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Some Scalar Difference Inequalities

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Abstract

In this paper we prove some scalar difference inequalities using the concept of under and over function with respect to solution of initial value problem.

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1 Introduction

In the recent year the theory and applications of difference equations are found to more useful in the engineering field. Agarwal [1], Kelley and Peterson [7] developed the theory of difference equations and difference inequalities. Some comparison results are obtained by K.L. Bondar, V.C. Borkar, S.T. Patil [2,3,4] and Eloe [6]. Some differential and integral inequalities are given in [10].

Let $J_0 = \{t_0, t_1, ..., t_0 + a\},\ t_0 \in R$ and E be an open subset of R.

Consider the difference equation

$$\Delta u(t) = g(t, u(t));\ \ u(t_0) = u_0 \tag{1}$$

where $u_0 \in E$, $t \in J$, $u : J \to R$ with $u(t) \in E$ and $g : J \times E \to R$.
A function $\phi : J \to R$ is said to be a solution of initial value problem (1), if it satisfies

$$\Delta\phi(t) = g(t, \phi(t));\ \ \phi(t_0) = u_0$$

The initial value problem is equivalent to the problem

$$u(t) = u_0 + \sum_{s=t_0}^{t-1} g(s, u(s)).$$

By summation convetion $\sum_{s=t_0}^{t_0-1} g(s,u(s)) = 0$ and so $u(t)$ given above is the solution of (1).

Now we define the under and over functions of (1).

Definition 1.1 *A function $v : J \to R$ satisfying the difference inequality*

$$\Delta v(t) < g(t, v(t)),\ t \in J$$

is said to an under function with respect to initial value problem (1).

Definition 1.2 *A function $w : J \to R$ satisfying the difference inequality*

$$\Delta w(t) > g(t, w(t)),\ t \in J$$

is said to an over function with respect to initial value problem (1).

In the present paper, some inequalities are obtained corresponding to the solution of initial value problem (1).

2 Main Results

Theorem 2.1 *Let $v, w : J \to R$ and $t \in J$. Suppose that for $t \in J$,*

$$v(t_0) < w(t_0), \tag{2}$$

and for $t > t_0$, the inequalities

$$\Delta v(t) \le g(t, v(t)), \tag{3}$$

$$\Delta w(t) > g(t, w(t)) \tag{4}$$

hold. Then

$$v(t) < w(t),\ t \in J. \tag{5}$$

Proof: Suppose (5) is false, then the set

$$Z = \{t \in J : w(t) \le v(t)\}$$

is nonempty. Defining $t_1 = \inf Z$, it is clear from (2) that $t_0 < t_1$. Furthermore

$$v(t_1) = w(t_1) \tag{6}$$

and

$$v(t) \le w(t),\ t \in [t_0, t_1]. \tag{7}$$

Using (6) and (7), we obtain,

$$v(t_1+1)-v(t_1) \geq w(t_1+1)-w(t_1)$$
$$i.e.\ \Delta v(t_1) \geq \Delta w(t_1). \tag{8}$$

The inequalities (3), (4) and (8) togegher with (6) gives us,

$$g(t_1, v(t_1)) \geq \Delta v(t_1) \geq \Delta w(t_1) > g(t_1, w(t_1))$$

which is contradition as $v(t_1) = w(t_1)$. Thus Z is empty. Hence the statement (5) is true.

Remark 2.1: It is obvious from the proof that the inequalities (3) and (4) can also be repplaced by

$$\Delta v(t) < g(t, v(t)),$$
$$\Delta w(t) \geq g(t, w(t))$$

respectively.

We require following simple lemmas.

Lemma 2.2 *Suppose that* $u : J \to R$ *and* $\Delta u(t) \leq 0$ *for* $t \in J - S$, S *is an atmost countable subset of* J*. Then* $u(t)$ *is nonincreasing in* t *on* J*.*

Proof: We have $\Delta u(t) \leq 0$ for $t \in J - S$, i.e. $u(t+1) \leq u(t)$ for $t \in J - S$. Thus $u(t)$ in nonincreasing for $t \in J - S$. Therefore $u(t+1) \leq u(t)$ for $t \in J$ also. i.e.$u(t)$ is nonincreasing in t on J.

Lemma 2.3 *Let* $u, v : J \to R$, S *be an atmost countable subset of* J *and* $\Delta v(t) \leq w(t)$ *for* $t \in J - S$*. Then* $\Delta v(t) \leq w(t)$ *for* $t \in J$*.*

Proof: Define the function

$$m(t) = v(t) - \sum_{s=t_0}^{t-1} w(s).$$

It then follows from the assumption that

$$\Delta m(t) = \Delta v(t) - w(t) \leq 0,\ \ t \in J - S.$$

Hence by Lemma (2.2), $m(t)$ is nonincreasing in t on J. Consequently,

$$\Delta m(t) = \Delta v(t) - w(t) \leq 0,\ \ t \in J.$$

Hence proved.

Remark: In the light of Lemma 2.3, it is clear that Theorem 2.1 remains true when the inequalities (3) and (4) hold for $t \in J - S$.

It will now be shown that any solution of initial value problem (1) can be bracketed between under and over functions.

Theorem 2.4 *Let $v(t), w(t)$ be under and over functions with respect to initial value problem (1), respectively, on J. If $u(t)$ is any solution of (1) existing on J such that*

$$v(t_0) = u_0 = w(t_0), \tag{9}$$

then

$$v(t) < u(t) < w(t),\ t \in J. \tag{10}$$

Proof: Let $w(t)$ and $u(t)$ be an over function and a solution of (1), respectively. Let $m(t) = w(t) - u(t)$. Then

$$\begin{aligned} \Delta m(t_0) &= \Delta w(t_0) - \Delta u(t_0) \\ &> g(t_0, w(t_0)) - g(t_0, u(t_0)) \\ &= g(t_0, u_0) - g(t_0, u_0) \\ &= 0. \end{aligned}$$

Thus $m(t)$ is increasing for $t > t_0$. This implies that $u(t_0 + 1) < w(t_0 + 1)$. Furthermore

$$\Delta u(t) \leq g(t, u(t))$$

and

$$\Delta w(t) > g(t, w(t))$$

for $t > t_0$ in J. Thus from Theorem 2.1 we get,

$$u(t) < w(t),\ t \in J.$$

Similarly let $v(t)$ and $u(t)$ be an under function and a solution of (1). Let $n(t) = v(t) - u(t)$. Then

$$\begin{aligned} \Delta n(t_0) &= \Delta v(t_0) - \Delta u(t_0) \\ &< g(t_0, v(t_0)) - g(t_0, u(t_0)) \\ &= g(t_0, u_0) - g(t_0, u_0) \\ &= 0. \end{aligned}$$

Thus $n(t)$ is decreasing for $t > t_0$. This implies that $u(t_0 + 1) > v(t_0 + 1)$. Furthermore

$$\Delta u(t) \geq g(t, u(t))$$

and

$$\Delta v(t) < g(t, v(t))$$

for $t > t_0$ in J. Thus from Theorem 2.1 we get,

$$u(t) > v(t),\ t \in J.$$

Corollary 2.5 *Let $g_1, g_2 : E \to R$ and*

$$g_1(t, u) < g_2(t, u),$$

where $u : J \to R$. Let $u_1(t), u_2(t)$ be any two solutions of

$$\Delta u_1(t) = g_1(t, u),\ \Delta u_2(t) = g_2(t, u),$$

respectively, on J such that $u_1(t_0) < u_2(t_0)$. Then

$$u_1(t) < u_2(t),\ t \in J.$$

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