon, 0) = 0$. By our assumption about φ, we have $\varepsilon = 0$, which is a contradiction and it follows that $\{x_n\}$ is a Cauchy sequence in X.

Now, we show that T has a fixed point.

Since (X, d) is complete and $\{x_n\}$ is Cauchy, there exists $z \in X$ such that $\lim_{n\to\infty} x_n = z$.

Moreover, the continuity of T implies that

$$z = \lim_{n\to\infty} x_{n+1} = \lim_{n\to\infty} Tx_n = Tz,$$

and this proves that z is a fixed point for T.

In the next theorem, we assume that T is only dominating and X is regular.

Definition 3.6 *Let $(X, \preceq, d)$ be a partially ordered metric space. We say that X is regular if and only if the following hypothesis holds: if $\{z_n\}$ is a non-decreasing sequence in X with respect to $\preceq$ such that $z_n \to z \in X$, as $n \to \infty$, then $z_n \preceq z$, for all $n \in N$.*

Theorem 3.7 *Let $(X, \preceq, d)$ be a partially ordered complete metric space. Let $T : X \to X$ be a dominating P-contraction (not necessarily continuous) as in theorem 3.4. If X is regular, then T has a fixed point.*

Proof 3.8 *Reviewing the proof of Theorem 3.4 we only have to show that $Tz = z$. As $\{x_n\}$ is a nondecreasing sequence in X and $x_n \to z$, then, regularity of X yields that $x_n \preceq z$, for every $n \in N$, and, consequently, the contractive condition 2 yeilds that, for all $n \in N$,*

$$\begin{aligned} d(x_{n+1}, Tz) &= d(Tx_n, Tz) \\ &\leq \tfrac{1}{4}(d(x_n, Tx_n) + d(x_n, Tz) + d(z, Tx_n) + d(z, Tz)) \\ &\quad -\varphi(d(x_n, Tx_n), d(x_n, Tz), d(z, Tx_n), d(z, Tz)) \\ &\leq \tfrac{1}{4}(d(x_n, x_{n+1}) + d(x_n, Tz) + d(z, x_{n+1}) + d(z, Tz)). \end{aligned} \tag{16}$$

If $n \to \infty$,

$$d(z, Tz) \leq \frac{1}{4}(0 + d(z, Tz) + 0 + d(z, Tz)) - \varphi(0, d(z, Tz), 0, d(z, Tz)),$$

and hence

$$\varphi(0, d(z, Tz), 0, d(z, Tz)) \leq -\frac{1}{2}(d(z, Tz)) \leq 0,$$

and, therefore $d(Tz, z) = 0$. So, we have $Tz = z$.

The following simple example shows that conditions of theorems 3.4 and 3.7 are not sufficient for the uniqueness of fixed point.

Example 3.9 *Let $X = \{(0, 1), (1, 0), (1, 1)\} \subset R^2$ with the euclidean distance d. (X, d) is, obviously, a complete metric space. Moreover, we consider the order $\preceq$ in X given by $(x_1, y_1) \preceq (x_2, y_2) \Leftrightarrow x_1 \leq x_2, y_1 \leq y_2$.*

We define $\varphi(x, y, z, t) = \frac{x+y+z+t}{8}$. Also we consider $T : X \to X$ given by $T(0, 1) = (1, 1) = T(1, 1)$ and $T(1, 0) = (1, 0)$.

We can easily check that condition 2, is hold. Finally, Theorems 3.4 and 3.7 give us the existence of two fixed points for T (i.e., the points $(1, 0)$ and $(1, 1)$).

In the next theorem, we give sufficient conditions for the uniqueness of the fixed point.

Theorem 3.10 *Let all the conditions of theorems 3.4 and 3.7 be fulfilled and T is nondecreasing and let the following condition also holds:*

For arbitrary two points $x, y \in X$, there exists $w \in X$ such that w is comparable with both x and y.

Then the fixed point of T is unique.

Proof 3.11 *Let u and v be two fixed points of T, i.e., $Tu = u$ and $Tv = v$. Consider the following two cases.*

1. u and v are comparable. Then we can apply condition 2 and obtain

$$\begin{aligned} d(u, v) = d(Tu, Tv) &\leq \tfrac{1}{4}(d(u, Tu) + d(u, Tv) + d(v, Tu) + d(v, Tv)) \\ &\quad -\varphi(d(u, Tu), d(u, Tv), d(v, Tu), d(v, Tv)) \\ &= \tfrac{1}{4}(2d(u, v)) - \varphi(0, d(u, v), d(v, u), 0). \end{aligned} \tag{17}$$

So, $\varphi(0, d(u,v), d(v,u), 0) \leq -\frac{1}{2}(d(u,v)) \leq 0$. *Therefore,* $u = v$.

2. Suppose that u *and* v *are not comparable. Choose an element* $w \in X$ *comparable with both of them. Then also* $u = T^n u$ *is comparable with* $T^n w$ *for each* n.

Since T *is a weak P-contraction, we have*

$$\begin{aligned} d(u,T^n w) &= d(T^n u, T^n w) = d(TT^{n-1}u, TT^{n-1}w) \\ &\leq \tfrac{1}{4}(d(T^{n-1}u, TT^{n-1}u) + d(T^{n-1}u, TT^{n-1}w) \\ &+ d(T^{n-1}w, TT^{n-1}u) + d(T^{n-1}w, TT^{n-1}w)) \\ &- \varphi(d(T^{n-1}u, TT^{n-1}u), d(T^{n-1}u, TT^{n-1}w) \\ &, d(T^{n-1}w, TT^{n-1}u), d(T^{n-1}w, TT^{n-1}w)) \\ &= \tfrac{1}{4}(d(u,u) + d(u,T^n w) + d(T^{n-1}w, u) + d(T^{n-1}w, T^n w)) \\ &- \varphi(d(u,u), d(u,T^n w), d(T^{n-1}w, u), d(T^{n-1}w, T^n w)) \\ &= \tfrac{1}{4}(d(u,T^n w) + d(T^{n-1}w, u) + d(T^{n-1}w, T^n w)) \\ &- \varphi(0, d(u,T^n w), d(T^{n-1}w, u), d(T^{n-1}w, T^n w)) \\ &\leq \tfrac{1}{2}(d(u,T^n w) + d(T^{n-1}w, u)) \\ &- \varphi(0, d(u,T^n w), d(T^{n-1}w, u), d(T^{n-1}w, T^n w)) \\ &\leq \tfrac{1}{2}(d(u,T^n w) + d(T^{n-1}w, u)). \end{aligned} \tag{18}$$

As in theorem 3.4, we can prove that $d(T^{n-1}w, T^n w) \to 0$, *when* $n \to \infty$ *(Note that* $w \preceq Tw$ *and we can define* $w_{n+1} = T^n w$, *for* $n =$ *0, 1, 2, 3,* $\cdots$, *where* $T^0 w = w$.*).*

From the above inequality, we have $d(u, T^n w) \leq d(u, T^{n-1}w)$. *This proves that the nonnegative decreasing sequence* $d(u, T^n w)$ *is convergent. If*

$$\lim_{n\to\infty} d(u, T^n w) = r,$$

then, letting $n \to \infty$ *in 18 and from the lower semicontinuity of* φ *we obtain*

$$r \leq \frac{1}{2}(2r) - \varphi(0, r, r, 0) \leq r.$$

This gives us $\varphi(0, r, r, 0) = 0$, *and, by our assumption about* φ, $r = 0$.

Consequently, $\lim_{n\to\infty} d(u, T^n w) = 0$. *Analogously, it can be proved that* $\lim_{n\to\infty} d(v, T^n w) = 0$. *Finally, since the limit is unique, we have* $u = v$. *This finishes the proof.*

Remark 3.12 *Suppose that* $\alpha \in (0, \frac{1}{4})$. *If we consider in Theorem 3.4 (or in Theorems 3.7)* $\varphi : [0,\infty)^4 \to [0,\infty)$ *be defined by*

$$\varphi(a,b,c,d) = (\frac{1}{4} - \alpha)(a+b+c+d)$$

which, obviously, satisfies that $\varphi(a,b,c,d) = 0$ *if and only if* $a = b = c = d = 0$, *condition 2 can be rewritten as*

$$d(Tx,Ty) \leq \alpha(d(x,Tx) + d(x,Ty) + d(y,Tx) + d(y,Ty)),$$

and hence Theorem 3.4 (or 3.7) is holds, if T *be a P-contraction.*

Remark 3.13 *We mention that, the paper of Harjani et al [5] had a deep impact in our work. In fact, we considered the concept of weakly P-contractive mappings instead of weakly C-contractions.*

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