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The Solitary Wave Solutions of Zoomeron Equation

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Abstract

In this paper, we construct the explicit traveling wave solutions of a incognito evolution equation, that called Zoomeron equation, by using the $(\frac{G'}{G})$–expansion method. By using this method, new exact solutions involving parameters, expressed by three types of functions which are hyperbolic, trigonometric and rational function solutions, are obtained.

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1 Introduction

Nonlinear evolution equations (NLEEs) have been the subject of study in various branches of mathematical-physical sciences such as physics, biology, chemistry, etc. The analytical solutions of such equations are of fundamental importance since a lot of mathematical-physical models are described by NLEEs. Among the possible solutions to NLEEs, certain special form solutions may depend only on a single combination of variables such as solitons. In mathematics and physics, a soliton is a self reinforcing solitary wave, a wave packet or pulse, that maintains its shape while it travels at constant speed. Solitons are caused by a cancelation of nonlinear and dispersive effects in the medium. The term "dispersive effects" refers to a property of certain systems where the speed of the waves varies according to frequency. Solitons arise as the solutions of a widespread class of weakly nonlinear dispersive partial differential equations describing physical systems. The soliton phenomenon was first described by *John Scott Russell* (1808-1882) who observed a solitary wave in the Union Canal in Scotland. He reproduced the phenomenon in a wave tank and named it the "Wave of Translation"[1]. The soliton solutions are typically obtained by means of the inverse scattering transform [2] and owe their stability to the integrability of the field equations. In the past years, many other powerful and direct methods have been developed to find special solutions of nonlinear evolution equations (NEE(s)), such as the Backlund transformation [3], Hirota bilinear method [4], numerical methods [5] and the Wronskian determinant technique [6]. With the help of the computer software, some other algebraic method proposed, such as tanh/coth method [7], F–expanded method [8], homogeneous balance method [9], Jacobi elliptic function method [10], the Miura transformation [11], sine/cosine method [12] and Exp–function method [13]. But, most of the methods may sometimes fail or can only lead to a kind of special solution and the solution procedures become very complex as the degree of nonlinearity increases.

Recently, the $(\frac{G'}{G})$–expansion method, firstly introduced by Wang et al. [14], has become widely used to search for various exact solutions of NLEEs [14]–[19]. The value of the $(\frac{G'}{G})$–expansion method is that one treats nonlinear problems by essentially linear methods. The method is based on the explicit linearization of NLEEs for traveling waves with a certain substitution which leads to a second–order differential equation with constant coefficients. Moreover, it transforms a nonlinear equation to a simple algebraic computation.

Our aim in this letter is to present an application of the $(\frac{G'}{G})$–expansion method to Zoomeron equation that is solved by this method for the first time:

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0, \tag{1}$$

where $u(x,y,t)$ is the amplitude of the relevant wave mode. This equation is one of incognito evolution equation. According to our recent search, there are a few article about this equation. We only know that this equation was introduced by Calogero and Degasperis [20].

2 Description of the $(\frac{G'}{G})$–expansion method

The objective of this section is to outline the use of the $(\frac{G'}{G})$–expansion method for solving certain nonlinear partial differential equations (PDEs). Suppose we have a nonlinear PDE for $u(x,y,t)$, in the form

$$P(u, u_x, u_y, u_t, u_{xx}, u_{x,y}, u_{x,t}, u_{yy}, ...) = 0, \tag{2}$$

where P is a polynomial in its arguments, which includes nonlinear terms and the highest order derivatives. The transformation $u(x,y,t) = U(\xi), \xi = x - cy - \omega t$, reduces Eq. (2) to the ordinary differential equation (ODE)

$$P(U, U', -cU', -\omega U'', U'', -\omega U'', ...) = 0, \tag{3}$$

where $U = U(\xi)$, and prime denotes derivative with respect to ξ. We assume that the solution of Eq. (3) can be expressed by a polynomial in $(\frac{G'}{G})$ as follows:

$$U(\xi) = \sum_{n=1}^{m} \alpha_n \Big(\frac{G'}{G}\Big)^n + \alpha_0, \quad \alpha_m \neq 0. \tag{4}$$

where α_n, $n = 0, 1, 2, ..., m$, are constants to be determined later and $G(\xi)$ satisfies a second order linear ordinary differential equation (LODE):

$$\frac{d^2G(\xi)}{d\xi^2} + \lambda\frac{dG(\xi)}{d\xi} + \mu G(\xi) = 0. \tag{5}$$

where λ and μ are arbitrary constants. Using the general solutions of Eq. (5), we have

$$\frac{G'(\xi)}{G(\xi)} = \begin{cases} \frac{\sqrt{\lambda^2-4\mu}}{2}\left(\frac{C_1\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}{C_1\cosh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)+C_2\sinh(\frac{\sqrt{\lambda^2-4\mu}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu > 0, \\ \frac{\sqrt{4\mu-\lambda^2}}{2}\left(\frac{-C_1\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}{C_1\cos(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)+C_2\sin(\frac{\sqrt{4\mu-\lambda^2}}{2}\xi)}\right) - \frac{\lambda}{2}, & \lambda^2 - 4\mu < 0, \end{cases} \tag{6}$$

and it follows, from (4) and (5), that

$$U' = -\sum_{n=1}^{m} n\ \alpha_n\Big((\tfrac{G'}{G})^{n+1} + \lambda(\tfrac{G'}{G})^n + \mu(\tfrac{G'}{G})^{n-1}\Big), \tag{7}$$

$$U'' = \sum_{n=1}^{m} n\ \alpha_n\Big((n+1)(\tfrac{G'}{G})^{n+2} + (2n+1)\lambda(\tfrac{G'}{G})^{n+1} + n(\lambda^2+2\mu)(\tfrac{G'}{G})^n \tag{8}$$
$$+(2n-1)\lambda\mu(\tfrac{G'}{G})^{n-1} + (n-1)\mu^2(\tfrac{G'}{G})^{n-2}\Big),$$

(9)and so on, here the prime denotes the derivative with respective to ξ. To determine u explicitly, we take the following four steps:

Step 1. Determine the integer m by substituting Eq. (4) along with Eq. (5) into Eq. (3), and balancing the highest order nonlinear term(s) and the highest order partial derivative.

Step 2. Substitute Eq. (4) give the value of m determined in *Step 1*, along with Eq. (5) into Eq. (3) and collect all terms with the same order of $(\frac{G'}{G})$ together, the left-hand side of Eq. (3) is converted into a polynomial in $(\frac{G'}{G})$. Then set each coefficient of this polynomial to zero to derive a set of algebraic equations for $c, \omega, \lambda, \mu, \alpha_n$ for $n = 0, 1, 2, ..., m$.

Step 3. Solve the system of algebraic equations obtained in *Step 2*, for $c, \omega, \lambda, \mu, \alpha_0, ..., \alpha_m$ by use of Maple.

Step 4. Use the results obtained in above steps to derive a series of fundamental solutions $u(\xi)$ of Eq. (3) depending on $(\frac{G'}{G})$, since the solutions of Eq. (5) have been well known for us, then we can obtain exact solutions of Eq. (2).

3 Application on Zoomeron equation

In this section, we will use our method to find solutions to Zoomeron equation [20]:

$$\left(\frac{u_{xy}}{u}\right)_{tt} - \left(\frac{u_{xy}}{u}\right)_{xx} + 2(u^2)_{xt} = 0, \tag{10}$$

We would like to use our method to obtain more general exact solutions of Eq (10) by assuming the solution in the following frame:

$$u = U(\xi), \quad \xi = x - cy - \omega t, \tag{11}$$

where c, ω are constants. We substitute Eq. (11) into Eq. (10) and integrating twice with respect to ξ, by setting the second integration constant equal to zero, we obtain the following nonlinear ordinary differential equation

$$c(1-\omega^2)U'' - 2\omega U^3 - \mathcal{R}U = 0 \tag{12}$$

where $\mathcal{R}$ is integration constant.

According to *Step 1*, we get $m+2 = 3m$, hence $m = 1$. We then suppose that Eq. (12) has the following formal solutions:

$$U = \alpha_1(\frac{G'}{G}) + \alpha_0, \qquad \alpha_1 \neq 0, \tag{13}$$

where α_1 and α_0, are unknown to be determined later.

Substituting Eq. (13) into Eq. (12) and collecting all terms with the same order of $(\frac{G'}{G})$, together, the left-hand sides of Eq. (12) are converted into a polynomial in $(\frac{G'}{G})$. Setting each coefficient of each polynomial to zero, we derive a set of algebraic equations for $c, \omega, \lambda, \mu, \alpha_0$, and α_1, as follows:

$$(\frac{G'}{G})^0: \quad c\lambda\,\mu\,(\omega-1)\,(\omega+1)\,\alpha_1 + \alpha_0\,(2\,{\alpha_0}^2\omega + \mathcal{R}) = 0, \tag{14}$$

$$(\frac{G'}{G})^1: \quad 2\,\mu\,c + \lambda^2 c - 2\,\mu\,\omega^2 c - \lambda^2\omega^2 c - 6\,{\alpha_0}^2\omega - R = 0, \tag{15}$$

$$(\frac{G'}{G})^2: \quad 2\,\omega\,\alpha_0\alpha_1 + c\lambda\,(\omega-1)\,(\omega+1) = 0, \tag{16}$$

$$(\frac{G'}{G})^3: \quad \omega\,{\alpha_1}^2 + c\,(\omega-1)\,(\omega+1) = 0,$$

(17)and solving by use of Maple, we get the following results:

$$\left\{\mu = \frac{1}{4}\,\frac{c\lambda^2(1-\omega^2) + 2\,R}{c\,(1-\omega^2)},\ \alpha_0 = \mp\frac{1}{2}\,\frac{c\lambda\,(\omega^2-1)}{\omega\sqrt{\frac{c(1-\omega^2)}{\omega}}},\ \alpha_1 = \pm\sqrt{\frac{c(1-\omega^2)}{\omega}}\right\} \tag{18}$$

where $c \neq 0, \omega \neq 0, 1$, and λ are arbitrary constant and $\mathcal{R}$ is integration constant. Therefore, substitute the above case in (13), we get

$$U = \pm\sqrt{\frac{c(1-\omega^2)}{\omega}}\,(\frac{G'}{G}) \mp \frac{1}{2}\,\frac{c\lambda\,(\omega^2-1)}{\omega\sqrt{\frac{c(1-\omega^2)}{\omega}}}, \tag{19}$$

Substituting the general solutions (6) into Eq. (19), we obtain three types of traveling wave solutions of Eq. (10) in view of the positive, negative or zero of $\lambda^2 - 4\mu$.

When $\mathcal{D} = \lambda^2 - 4\mu = \frac{1}{2}\sqrt{\frac{2R}{c(\omega^2-1)}} > 0$, using the relationship (19), we obtain hyperbolic function solution $u_{\mathcal{H}}$, of Zoomeron equation (10) as follows:

$$u_{\mathcal{H}} = \pm\sqrt{\frac{c(1-\omega^2)}{\omega}}\left(\frac{\sqrt{\mathcal{D}}}{2}\Big(\frac{C_1\sinh(\frac{\sqrt{\mathcal{D}}}{2}\,\xi)+C_2\cosh(\frac{\sqrt{\mathcal{D}}}{2}\,\xi)}{C_1\cosh(\frac{\sqrt{\mathcal{D}}}{2}\,\xi)+C_2\sinh(\frac{\sqrt{\mathcal{D}}}{2}\,\xi)}\Big)-\frac{\lambda}{2}\right) \mp \frac{c\lambda\,(\omega^2-1)}{2\omega\sqrt{\frac{c(1-\omega^2)}{\omega}}}, \tag{20}$$

where $\xi = x - c\,y - \omega\,t$, and C_1, C_2, are arbitrary constants. It is easy to see that the hyperbolic solution (20) can be rewritten at $C_1^2 > C_2^2$, as follows

$$u_{\mathcal{H}}(x,y,t) = \mp\frac{1}{2}\sqrt{\frac{-2\mathcal{R}}{\omega}}\tanh\Big(-\frac{1}{2}\sqrt{\frac{2\mathcal{R}}{c(\omega^2-1)}}\,(x-cy-\omega t)-\eta_{\mathcal{H}}\Big), \tag{21a}$$

while at $C_1^2 < C_2^2$, one can obtain

$$u_{\mathcal{H}}(x,y,t) = \mp\frac{1}{2}\sqrt{\frac{-2\mathcal{R}}{\omega}}\coth\Big(-\frac{1}{2}\sqrt{\frac{2\mathcal{R}}{c(\omega^2-1)}}\,(x-cy-\omega t)-\eta_{\mathcal{H}}\Big), \tag{21b}$$

where $\eta_{\mathcal{H}} = \tanh^{-1}\left(\frac{C_1}{C_2}\right)$, and $c \neq 0, \omega \neq 0, 1$, are arbitrary and $\mathcal{R}$ is integration constant.

Now, when $\mathcal{D} = \lambda^2 - 4\mu = \frac{1}{2}\sqrt{\frac{2R}{c(\omega^2-1)}} < 0$, using the relationship (19), we obtain trigonometric function solution $U_{\mathcal{T}}$, of Zoomeron equation (10) as follows:

$$u_{\mathcal{T}} = \pm\sqrt{\frac{c(1-\omega^2)}{\omega}}\left(\frac{\sqrt{-\mathcal{D}}}{2}\Big(\frac{-C_1\sin(\frac{\sqrt{-\mathcal{D}}}{2}\,\xi)+C_2\cos(\frac{\sqrt{-\mathcal{D}}}{2}\,\xi)}{C_1\cos(\frac{\sqrt{-\mathcal{D}}}{2}\,\xi)+C_2\sin(\frac{\sqrt{-\mathcal{D}}}{2}\,\xi)}\Big)-\frac{\lambda}{2}\right) \mp \frac{c\lambda\,(\omega^2-1)}{2\omega\sqrt{\frac{c(1-\omega^2)}{\omega}}},$$

(22)where $\xi = x - cy - \omega t$, and C_1, C_2, are arbitrary constants. Similarity, it is easy to see that the trigonometric solution (22) can be rewritten at $C_1^2 > C_2^2$, and $C_1^2 < C_2^2$, as follows

$$u_{\mathcal{T}}(x,y,t) = \mp\frac{1}{2}\sqrt{\frac{-2\mathcal{R}}{\omega}}\tan\Big(-\frac{1}{2}\sqrt{\frac{-2\mathcal{R}}{c(\omega^2-1)}}\,(x-cy-\omega t)-\eta_{\mathcal{T}}\Big), \tag{23a}$$

and

$$u_{\mathcal{T}}(x,y,t) = \mp\frac{1}{2}\sqrt{\frac{-2\mathcal{R}}{\omega}}\cot\Big(-\frac{1}{2}\sqrt{\frac{-2\mathcal{R}}{c(\omega^2-1)}}(x-cy-\omega t)-\eta_{\mathcal{T}}\Big), \quad (23b)$$

respectively, where $\eta_{\mathcal{T}} = \tan^{-1}\left(\frac{C_1}{C_2}\right)$, and $c \neq 0, \omega \neq 0, 1$, are arbitrary and $\mathcal{R}$ is integration constant.

And finally, when $\mathcal{D} = \lambda^2 - 4\mu = 0$, we obtain following rational function solution of Zoomeron equation (10)

$$u_{rat}(x,y,t) = \mp\frac{c(\omega^2-1)C_2}{\omega\left(C_1 + C_2(x-cy-\omega t)\right)\sqrt{-\frac{c(\omega^2-1)}{\omega}}}. \quad (24)$$

where $C_1, C_2, c \neq 0$, and $\omega \neq 0, 1$, are arbitrary constants.

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