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Parameters Estimation of the Modified Weibull Distribution Based on Type I Censored Samples

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Abstract

Recently, Sarhan and Zaindin (2009) [10] introduced a new three parameter distribution called the Modified Weibull distribution (MWD) which is a general form for some well known and most commonly used distributions in reliability and life testing such as exponential, Rayleigh, linear failure rate and classic Weibull distribution [7]. They investigated and studied some essential properties of this new distribution. In this paper confidence estimation for the parameters of the MWD based on type I censored samples with replacements and without replacements is developed. For illustrative purpose, the results obtained are applied on sets of simulated data. A real data set is analyzed to see how the model works in practice.

Mathematics Subject Classification: 60K10, 60K05, 62F10

Keywords: Modified Weibull distribution, maximum likelihood estimation, likelihood ratio, confidence region, linear failure rate, Weibull, Rayleigh, exponential distribution

1 Introduction

A new life time distribution named a modified Weibull distribution (MWD) is recently proposed by Sarhan and Zaindin [10] and Zaindin and Sarhan [12]. This distribution is a general form for some most commonly used distributions

in survival analysis of technical products, such as exponential, Rayleigh, linear failure rate and Weibull distribution.
It is known that the exponential distribution has a constant failure rate, the Rayleigh distribution has an increasing failure rate, the linear failure rate distribution has a non-increasing hazard failure rate and the Weibull distribution may have an increasing or decreasing failure rate. When it has an increasing failure rate, it starts from origin. This condition is a limitation of the application for some real reliability problems. The modified Weibull distribution introduced by Sarhan and Zaindin [10] and Zaindin and Sarhan [12] is a very important distribution; then it can be used to describe several reliability models. It has three parameters, one scale parameter α and two shape parameters β and γ. They used MWD(α, β, γ) to denote the modified Weibull distribution with three parameters α, β, γ. The probability density function pdf and the cumulative density function cdf of the MWD(α, β, γ) has respectively the following form:

$$f(x, \theta) = (\alpha + \beta\gamma x^{\gamma-1}) \exp(-\alpha x - \beta x^{\gamma}), x > 0,$$

where $\theta = (\alpha, \beta, \gamma), \gamma > 0, \alpha, \beta \geq 0$ such that $\alpha + \beta > 0$.

$$F(x, \theta) = 1 - \exp(-\alpha x - \beta x^{\gamma}), x > 0.$$

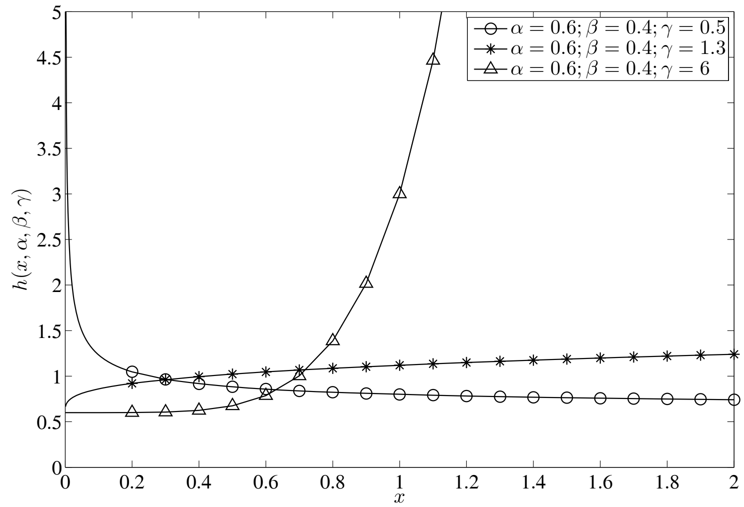
The survival function of the MWD(α, β, γ) takes the following form:

$$S(x, \theta) = \exp(-\alpha x - \beta x^{\gamma}), x > 0,$$

and the hazard rate function of MWD(α, β, γ) is

$$h(x, \theta) = \alpha + \beta\gamma x^{\gamma-1}, x > 0.$$

We can see that the hazard function is constant if $\gamma = 1$, increasing if $\gamma > 1$ and decreasing if $\gamma < 1$. In the case of $\beta = 0$ we obtain the exponential distribution with parameter α, by putting $\alpha = 0$ and $\gamma = 2$ we find the Rayleigh distribution with parameter β, if $\gamma = 2$ a linear failure distribution with parameter α and β is obtained and when $\alpha = 0$ the basic 2-parameter Weibull distribution with parameters β and γ will be derived. Figure 1 show different patterns of the hazard rate function of MWD(α, β, γ) for different parameters values.



The hazard rate function

The main objective of this article is the development of confidence estimations for the parameters of the MWD(α, β, γ) for type I censored samples without and with replacements. Confidence regions for parameters of life time distributions are of interest in the view of theory and application. If the parameters of the lifetime distribution are estimated it is possible to calculate for instance, the reliability of the product and the number of needed spare. Therefore it is useful to find simultaneous confidence regions of the parameters estimations. In the analyzing of life time data very often censored samples are observed. In these cases it is impossible to find the exact distribution of the point estimator and therefore asymptotic methods are used for confidence estimation. This is in compatible with the fact that in practice mostly small samples are observed. In this research confidence estimations will be found by using the asymptotic normality of the maximum likelihood estimators, ML and the asymptotic χ^2–distribution of the likelihood ratio statistic.

2 Type I censored samples without replacement

2.1 M L estimators of the parameters

Let X be the random time to the failure with survival function $S_X(x, \theta)$ and density function $f_X(x, \theta)$. In the case of type I censored without replacement N items are independently observed and the observation of the i-th item $(i = 1, \cdots, N)$ is censored at the time T_i.

The likelihood function from one observation has the form [2]

$$L_i(m_i, \delta_i) = (f_X(x_i, \theta))^{\delta_i} (S_X(T_i, \theta))^{1-\delta_i},$$

with $m_i = \min(x_i, T_i)$ and $\delta_i = \begin{cases} 1 & \text{if a failure was observed,} \\ 0 & \text{if the observation was censored at time} T_i. \end{cases}$
If all T_i are equal: $T_i = T, i = 1, \cdots, N$ the likelihood function has the form:

$$L(m, \delta) = \left\{ \prod_{i=1}^{n} f_X(x_i, \theta) \right\} (S_X(T, \theta))^{N-n},$$

with n=n(T) is a random number of observed failures. It follows then

$$L(m, \theta) = \left\{ \prod_{i=1}^{n} \left(\alpha + \beta\gamma x_i^{\gamma-1}\right) \exp\left(-\alpha x_i - \beta x_i^{\gamma}\right) \right\} \left[\exp\left(-\alpha T - \beta T^{\gamma}\right)\right]^{N-n}.$$

We obtain then the following log-likelihood function:

$$\ln L(x, \theta) = \sum_{i=1}^{n} [\ln(\alpha + \beta\gamma x_i^{\gamma-1}) - \alpha x_i - \beta x_i^{\gamma}] + (N - n)(-\alpha T - \beta T^{\gamma}).$$

If we calculate the first partial derivatives of the log-likelihood function $\ln L(x, \theta)$ with respect to the parameters α, β, γ and equating each to zero, we get then the following system of nonlinear equations of α, β, γ.

$$\sum_{i=1}^{n} \frac{1}{\alpha + \beta\,\gamma\, x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i - (N - n)T = 0$$
$$\sum_{i=1}^{n} \frac{\gamma x_i^{\gamma-1}}{\alpha + \beta\,\gamma\, x_i^{\gamma-1}} - \sum_{i=1}^{n} x_i^{\gamma} - (N - n)T^{\gamma} = 0$$
$$\beta \sum_{i=1}^{n} \frac{x_i^{\gamma-1}(1 + \gamma \ln x_i)}{\alpha + \beta\,\gamma\, x_i^{\gamma-1}} - \beta \sum_{i=1}^{n} x_i^{\gamma} \ln x_i - \beta\,(N - n)T^{\gamma} \ln T = 0.$$

To find out the maximum likelihood estimators of α, β, γ we have to solve the above system of non linear equations with respect to α, β, γ. As it seems, this system has no closed form solution in α, β, γ. Then we have to use a numerical technique method, such as optimization with constraints method, to obtain the solution. We solve the system in three variables α, β, γ , by using the trust-region-reflective optimization method. The objective function to minimize is the Euclidean norm of vector composed by the system equations. The bound constraints are considered allow us, one hand to select the solution, and the second hand to control the error. In order to accelerate the convergence the gradient and the hessian of objective function are explicitly determined.

2.2 Asymptotic Normality of the maximum likelihood estimators

For type I censored samples we can use asymptotic method if the number N of items tends to infinity. We know that if X is a random variable with a three parametric density function $f_X(x, \theta)$, twice continuous differentiable with respect to the parameters α, β, γ and with the expectation $E(X) > 0$. Assume that the third absolute moment of X exists. If the maximum likelihood estimators of the parameters in the case of N independent and identically distributed realizations are asymptotically normal distributed, then the parameter estimators in the case of type I censored samples are asymptotically normal distributed (Kahle (1996)) [6]. In Borgan (1984) [3] and Svenson (1990) [11] conditions for the asymptotic normality are given for type I censored samples with and without replacements. It is easy to prove, that these conditions are fulfilled under the above mentioned assumptions.

2.3 Estimation of the Fisher information and asymptotic confidence bounds

Because the MLE of the vector $\theta = (\alpha, \beta, \gamma)$ is not obtained in closed form, it is not possible to derive the exact distribution of the MLE. In this paragraph, we derive the approximation of the Fisher information matrix needed to define confidence intervals of the parameters based on the asymptotic distributions of their MLE. Let $I(\theta)$ be the Fisher information matrix of the vector of unknown parameters $\theta = (\theta_1, \theta_2, \theta_2)$. Let $\theta_1 = \alpha$,$\theta_2 = \beta$ and $\theta_3 = \gamma$. The elements of the 3×3 matrix $I(\theta)$, $I_{rs}(\theta), r, s \in \{1, 2, 3\}$, can be approximated by $\widehat{I_{rs}(\hat{\theta})} = -\frac{\partial^2 \ln L(x,\hat{\theta})}{\partial\theta_r \partial\theta_s}$.
In the following we will find the second partial derivatives of the function $\ln L(x, \theta)$, for the calculus of the observed information matrix of α, β and γ.

$$
\begin{aligned}
-\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha^2} &= \sum_{i=1}^{n} \frac{1}{(\alpha + \beta\,\gamma\,x_i^{\gamma-1})^2} \\
-\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\beta} &= \sum_{i=1}^{n} \frac{\gamma\,x_i^{\gamma-1}}{(\alpha + \beta\,\gamma\,x_i^{\gamma-1})^2} \\
-\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\gamma} &= \sum_{i=1}^{n} \frac{\beta\,x_i^{\gamma-1}(1 + \gamma\,\ln x_i)}{(\alpha + \beta\,\gamma\,x_i^{\gamma-1})^2} \\
-\frac{\partial^2 \ln L(x,\theta)}{\partial\beta^2} &= \sum_{i=1}^{n} \frac{\gamma^2\,x_i^{2(\gamma-1)}}{(\alpha + \beta\,\gamma\,x_i^{\gamma-1})^2} \\
-\frac{\partial^2 \ln L(x,\theta)}{\partial\beta\partial\gamma} &= -\sum_{i=1}^{n} \frac{\alpha\,x_i^{\gamma-1}(1 + \gamma\,\ln x_i)}{(\alpha + \beta\,\gamma\,x_i^{\gamma-1})^2} + \sum_{i=1}^{n} x_i^{\gamma}\,\ln x_i + (N-n)\,T^{\gamma}\,\ln T \\
-\frac{\partial^2 \ln L(x,\theta)}{\partial\gamma^2} &= -\beta \sum_{i=1}^{n} \frac{x_i^{\gamma-1}(\alpha\,\gamma\,\ln^2 x_i + 2\,\alpha\,\ln x_i - \beta\,x_i^{\gamma-1})}{(\alpha + \beta\,\gamma\,x_i^{\gamma-1})^2} \\
&\quad + \beta \sum_{i=1}^{n} x_i^{\gamma}\,\ln^2 x_i + \beta\,(N-n)\,T^{\gamma}\,\ln^2 T.
\end{aligned}
$$

It follows then, the observed information matrix I given by:

$$I = \begin{pmatrix} -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\alpha^2} & -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\beta} & -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\gamma} \\ -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\beta\partial\alpha} & -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\beta^2} & -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\beta\partial\gamma} \\ -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\gamma\partial\alpha} & -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\gamma\partial\beta} & -\dfrac{\partial^2 \ln L(x,\theta)}{\partial\gamma^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

We can approximate then the variance-covariance matrix, we obtain:

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = I^{-1}.$$

Miller (1981) [9] proved that asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ is given by

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix} \sim N\left(\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} \right). \tag{1}$$

If we replace the parameters α, β, γ by the corresponding MLE's, we get then an estimate of V, denoted by $\hat{V}$ and defined as follows:

$$\hat{V} = \begin{pmatrix} \widehat{I_{11}} & \widehat{I_{12}} & \widehat{I_{13}} \\ \widehat{I_{21}} & \widehat{I_{22}} & \widehat{I_{23}} \\ \widehat{I_{31}} & \widehat{I_{32}} & \widehat{I_{33}} \end{pmatrix}^{-1}, \text{ where } \widehat{I_{ij}} = I_{ij} \text{ if we replace } (\alpha, \beta, \gamma) \text{ by } (\hat{\alpha}, \hat{\beta}, \hat{\gamma}). \tag{2}$$

If we use the equation (1), approximate $100(1-\nu)\%$ confidence intervals for the parameters α, β, γ are given, respectively as:

$$\hat{\alpha} \pm z_{\frac{\nu}{2}} \sqrt{\widehat{V_{11}}}, \quad \hat{\beta} \pm z_{\frac{\nu}{2}} \sqrt{\widehat{V_{22}}}, \quad \hat{\gamma} \pm z_{\frac{\nu}{2}} \sqrt{\widehat{V_{33}}},$$

where $z_{\frac{\nu}{2}}$ is the upper $\frac{\nu}{2}$–th percentile of the standard normal distribution.

2.4 Simultaneous confidence region based on the likelihood ratio

To improve the confidence regions for small samples it is very useful to construct confidence regions based on the likelihood ratio. It is known that under regularity condition [5] the log-likelihood ratio $q = 2\left\{\ln L(x, \hat{\theta}) - \ln(x, \theta)\right\}$ converges in distribution to a central χ^2–distribution with 3 degrees of freedom, where $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ is the maximum likelihood estimator of the unknown

parameter of interest $\theta = (\alpha, \beta, \gamma)$.
The simultaneous confidence region is defined by the inequality $q \leq \chi^2_{1-\nu,3}$, where $\chi^2_{1-\nu,3} = -2\ln\nu$ is the $(1-\nu)$ quantil of the χ^2–distribution with 3 degrees of freedom.
The likelihood ratio takes then the form $q = 2\left\{\ln L(x,\hat{\theta}) - \ln(x,\theta)\right\} = -2\ln\nu$.
We obtain then the following simultaneous confidence region:

$$\sum_{i=1}^{n} \ln\left(\frac{\hat{\alpha} + \hat{\beta}\,\hat{\gamma}\,x_i^{\hat{\gamma}-1}}{\alpha + \beta\,\gamma\,x_i^{\gamma-1}}\right) + (\alpha - \hat{\alpha})\sum_{i=1}^{n} x_i + \beta \sum_{i=1}^{n} x_i^{\gamma} - \hat{\beta}\sum_{i=1}^{n} x_i^{\hat{\gamma}} + (N-n)(T(\alpha-\hat{\alpha}) + \beta\,T^{\gamma} - \hat{\beta}\,T^{\hat{\gamma}}) = -\ln\nu.$$

Remark: If we fix $\gamma = \tilde{\gamma}$ the log likelihood ratio $q = 2\left\{\ln L(m,\hat{\theta}) - \ln(m,\theta)\right\}$ converges in distribution to a central χ^2–distribution with 2 degrees of freedom, where $\hat{\theta} = (\hat{\alpha}, \hat{\beta})$ is the maximum likelihood estimator of the unknown parameter of interest $\theta = (\alpha, \beta)$. The simultaneous confidence region is defined by the inequality $q \leq \chi^2_{1-\nu,2}$, where $\chi^2_{1-\nu,2} = -2\ln\nu$ is the $(1-\nu)$ quantil of the χ^2–distribution with 2 degrees of freedom. We obtain then the following simultaneous confidence region:

$$\sum_{i=1}^{n} \ln\left(\frac{\hat{\alpha} + \hat{\beta}\,\tilde{\gamma}\,x_i^{\tilde{\gamma}-1}}{\alpha + \beta\,\tilde{\gamma}\,x_i^{\tilde{\gamma}-1}}\right) + (\alpha - \hat{\alpha})\sum_{i=1}^{n} x_i + (\beta - \hat{\beta})\sum_{i=1}^{n} x_i^{\tilde{\gamma}} + (N-n)\left(T(\alpha-\hat{\alpha}) + T^{\tilde{\gamma}}\,(\beta-\hat{\beta})\right) = -\ln\nu.$$

By fixing $\beta = \tilde{\beta}$ we obtain the following simultaneous confidence region:

$$\sum_{i=1}^{n} \ln\left(\frac{\hat{\alpha} + \tilde{\beta}\,\hat{\gamma}\,x_i^{\hat{\gamma}-1}}{\alpha + \tilde{\beta}\,\gamma\,x_i^{\gamma-1}}\right) + (\alpha - \hat{\alpha})\sum_{i=1}^{n} x_i + \tilde{\beta}\left(\sum_{i=1}^{n} x_i^{\gamma} - \sum_{i=1}^{n} x_i^{\hat{\gamma}}\right) + (N-n)T(\alpha-\hat{\alpha}) + \tilde{\beta}\,(T^{\gamma} - T^{\hat{\gamma}}) = -\ln\nu.$$

By fixing $\alpha = \tilde{\alpha}$ we obtain then the following simultaneous confidence region:

$$\sum_{i=1}^{n} \ln\left(\frac{\tilde{\alpha} + \hat{\beta}\,\hat{\gamma}\,x_i^{\hat{\gamma}-1}}{\tilde{\alpha} + \beta\,\gamma\,x_i^{\gamma-1}}\right) + \beta\sum_{i=1}^{n} x_i^{\gamma} - \hat{\beta}\sum_{i=1}^{n} x_i^{\hat{\gamma}} + (N-n)(\beta\,T^{\gamma} - \hat{\beta}\,T^{\hat{\gamma}}) = -\ln\nu.$$

3 Type I censored with replacement

3.1 M L estimators of the parameters

We observe now, N independent items, after each failure the item is immediately replaced by a new one and the observation continued up to the time

$T_i, i = 1, \cdots, N$. We have then N independent observations of a renewal process each up to time $T_i, i = 1, \ldots, N$. The likelihood function for one realisation of this process, can be expressed as [8]:

$$L_i(x, \theta) = \left(\prod_{j=1}^{d_i} f_X(x_{ij}, \theta) \right) S_X(R_i, \theta) \,,$$

where d_i is the number of failures of the i-th realization of the process, $x_i = (x_{i1}, \cdots, x_{id_i})$ denotes the distance between failures, $S_X(R_i, \theta)$ is the survival function of the MWD(α, β, γ) and

$$R_i = T_i - \sum_{j=1}^{d_i} x_{ij}$$

is the rest-time of the observation. The likelihood function for such renewal process is given by:

$$L(x, \theta) = \prod_{i=1}^{N} \left\{ \prod_{j=1}^{d_i} \left(\alpha + \beta\gamma x_{ij}^{\gamma-1} \right) \exp\left(-\alpha x_{ij} - \beta x_{ij}^{\gamma} \right) \right\} \exp\left(-\alpha R_i - \beta R_i^{\gamma} \right).$$

The log-likelihood function can be expressed then as:

$$\ln L(x, \theta) = \sum_{i=1}^{N} \sum_{j=1}^{d_i} \ln\left(\alpha + \beta\gamma x_{ij}^{\gamma-1} \right) - \sum_{i=1}^{N} \sum_{j=1}^{d_i} \ln\left(\alpha x_{ij} + \beta x_{ij}^{\gamma} \right) - \sum_{i=1}^{N} \left(\alpha R_i + \beta R_i^{\gamma} \right).$$

To find out the maximum likelihood estimators, we have to solve the system of non linear equations with respect to α, β, γ. This system has no closed form solution in α, β, γ. Then it is necessary to use a numerical technique method to solve the following system:

$$\sum_{i=1}^{N} \sum_{j=1}^{d_i} \frac{1}{\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1}} - \sum_{i=1}^{N} \sum_{j=1}^{d_i} x_{ij} - \sum_{i=1}^{N} R_i = 0$$
$$\sum_{i=1}^{N} \sum_{j=1}^{d_i} \frac{\gamma x_{ij}^{\gamma-1}}{\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1}} - \sum_{i=1}^{N} \sum_{j=1}^{d_i} x_{ij}^{\gamma} - \sum_{i=1}^{N} R_i^{\gamma} = 0$$
$$\beta \sum_{i=1}^{N} \sum_{j=1}^{d_i} \frac{x_{ij}^{\gamma-1}(1 + \gamma \ln x_{ij})}{\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1}} - \beta \sum_{i=1}^{N} \sum_{j=1}^{d_i} x_{ij}^{\gamma} \ln x_{ij} - \beta \sum_{i=1}^{N} R_i^{\gamma} \ln R_i = 0.$$

The problem is reduced to finding the minimum of constrained nonlinear multivariable norm function. The Trust-Region-Reflective algorithm [4] is used to effectuate the minimization. And the gradient and Hessian are calculated in order to accelerate the convergence of the algorithm.

3.2 Asymptotic confidence bounds

In this section we have to derive the approximate confidence intervals of the parameters α, β, γ in the case of type I censored samples with replacement, based on the asymptotic distributions of the MLE of the unknown parameters α, β, γ.

We now derive the observed Fisher information matrix for the parameters α, β and γ. At follows we will find the second partial derivatives of the function $\ln L(x, \theta)$ as:

$$-\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha^2} = \sum_{i=1}^{N}\sum_{j=1}^{d_i} \frac{1}{(\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1})^2}$$

$$-\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\beta} = \sum_{i=1}^{N}\sum_{j=1}^{d_i} \frac{\gamma\, x_{ij}^{\gamma-1}}{(\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1})^2}$$

$$-\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\gamma} = \sum_{i=1}^{N}\sum_{j=1}^{d_i} \frac{\beta\, x_{ij}^{\gamma-1}(1 + \gamma\, \ln x_{ij})}{(\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1})^2}$$

$$-\frac{\partial^2 \ln L(x,\theta)}{\partial\beta^2} = \sum_{i=1}^{N}\sum_{j=1}^{d_i} \frac{\gamma^2\, x_{ij}^{2(\gamma-1)}}{(\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1})^2}$$

$$-\frac{\partial^2 \ln L(x,\theta)}{\partial\beta\partial\gamma} = -\sum_{i=1}^{N}\sum_{j=1}^{d_i} \frac{\alpha\, x_{ij}^{\gamma-1}(1 + \gamma\, \ln x_{ij})}{(\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1})^2} + \sum_{i=1}^{N}\sum_{j=1}^{d_i} x_{ij}^{\gamma}\, \ln x_{ij} + \sum_{i=1}^{N} R_i^{\gamma} \ln R_i$$

$$-\frac{\partial^2 \ln L(x,\theta)}{\partial\gamma^2} = -\beta \sum_{i=1}^{N}\sum_{j=1}^{d_i} \frac{x_{ij}^{\gamma-1}(\alpha\,\gamma\, \ln^2 x_{ij} + 2\,\alpha\, \ln x_{ij} - \beta\, x_{ij}^{\gamma-1})}{(\alpha + \beta\,\gamma\, x_{ij}^{\gamma-1})^2} + \beta \sum_{i=1}^{N}\sum_{j=1}^{d_i} x_{ij}^{\gamma}\, \ln^2 x_{ij} + \beta \sum_{i=1}^{N} R_i^{\gamma} \ln^2 R_i.$$

The observed information matrix I for this model is given then by

$$I = \begin{pmatrix} -\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha^2} & -\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\beta} & -\frac{\partial^2 \ln L(x,\theta)}{\partial\alpha\partial\gamma} \\ -\frac{\partial^2 \ln L(x,\theta)}{\partial\beta\partial\alpha} & -\frac{\partial^2 \ln L(x,\theta)}{\partial\beta^2} & -\frac{\partial^2 \ln L(x,\theta)}{\partial\beta\partial\gamma} \\ -\frac{\partial^2 \ln L(x,\theta)}{\partial\gamma\partial\alpha} & -\frac{\partial^2 \ln L(x,\theta)}{\partial\gamma\partial\beta} & -\frac{\partial^2 \ln L(x,\theta)}{\partial\gamma^2} \end{pmatrix} = \begin{pmatrix} I_{11} & I_{12} & I_{13} \\ I_{21} & I_{22} & I_{23} \\ I_{31} & I_{32} & I_{33} \end{pmatrix}.$$

As next we can approximate the variance-covariance matrix V as the inversion of the observed information matrix I.

$$V = \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix} = I^{-1}.$$

It follows then that the asymptotic distribution of the MLE $(\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ is given by [9]:

$$\begin{pmatrix} \hat{\alpha} \\ \hat{\beta} \\ \hat{\gamma} \end{pmatrix} \sim N\left(\begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix}, \begin{pmatrix} V_{11} & V_{12} & V_{13} \\ V_{21} & V_{22} & V_{23} \\ V_{31} & V_{32} & V_{33} \end{pmatrix}\right). \tag{3}$$

If we replace the parameters α, β, γ by the corresponding MLE's, we get then an estimate of the variance-covariance matrix V, denoted by $\hat{V}$ and defined as follows:

$$\hat{V} = \begin{pmatrix} \widehat{I_{11}} & \widehat{I_{12}} & \widehat{I_{13}} \\ \widehat{I_{21}} & \widehat{I_{22}} & \widehat{I_{23}} \\ \widehat{I_{31}} & \widehat{I_{32}} & \widehat{I_{33}} \end{pmatrix}^{-1}, \text{ where } \widehat{I_{ij}} = I_{ij} \text{ if we replace } (\alpha, \beta, \gamma) \text{ by } (\hat{\alpha}, \hat{\beta}, \hat{\gamma}). \tag{4}$$

By using the equation (3), we can approximate $100(1-\nu)\%$ confidence intervals for the parameters α, β, γ respectively as:

$$\hat{\alpha} \pm z_{\frac{\nu}{2}} \sqrt{\widehat{V_{11}}}, \quad \hat{\beta} \pm z_{\frac{\nu}{2}} \sqrt{\widehat{V_{22}}}, \quad \hat{\gamma} \pm z_{\frac{\nu}{2}} \sqrt{\widehat{V_{33}}},$$

where $z_{\frac{\nu}{2}}$ is the upper $\frac{\nu}{2}$–th percentile of the standard normal distribution.

3.3 Simultaneous confidence region based on the likelihood ratio

As in paragraph (2.4) the likelihood ratio takes then the form:
$q = 2\left\{\ln L(x, \hat{\theta}) - \ln(x, \theta)\right\} = -2\ln\nu$, where $\hat{\theta} = (\hat{\alpha}, \hat{\beta}, \hat{\gamma})$ is the maximum likelihood estimator of the unknown parameter of interest $\theta = (\alpha, \beta, \gamma)$.
The following simultaneous confidence region will be obtained:

$$\sum_{i=1}^{N}\sum_{j=1}^{d_i} \ln\left(\frac{\hat{\alpha} + \hat{\beta}\,\hat{\gamma}\,x_{ij}^{\hat{\gamma}-1}}{\alpha + \beta\,\gamma\,x_{ij}^{\gamma-1}}\right) + \sum_{i=1}^{N}\sum_{j=1}^{d_i}\left[(\alpha - \hat{\alpha})\,x_{ij} + \left(\beta\,x_{ij}^{\gamma} - \hat{\beta}\,x_{ij}^{\hat{\gamma}}\right)\right]$$
$$+ \sum_{i=1}^{N}\left[(\alpha - \hat{\alpha})R_i + \left(\beta\,R_i^{\gamma} - \hat{\beta}\,R_i^{\hat{\gamma}}\right)\right] = -\ln\nu.$$

Remark:

1. If we fix $\gamma = \tilde{\gamma}$ we obtain then the following simultaneous confidence region:

$$\sum_{i=1}^{N}\sum_{j=1}^{d_i} \ln\left(\frac{\hat{\alpha} + \hat{\beta}\,\tilde{\gamma}\,x_{ij}^{\tilde{\gamma}-1}}{\alpha + \beta\,\tilde{\gamma}\,x_{ij}^{\tilde{\gamma}-1}}\right) + \sum_{i=1}^{N}\sum_{j=1}^{d_i}\left[(\alpha - \hat{\alpha})\,x_{ij} + \left(\beta - \hat{\beta}\right)\,x_{ij}^{\tilde{\gamma}}\right] +$$
$$\sum_{i=1}^{N}\left[(\alpha - \hat{\alpha})R_i + \left(\beta - \hat{\beta}\right)\,R_i^{\tilde{\gamma}}\right] = -\ln\nu.$$

2. By fixing $\beta = \tilde{\beta}$ we obtain the following simultaneous confidence region:

$$\sum_{i=1}^{N}\sum_{j=1}^{d_i} \ln\left(\frac{\hat{\alpha}+\tilde{\beta}\,\hat{\gamma}\,x_{ij}^{\hat{\gamma}-1}}{\alpha+\tilde{\beta}\,\gamma\,x_{ij}^{\gamma-1}}\right) + \sum_{i=1}^{N}\sum_{j=1}^{d_i}\left[(\alpha-\hat{\alpha})\,x_{ij}+\tilde{\beta}\,\left(x_{ij}^{\gamma}-x_{ij}^{\hat{\gamma}}\right)\right] + \sum_{i=1}^{N}\left[(\alpha-\hat{\alpha})R_i+\tilde{\beta}\,\left(R_i^{\gamma}-R_i^{\hat{\gamma}}\right)\right] = -\ln\nu$$

3. By fixing $\alpha = \tilde{\alpha}$ we obtain then the following simultaneous confidence region:

$$\sum_{i=1}^{N}\sum_{j=1}^{d_i} \ln\left(\frac{\tilde{\alpha}+\hat{\beta}\,\hat{\gamma}\,x_{ij}^{\hat{\gamma}-1}}{\tilde{\alpha}+\beta\,\gamma\,x_{ij}^{\gamma-1}}\right) + \sum_{i=1}^{N}\sum_{j=1}^{d_i}\left(\beta\,x_{ij}^{\gamma}-\hat{\beta}\,x_{ij}^{\hat{\gamma}}\right) + \sum_{i=1}^{N}\left(\beta\,R_i^{\gamma}-\hat{\beta}\,R_i^{\hat{\gamma}}\right) = -\ln\nu.$$

4 Numerical illustration and simulation

In this section numerical study is considered to apply the previous theoretical results to simulated lifetime data. This section is devoted to introduce numerical results based on a large simulation study. The simulation has been made by writing some computer programs with Matlab 7.

4.1 Type I censored samples without replacement

To conduct a computer simulation based on type I censored samples based on MWD we need to generate samples from the population MWD, so this enhances to develop an algorithm to archive this purpose. Thus the algorithm is a following:

Step 1. Generate a uniform distributed random variable U on (0,1)

Step 2. Set $U = F(x,\theta) = 1 - \exp(-\alpha x - \beta x^{\gamma})$, where $\theta = (\alpha,\beta,\gamma), x > 0, \gamma \geq 0, \alpha, \beta \geq 0$ such that $\alpha + \beta > 0$ and solve the following equation for $x > 0$, $\beta x^{\gamma} + \alpha x + \ln(1-U) = 0$, we obtain then x_i.

Step 3. Compare x_i with T, if $x_i < T$, then we have a failure and if $x_i \geq T$ then the item is censored at time T.

Step 4. Repeat step $1-2-3$, N times.

We obtain then the vector of failure times $X = (x_1, \cdots, x_N)$ and $(N-n)$ censored items at time T where N is given and $n = n(T)$ is a random number of observed failures.

In the following we present practical applications of theoretical results discussed in the preceding sections.

Example 1 for fixed γ
The following example illustrate confidence estimation using a small type I censored sample with size $N = 10$ censored at time $T = 0.8$ and with fixed $\gamma = \tilde{\gamma} = 2$. Let $\alpha = 0.500$, $\beta = 1.100$ The observed times of failures are:

i	1	2	3	4	5
x_i	0.001	0.773	0.668	0.612	0.330

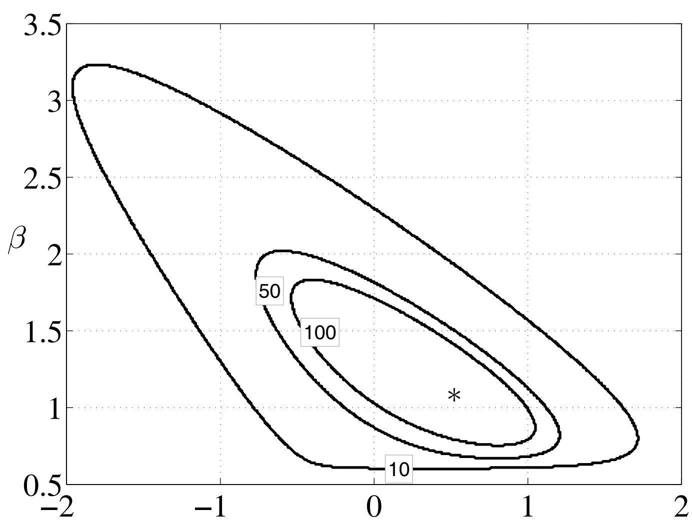
And the number of observed failures is $n = 5$. The parameter estimators are $\hat{\alpha} = 0.470$ and $\hat{\beta} = 1.050$. The estimations of the Fisher information are

$$\widehat{I_{11}}(\hat{\theta}) = 0.216, \quad \widehat{I_{22}}(\hat{\theta}) = 0.702, \quad \widehat{I_{12}}(\hat{\theta}) = -0.189.$$

Substituting the MLE of unknown parameters in the equation 4, we get then estimation of the variance covariance matrix as:

$$\hat{V} = \begin{pmatrix} 6.059 & 1.629 \\ 1.629 & 1.863 \end{pmatrix}$$

Therefore, the approximation 95% two side confidence intervals of the parameters α and β are $[-4.335, 5.292]$, $[-1.625, 3.725]$ respectively. We obtain then in figure 4.1 the confidence region of the estimators $\hat{\alpha}$ and $\hat{\beta}$ based on the likelihood ratio statistic.



Simultaneous confidence region for N=10,50,100.

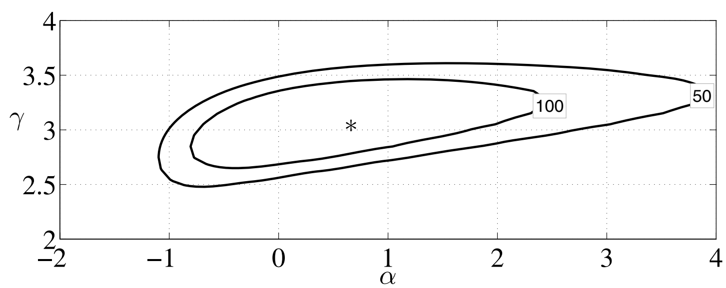
Example 2 for fixed β
The following example illustrate confidence estimation using a large type I censored sample with size $N = 50$ censored at time $T = 0.8$ and for fixed $\beta = \tilde{\beta} = 0.8$. Let $\alpha = 0.6, \gamma = 3.0$. The observed times of failures are:

i	1	2	3	4	5	6	7	8	9	10
x_i	0.485	0.315	0.099	0.601	0.559	0.299	0.638	0.325	0.238	0.718
i	11	12	13	14	15	16	17	18	19	20
x_i	0.140	0.597	0.528	0.495	0.730	0.574	0.620	0.643	0.626	0.708
i	21	22	23	24	25	26	27	28	29	30
x_i	0.327	0.765	0.372	0.596	0.585	0.470	0.397	0.403	0.657	0.583
i	31	32	33	34						
x_i	0.196	0.500	0.072	0.719						

And the number of observed failures is $n = 34$. The parameter estimators are $\hat{\alpha} = 0.6$ and $\hat{\gamma} = 3.0$. The estimations of the Fisher information are

$$\widehat{I_{11}}(\hat{\theta}) = 30.923, \quad \widehat{I_{22}}(\hat{\theta}) = 1.442, \quad \widehat{I_{12}}(\hat{\theta}) = -99.564$$

We obtain then in figure 4.1 the confidence region of the estimators $\hat{\alpha}$ and $\hat{\gamma}$ based on the likelihood ratio statistic.



Simultaneous confidence region for N=50,100.

We remark that for the case $N = 10$ the simultaneous confidence region is not closed.

Example 3 for fixed α
The following example illustrate confidence estimation using a type I censored sample with size $N = 10$ censored at time $T = 1$ and for fixed $\alpha = \tilde{\alpha} = 0.1$. Let $\beta = 2.2$, $\gamma = 1$. The observed times of failures are:

i	1	2	3	4	5	6	7
x_i	0.276	0.103	0.756	0.559	0.488	0.802	0.220

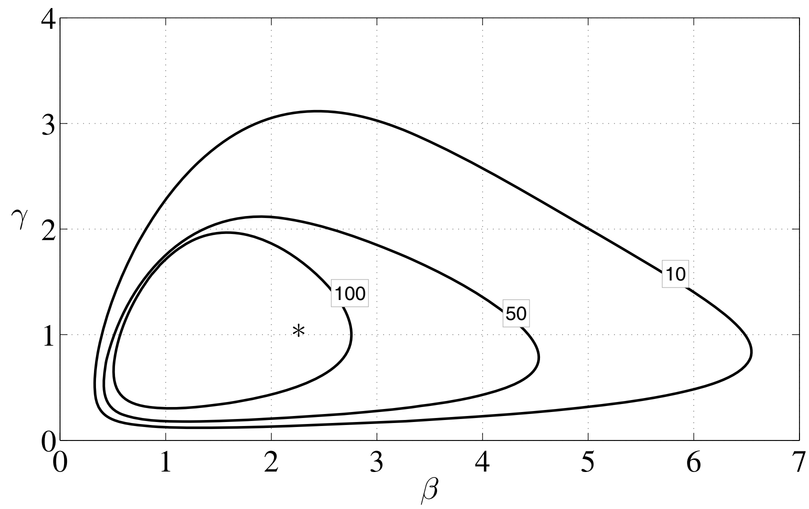
The parameter estimators are $\hat{\beta} = 2.187, \hat{\gamma} = 0.995$. The estimations of the Fisher information are

$$\widehat{I_{11}}(\hat{\theta}) = 1.023, \quad \widehat{I_{22}}(\hat{\theta}) = 0.123, \quad \widehat{I_{12}}(\hat{\theta}) = 0.184.$$

Substituting the MLE of the unknown parameters in the equation 4, we get then estimation of variance covariance matrix as:

$$\hat{V} = \begin{pmatrix} 1.338 & -2.000 \\ -2.000 & 11.094 \end{pmatrix}.$$

We obtain then in figure 4.1 the confidence region of the estimators $\hat{\beta}$ and $\hat{\gamma}$ based on the likelihood ratio statistic.



Simultaneous confidence region for N=10,50,100.

4.2 Type I censored samples with replacement

In the following we present practical applications of theoretical results discussed in the preceding sections with three examples.

Example 1 for fixed γ

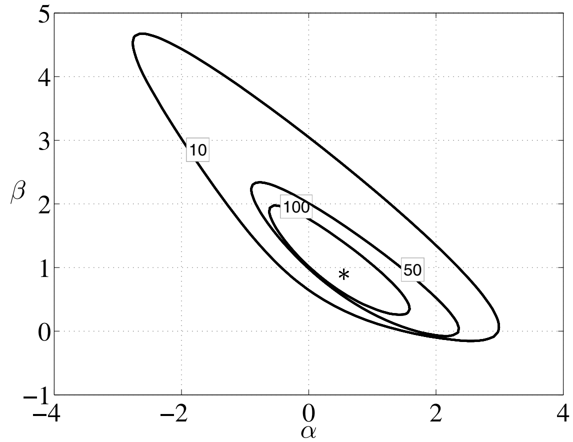
The following example illustrate confidence estimation using a type I censored sample with size $N = 10$ censored at time $T = 0.9$ by given $\gamma = \tilde{\gamma} = 1.6$. Let $\alpha = 0.400$, $\beta = 0.800$. The number of failures d_i and the rest time R_i of the i–th realization are given in the following table:

i	1	2	3	4	5	6	7	8	9	10
d_i	2	1	3	1	1	0	2	1	1	1
R_i	0.079	0.860	0.087	0.168	0.543	0.900	0.440	0.755	0.765	0.108

The parameter estimators are $\hat{\alpha} = 0.450$, $\hat{\beta} = 0.830$. The estimations of the Fisher information are:

$$\widehat{I_{11}}(\hat{\theta}) = 0.065, \quad \widehat{I_{22}}(\hat{\theta}) = 0.173, \quad \widehat{I_{12}}(\hat{\theta}) = -0.006.$$

We obtain then in figure 4.2 the confidence region of the estimators $\hat{\alpha}$ and $\hat{\beta}$ based on the likelihood ratio statistic.



Simultaneous confidence region for N=10,50,100.

Substituting the MLE of unknown parameters in the equation 4, we get then estimation of variance covariance matrix as

$$\hat{V} = \begin{pmatrix} 15.441 & 0.540 \\ 0.540 & 5.790 \end{pmatrix}.$$

Example 2 for fixed β

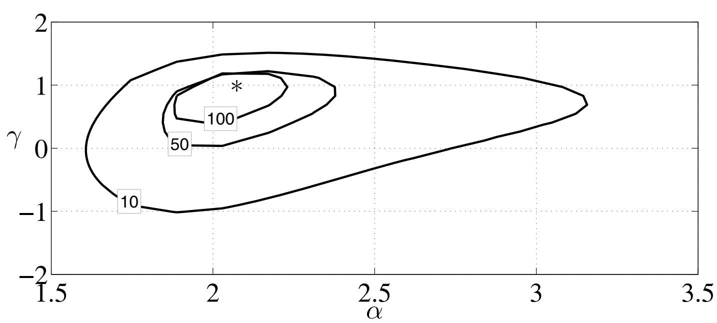
The following example illustrate confidence estimation using a type I censored sample with size $N = 10$ censored at time $T = 1.5$ and for fixed $\beta = \tilde{\beta} = 1$. Let $\alpha = 2.1$, $\gamma = 0.9$. The number of failures d_i and the rest time R_i of the i–th realization are given in the following table:

i	1	2	3	4	5	6	7	8	9	10
d_i	5	5	5	2	6	1	1	7	1	3
R_i	0.023	0.549	0.152	1.229	0.228	0.043	0.174	0.075	1.468	0.160

The parameter estimators are $\hat{\alpha}$=2.053, $\hat{\gamma}$=0.919. The estimations of the Fisher information are:

$$\widehat{I_{11}}(\hat{\theta}) = 0.271, \quad \widehat{I_{22}}(\hat{\theta}) = 0.057, \quad \widehat{I_{12}}(\hat{\theta}) = 10^{-4}.$$

We obtain then in figure 4.2 the confidence region of the estimators $\hat{\alpha}$ and $\hat{\gamma}$ based on the likelihood ratio statistic.



Simultaneous confidence region for N=10,50,100.

Substituting the MLE of unknown parameters in the equation 4, we get then estimation of variance covariance matrix as:

$$\hat{V} = \begin{pmatrix} 3.692 & -0.004 \\ -0.004 & 17.674 \end{pmatrix}.$$

Example 3 for fixed α

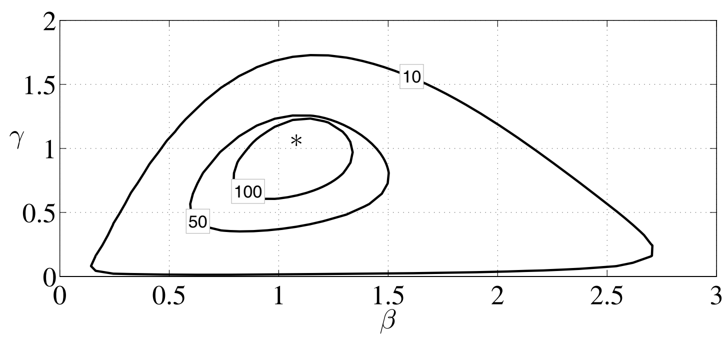
The following example illustrate confidence estimation using a type I censored sample with size $N = 10$ censored at time $T = 1.2$. $\alpha = \tilde{\alpha} = 0.5$. Let $\beta = 1.1$, $\gamma = 1$. The number of failures d_i and the rest time R_i of the i–th realization are given in the following table:

i	1	2	3	4	5	6	7	8	9	10
d_i	0	2	1	0	1	2	4	2	0	2
R_i	1.2	0.718	0.924	1.200	0.796	1.073	0.090	0.460	1.200	0.218

The parameter estimators are $\hat{\beta} = 1.050$, $\hat{\gamma} = 1.030$. The estimations of the Fisher information are

$$\widehat{I_{11}}(\hat{\theta}) = 0.174, \quad \widehat{I_{22}}(\hat{\theta}) = 0.095, \quad \widehat{I_{12}}(\hat{\theta}) = 0.005.$$

We obtain then in figure 4.2 the confidence region of the estimators $\hat{\beta}$ and $\hat{\gamma}$ based on the likelihood ratio statistic.



Simultaneous confidence region for N=10,50,100.

Substituting the MLE of unknown parameters in the equation 4, we get then estimation of variance covariance matrix as:

$$\hat{V} = \begin{pmatrix} 5.763 & -0.275 \\ -0.275 & 10.503 \end{pmatrix}.$$

5 Application

In this section we provide a data analysis to investigate how the MWD model works in practice and to illustrate the modeling and estimation procedure. We present a practical application of the theoretical results considered in the preceding sections. The failure time data have been obtained from [1] giving the lifetime of 50 devices. This data are presented in table 1.

0.1	0.2	1	1	1	1	1	2	3 6	7	11	12	18	18	18	18	18
21	32	36	40	45	46	47	50	55 60	63	63	67	67	67	67	72	75
79	82	82	83	84	84	84	85	85 85	85	85	86	86				

Failure time data from Aarset

We assume a type 1 censored sample without replacement. If we choose $T = 26.4$ we obtain the following number of observed failure $n = 19$. To analyze the data we have used the following distributions: Exponential distribution (ED), Rayleigh distribution (RD), Weibull distribution (WD), linear failure rate distribution (LFRD) and modified Weibull distribution (MWD). Also we compared these distributions to fit the data by using the mean square of the difference between the empirical cdf and the fitted cdf, denoted MSD:

$$MSD = \frac{1}{n}\sum_{i=1}^{n}(\hat{F}_i - FE_i)^2.$$

Where $\hat{F}_i$ and FE_i are the empirical and the estimated cdf calculated at x_i. The estimated cdf calculated by replacing the parameters of the model adopted with their estimations. The obtained results are given in table 2.

The model	**MLE of the parameter(s)**	**MSD**
WD (β, γ)	$\hat{\beta} = 0.0078$, $\hat{\gamma} = 0.2993$	0.4069
LFRD (α, β)	$\hat{\alpha} = -0.0002$, $\hat{\beta} = 0.0008$	0.2795
ED (α)	$\hat{\alpha} = 0.0195$	0.2265
RD (β)	$\hat{\beta} = NaN$	—
MWD (α, β, γ)	$\hat{\alpha} = 0.0133$, $\hat{\beta} = 0.0438$, $\hat{\gamma} = 0.2911$	0.2160

Results

The results shown in table 5 indicate that the model of the MWD fits the given data better than all other models tested above.
Substituting the MLE of the unknown parameters in (2), we obtain then estimation of the variance covariance matrix $\hat{V}$ as

$$\hat{V} = \begin{pmatrix} 4.0132 \times 10^{-5} & -1.4160 \times 10^{-4} & -5.7391 \times 10^{-4} \\ -1.4160 \times 10^{-4} & 1.1688 \times 10^{-3} & 1.8814 \times 10^{-3} \\ -5.7391 \times 10^{-4} & 1.8814 \times 10^{-3} & 2.7716 \times 10^{-2} \end{pmatrix}.$$

Therefore, if we use the equation (1), the approximation 95% two side confidence intervals of the parameters α , β and γ are $[0.0883 \times 10^{-2}, 0.0257]$, $[-0.0232, 0.1108]$ and $[-0.0352, 0.6174]$ respectively.

6 Conclusion

In this paper we discussed the parameter estimation of the MWD(α, β, γ) based on type I censored samples with replacements and without replacements. The maximum likelihood estimators and the confidence regions based on the likelihood ratio statistics are obtained. Based on the previous analysis we can remark that the simultaneous confidence region based on the likelihood ratio statistics will be smaller with increasing sample size N in both cases with replacements and without replacements. Furthers, simulation studies are developed to investigate the parameter estimations of the MWD. Confidence estimations based on the asymptotic χ^2–distribution of the likelihood ratio statistics will be found. Based on the MSD criteria, we found that the MWD model concord the data better than those compared distributions.

References

[1] M. V. Aarset, How to identify bathtub hazard rate. *IEEE transaction of reliability* **36** (1987), 106 - 108.

[2] P. K. Andersen and O. Borgan, Counting process models for life history data. *Scandinavian Journal of Statistics* **12** (1985), 97 - 158.

[3] O. Borgan, Maximum likelihood estimation in parametric counting process models, with applications to censored failure time data. *Scandinavian Journal of Statistics* **11** (1984), 1-16.

[4] T.F. Coleman and Y. Li, An Interior, Trust Region Approach for Nonlinear Minimization Subject to Bounds. *SIAM Journal on Optimization*, (1996), 418-445.

[5] G. M. Corderio, On the corrections to the likelihood ratio statistics. *Biometrika* **74** (1987), 265-274.

[6] W. Kahle, Estimation of the parameters of the Weibull distribution for censored samples. *Metrika* **44** (1996), 27-40.

[7] J. F. Lawless, *Statistical models and methods for lifetime Data,* John Wiley and Sons,New York, 2003.

[8] R. S. Lipster and A. N. Shiryayev *Statistics of random processes II,* Springer-Verlag, New York, 2000.

[9] J. R. G. Miller, *Survival Analysis*. John Wiley, New York 1981.

[10] A. M. Sarhan and M. Zaindin, Modified Weibull distribution, *Applied Sciences* **11** (2009), 123-136.

[11] A. Svenson, Asymptotic estimation in counting processes with parametric intensities based on one realization, *Scandinavian Journal of Statistics* **17** (1990), 23-33.

[12] M. Zaindin and A. M. Sarhan, Parameters estimation of the modified Weibull distribution, *Applied Mathematical Sciences* **11** (2009), 541-550.

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