Applied Mathematical Sciences, Vol. 5, 2011, no. 56, 2775 - 2779

Integral Convergence Related to Weak Convergence of Measures

Xiangfeng Yang

Department of Mathematics, Tulane University
xyang2@wave.tulane.edu

Abstract

We consider probability measures μ_n, μ on a metric space X such that μ_n weakly converges to μ. The following convergence

$$\lim_{n\to\infty} \int_X f_n(x)\mu_n(dx) = \int_X f(x)\mu(dx)$$

is proved under some restrictions on real valued functions f_n and f which are measurable, not necessarily continuous nor bounded.

Mathematics Subject Classification: 60F05

Keywords: weak convergence; integral convergence

1 Introduction

Recall that a sequence of probability measures $\{\mu_n, n = 1, 2, \cdots\}$ on a σ-algebra $\mathcal{B}_X$ of Borel subsets of metric space X *weakly converges* to a probability measure μ on $\mathcal{B}_X$ if for any bounded continuous function f on X,

$$\lim_{n\to\infty} \int_X f(x)\mu_n(dx) = \int_X f(x)\mu(dx).$$

It is often that we need to deal with the following convergence

$$\lim_{n\to\infty} \int_X f_n(x)\mu_n(dx) = \int_X f(x)\mu(dx), \tag{1}$$

for some f_n and f. Limit (1) can be trivially derived if f_n uniformly converges to f on whole X. But the condition that uniform convergence of f_n to f on whole space X is so strong that it can be hardly applied.

This short note presents several criterions for (1) in which uniform convergence of f_n to f is assumed only on every compact (or bounded) subset of X (see Theorem 2.2 and Theorem 2.1). Section 2 also gives a theorem for which (1) holds where a condition $\mu(E) = 0$ is assumed instead of uniform convergence of f_n to f. The proofs are shown in section 3.

2 Main Results

In this section and in sequel, we consider a metric space $(X, \mathcal{B}_X)$ where $\mathcal{B}_X$ is the Borel σ-algebra generated by all open sets, and probability measures μ_n, μ on this metric space with μ_n weakly converging to μ.

Theorem 2.1. *Assume $f_n(x), f(x)$ are real valued measurable functions defined on X such that*

(i) $f_n(x)$ converge to $f(x)$ uniformly on every bounded set as $n \to \infty$;

(ii) $f_n(x)$ are bounded on every bounded set;

(iii) $\lim_{C\to+\infty} \sup_n \int_{\{|f_n(x)|>C\}} |f_n(x)|\mu_n(dx) = 0$;

(iv) $f(x)$ is continuous almost everywhere with respect to μ.

Then

$$\lim_{n\to\infty} \int_X f_n(x)\mu_n(dx) = \int_X f(x)\mu(dx).$$

Theorem 2.2. *Assume $f_n(x), f(x)$ are real valued measurable functions defined on X such that*

(I) $f_n(x)$ converge to $f(x)$ uniformly on every compact set as $n \to \infty$;

(II) $f_n(x)$ are bounded on every compact set;

(III) $\lim_{C\to+\infty} \sup_n \int_{\{|f_n(x)|>C\}} |f_n(x)|\mu_n(dx) = 0$;

(IV) $f(x)$ is continuous almost everywhere with respect to μ.

Then

$$\lim_{n\to\infty} \int_X f_n(x)\mu_n(dx) = \int_X f(x)\mu(dx).$$

Theorem 2.3. *Assume $f_n(x), f(x)$ are real valued measurable functions defined on X with $\lim_{C\to+\infty} \sup_n \int_{|f_n(x)|>C} |f_n(x)|\mu_n(dx) = 0$. Define the set*

$$E = \left\{ x \in X : \text{ there are } x_n \in X \text{ with } \lim_{n\to\infty} x_n = x, \text{ but } \lim_{n\to\infty} f_n(x_n) \neq f(x) \right\}.$$

If $\mu(E) = 0$, then

$$\lim_{n\to\infty} \int_X f_n(x)\mu_n(dx) = \int_X f(x)\mu(dx).$$

3 Proofs

Let's first prove Theorem 2.3. Essentially, Theorem 2.1 and Theorem 2.2 are two special cases of Theorem 2.3 (see the proofs of these two theorems at the end of this section). Before the proof, we recall two results in [1].

Lemma 3.1 (Theorem 5.2 in [1]). *Let P_n, P be probability measures defined on on metric space $(X, \mathcal{B}_X)$ such that P_n weakly converges to P, and*

$h(x)$ be a real valued, bounded, measurable function with property that it is continuous almost surely with respect to P. then

$$\lim_{n\to\infty}\int_X h(x)P_n(dx) = \int_X h(x)P(dx). \tag{2}$$

Lemma 3.2 (Theorem 5.5 in [1]). *Let P_n, P be probability measures defined on metric space $(X, \mathcal{B}_X)$ such that P_n weakly converges to P, and $h_n(x), h(x)$ be functions with values in a separable metric space $(X', \mathcal{B}_{X'})$ satisfying $P(E) = 0$, where*

$$E = \left\{x \in X : \text{ there are } x_n \in X \text{ with } \lim_{n\to\infty} x_n = x, \text{ but } \lim_{n\to\infty} h_n(x_n) \neq h(x)\right\}.$$

Then on metric space $(X', \mathcal{B}_{X'})$,

$$P_n \circ h_n^{-1} \text{ weakly converges to } P \circ h^{-1}.$$

Proof of Theorem 2.3. From conditions of theorem and Lemma 3.2, we first get

$$\mu_n \circ f_n^{-1} \text{ weakly converges to } \mu \circ f^{-1} \text{ on real line.} \tag{3}$$

Claim 1: any constant $C > 0$ (except for countably many C)

$$(a). \quad \lim_{n\to\infty}\int_{\{|f_n(x)|\le C\}} f_n(x)\mu_n(dx) = \int_{\{|f(x)|\le C\}} f(x)\mu(dx);$$

$$(b). \quad \lim_{n\to\infty}\int_{\{|f_n(x)|\le C\}} |f_n(x)|\mu_n(dx) = \int_{\{|f(x)|\le C\}} |f(x)|\mu(dx).$$

To prove part (a) of Claim 1, invent a real valued function g defined on real,

$$g(y) = \begin{cases} y & \text{if } |y| \le C, \\ 0 & \text{if } |y| > C, \end{cases}$$

and from (2) and (3), if $\mu \circ f^{-1}\{-C, C\} = 0$, then

$$\lim_{n\to\infty}\int_R g(y)\mu_n \circ f_n^{-1}(dy) = \int_R g(y)\mu \circ f^{-1}(dy),$$

which is the same as

$$\lim_{n\to\infty}\int_X g\left(f_n(x)\right)\mu_n(dx) = \int_X g\left(f(x)\right)\mu(dx),$$

and this equals to, by definition of g (and the fact the f_n, f are real valued),

$$\lim_{n\to\infty}\int_{\{|f_n(x)|\le C\}} f_n(x)\mu_n(dx) = \int_{\{|f(x)|\le C\}} f(x)\mu(dx).$$

Thus part (a) of Claim 1 is proved for such C with $\mu \circ f^{-1}\{-C, C\} = 0 = \mu\{x : |f(x)| = C\}$, from which we get that the set of C such that above doesn't hold is at most countable because of finiteness of measure μ. Now for part (b) of Claim 1, the proof copies that of part (a) but with a new real function

$$g(y) = \begin{cases} |y| & \text{if } |y| \le C, \\ 0 & \text{if } |y| > C. \end{cases}$$

Claim 2:

$$\int_X |f(x)|\mu(dx) < +\infty.$$

To see this, we use part (b) of Claim 1.

$$\begin{aligned} \int_{\{|f(x)| \le C\}} |f(x)|\mu(dx) &= \lim_{n\to\infty} \int_{\{|f_n(x)| \le C\}} |f_n(x)|\mu_n(dx) \\ &\le \sup_n \int_X |f_n(x)|\mu_n(dx). \end{aligned} \tag{4}$$

Now we show $\sup_n \int_X |f_n(x)|\mu_n(dx) < +\infty$. From condition

$$\lim_{C\to+\infty} \sup_n \int_{\{|f_n(x)| > C\}} |f_n(x)|\mu_n(dx) = 0,$$

we can choose some C_0 such that $\sup_n \int_{\{|f_n(x)| > C_0\}} |f_n(x)|\mu_n(dx) \le 1$. Then

$$\begin{aligned} &\sup_n \int_X |f_n(x)|\mu_n(dx) \\ &\quad \le \sup_n \int_{\{|f_n(x)| \le C_0\}} |f_n(x)|\mu_n(dx) + \sup_n \int_{\{|f_n(x)| > C_0\}} |f_n(x)|\mu_n(dx) \\ &\quad \le C_0 + 1 < +\infty. \end{aligned}$$

Now we take limit $\lim_{C\to\infty}$ for (4), which proves Claim 2 since $f(x)$ is almost everywhere finite with respect to μ (since it follows from the fact $f(x)$ is real valued that $\mu\{x : |f(x)| = \infty\} = 0$).

$$\begin{aligned} &\left| \int_X f_n(x)\mu_n(dx) - \int_X f(x)\mu(dx) \right| \\ &\le \left| \int_{\{|f_n(x)| > C\}} f_n(x)\mu_n(dx) - \int_{\{|f(x)| > C\}} f(x)\mu(dx) \right| \\ &\qquad + \left| \int_{\{|f_n(x)| \le C\}} f_n(x)\mu_n(dx) - \int_{\{|f(x)| \le C\}} f(x)\mu(dx) \right| \\ &\to 0, \ \text{as } n, C \to \infty, \end{aligned}$$

where $\lim_{C\to\infty}\left|\int_{\{|f_n(x)|>C\}} f_n(x)\mu_n(dx)\right| = 0$ is from assumptions of theorem, and $\lim_{C\to\infty}\left|\int_{\{|f(x)|>C\}} f(x)\mu(dx)\right| = 0$ is from Claim 2 by using absolute continuity of integral since $\lim_{C\to\infty}\mu\{|f(x)| > C\} = 0$. The term in the second absolute value sign converges to zero as $n \to \infty$ because of part (a) of Claim 1. □

Proof of Theorem 2.1. We want to prove that condition $\mu(E) = 0$ in Theorem 2.3 is satisfied provided (i),(ii) and (iv) are given. Recall set E,

$$E = \left\{x \in X: \text{ there are } x_n \in X \text{ with } \lim_{n\to\infty} x_n = x, \text{ but } \lim_{n\to\infty} f_n(x_n) \neq f(x)\right\},$$

for any $x_0 \in E$, we claim $x_0 \in$ {discontinuity points of f}. If not, we consider bounded set $\{x \in X : \text{dist}(x_0, x) \leq 1\}$, since f_n are bounded and converge to f uniformly on this bounded set and f is continuous at x_0, then for any sequence $x_n \to x_0$, we have $f_n(x_n) \to f(x_0)$, so $x_0 \notin E$, and this is a contradiction. Thus

$$\mu(E) \leq \mu\{\text{discontinuity points of } f\} = 0.$$

□

Proof of Theorem 2.2. We want to use (I), (II) and (IV) to prove $\mu(E) = 0$. Again, for any $x_0 \in E$, we claim $x_0 \in$ {discontinuity points of f}. Argue by contradiction, assume x_0 is a continuity point of f. Since $x_0 \in E$, there exist $x_n \to x_0$, but $\lim_{n\to\infty} f_n(x_n) \neq f(x_0)$. Now we construct a compact set $\{x_0, x_1, x_2, \cdots\}$. On this compact set, f_n are bounded and converge to f uniformly, and f is continuous at x_0, thus $\lim_{n\to\infty} f_n(x_n) = f(x_0)$, which gives a contradiction. □

References

[1] P. Billingsley, *Convergence of Probability Measures*, John Wiley & Sons, 1968.

Received: February, 2011