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Mathematical Modeling of Rwanda's Population Growth

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Abstract

Rwanda is a small landlocked African country located in Central Africa. It borders Democratic Republic of Congo, Uganda, Tanzania and Burundi. It has a total area of 26,338 square kilometers comprising 24,668 square kilometers of land and 1,670 square kilometers of water. Most of the country is savanna grassland with the population predominantly rural. It is a country characterized with a high population density. This paper centers on the application of the logistic growth model to model the population growth of Rwanda using data from the years 1980 to 2008 (inclusive). We have determined the carrying capacity and the vital coefficients governing the population growth of Rwanda. The results from the predictions show that the carrying capacity for the population of Rwanda is 77208025.64 and that the vital coefficients a and b are 0.03 and $3.88566419 \times 10^{-10}$, respectively. Thus the population growth rate of Rwanda, according to this model, is 3% per annum. This approximated population growth rate compares well with statistically predicted values in literature. Based on this model we also predict that the population of Rwanda will be 38604012.82 in the year 2067. The data used were collected from National Institute of Statistics of Rwanda (NISR) and International Data Base (IDB). It was analyzed using MATLAB and Statistical Package for Social Science (SPSS) and it accurately fitted the logistic growth curve.

Mathematics Subject Classification: 92D25

Keywords: Logistic growth model, Carrying capacity, Vital coefficients, Population growth rate

1 Introduction

The population projection has become one of the most important problems in the world. Population sizes and growth in a country directly influence the situation of economy, policy, culture, education and environment of that country and determine exploring and cost of natural resources. No one wants to wait until those resources are exhausted because of population explosion. Population has been a controversial subject for ages. Charles Darwin[2] once said, in the struggle for life number gives the best insurance to win. The Bible (Genesis 22:17) records that when God wanted to boost the elected ones, he promised that they would become more numerous than the grains of sand on the sea shore (i.e.,$>> 10^{12}$). Every government and collective sectors always require accurate idea about the future size of various entities like population, resources, demands and consumptions for their planning activities. To obtain this information, the behavior of the connected variables is analyzed based on the previous data by the statisticians and mathematicians at first and using the conclusions drawn from the analysis they make future projections of the variable aimed. There are enormous concerns about the consequences of human population growth for social, environment and economic development. Intensifying all these problems is population growth. Mathematical modeling is a broad interdisciplinary science that uses mathematical and computational techniques to model and elucidate the phenomena arising in life sciences. Thus it is a process of mimicking reality by using the language of mathematics. Many people examine population growth through observation, experimentation or through mathematical modeling. Mathematical models can take many forms, including but not limited to dynamical systems, statistical models and differential equations. These and other types of models can overlap, with a given model involving a variety of abstract structures. In this paper we model the population growth of Rwanda using Verhulst model (logistic growth model). The use of the logistic growth model is widely established in many fields of modeling and forecasting [1]. First order differential equations govern the growth of various species. At first glance it would seem impossible to model the growth of a species by a differential equation since the population of any species always changes by integer amounts. Hence the population of any species can never be a differentiable function of time. However if a given population is very large and it is suddenly increased by one, then the change is very small compared to the given population [5]. Thus we make the approximation that large populations change continuously and even differentiably with time. The projections of future population are normally based on present population. Rwanda is a small country and one of the most densely populated countries in Africa. It has a very high population growth rate which according to information from International Data Base (IDB) was approximately 2.8% per annum in 2006 and 2.9% in 2007, 2008 and 2009. Ideally if the population continues to grow without bound, nature will take over and the death rate will rise to solve the problem. Unfortunately, this is not the most attractive scenario: instead the birth rate would rather be controlled in order to reduce population growth. In this paper, we will determine the carrying capacity and the vital coefficients governing the population growth of Rwanda. Farther this paper gives an insight on how to determine the carrying capacity and the vital coefficients governing population growth.

2 Methodology

A research is best understood as a process of arriving at dependant solutions to the problems through the systematic collection, analysis and interpretation of data. In relation to this paper, secondary classified yearly population data of Rwanda from 1980-2008 (inclusive) were collected from International Data Base (IDB) and National Institute of Statistics of Rwanda (NISR). The Logistic growth mathematical model was used to compute predicted population values employing MATLAB. The Statistical Package for Social Sciences (SPSS) was also used to plot the graph of the classified yearly population data of Rwanda from 1980 to 2008 (inclusive).

3 Development of the model

In 1798 the Englishman Thomas R. Malthus [4], proposed a mathematical model of population growth. Let $N(t)$ denote the population of a given species at time t and let a denote the difference between its birth rate and death rate. If this population is isolated, then $\frac{d}{dt}N(t)$, the rate of change of the population, equals $aN(t)$ where a is a constant that does not change with either time or population. The differential equation governing population growth in this case is

$$\frac{d}{dt}N(t) = aN(t) \tag{1}$$

where, t represents the time period and a, referred to as the Malthusian factor, is the multiple that determines the growth rate.
This equation is a non-homogeneous linear first order differential equation known as Malthusian law of population growth. $N(t)$ takes on only integral values and it is a discontinuous function of t. However $N(t)$ may be approximated by a continuous and differentiable function as soon as the number of individuals is large enough [7].
The solution of equation (3.1) is

$$N(t) = N_0 e^{at} \tag{2}$$

Hence any species satisfying the Malthusian law of population growth grows exponentially with time. This model is often referred to as *The Exponential Law* and is widely regarded in the field of population ecology as the first principle of population Dynamics. At best, it can be described as an approximate physical law as it is generally acknowledged that nothing can grow at a constant rate indefinitely. As population increases in size, the environment's ability to support the population decreases. As the population increases per capita food availability decreases, waste products may accumulate and birth rates tend to decline while death rates tend to increase. Thus it seems reasonable to consider a mathematical model which explicitly incorporates the idea of carrying capacity (limiting value). A Belgian Mathematician Verhulst [6], showed that the population growth not only depends on the population size but also on how far this size is from its upper limit i.e. its carrying capacity (maximum supportable population). He modified Malthus's Model to make the population size proportional to both the previous population and a new term

$$\frac{a - bN(t)}{a} \tag{3}$$

where a and b are called the vital coefficients of the population. This term reflects how far the population is from its maximum limit. However, as the population value grows and gets closer to $\frac{a}{b}$, this new term will become very small and get to zero, providing the right feedback to limit the population growth. Thus the second term models the competition for available resources, which tends to limit the population growth. So the modified equation using this new term is:

$$\frac{d}{dt}N(t) = \frac{aN(t)(a - bN(t))}{a} \tag{4}$$

This is a nonlinear differential equation unlike equation (3.1) in the sense that one cannot simply multiply the previous population by a factor. In this case the population $N(t)$ on the right of equation (3.4) is being multiplied by itself. This equation is known as the logistic law of population growth. Putting $N = N_0$ for $t = 0$, where N_0 represents the population at some specified time, $t = 0$, equation (3.4) becomes

$$\frac{d}{dt}N = aN - bN^2. \tag{5}$$

Separating the variables in equation (3.5) and integrating, we obtain $\int \frac{1}{a}(\frac{1}{N} + \frac{b}{a-bN})dN = t + c$, so that

$$\frac{1}{a}(\log N - \log(a - bN)) = t + c. \tag{6}$$

Using $t = 0$ and $N = N_0$ we see that $c = \frac{1}{a}(\log N_0 - \log(a - bN_0))$. Equation (3.6) becomes

$\frac{1}{a}(\log N - \log(a - bN)) = t + \frac{1}{a}(\log N_0 - \log(a - bN_0))$. Solving for N yields

$$N = \frac{\frac{a}{b}}{1 + (\frac{\frac{a}{b}}{N_0} - 1)e^{-at}} \tag{7}$$

If we take the limit of equation (3.7) as $t \to \infty$, we get (since $a > 0$)

$$N_{max} = \lim_{t\to\infty} N = \frac{a}{b} \tag{8}$$

Suppose that at time $t = 1$ and $t = 2$, the values of N are N_1 and N_2 respectively, then from equation (3.7), we obtain

$$\frac{b}{a}(1 - e^{-a}) = \frac{1}{N_1} - \frac{e^{-a}}{N_0}, \quad \frac{b}{a}(1 - e^{-2a}) = \frac{1}{N_2} - \frac{e^{-2a}}{N_0} \tag{9}$$

Dividing the members of the second equation in relation (3.9) by corresponding members of the first equation to eliminate $\frac{b}{a}$ we get

$$1 + e^{-a} = \frac{\frac{1}{N_2} - \frac{e^{-2a}}{N_0}}{\frac{1}{N_1} - \frac{e^{-a}}{N_0}} \tag{10}$$

so that

$$e^{-a} = \frac{N_0(N_2 - N_1)}{N_2(N_1 - N_0)} \tag{11}$$

Putting this value of e^{-a} into the first equation in equation (3.9), we obtain

$$\frac{b}{a} = \frac{N_1^2 - N_0 N_2}{N_1(N_0 N_1 - 2N_0 N_2 + N_1 N_2)} \tag{12}$$

Thus the limiting value of N is given by

$$N_{max} = \lim_{t \to \infty} N = \frac{a}{b} = \frac{N_1(N_0 N_1 - 2N_0 N_2 + N_1 N_2)}{N_1^2 - N_0 N_2} \tag{13}$$

4 Results

Based on the population from 1980 to 2008 (inclusive) in Table 1 below, let $t = 0, 1, 2$ correspond to the years 1980, 1981 and 1982 respectively. Then N_0, N_1, N_2 are respectively 5139838, 5308986 and 5483277.

Table 1: Actual values of population

Year	Actual Population		Year	Actual Population [1]
1980	5139838		1995	5472977
1981	5308986		1996	6546456
1982	5483277		1997	7647419
1983	5663174		1998	8041839
1984	5834293		1999	8222350
1985	5986589		2000	8398413
1986	6161714		2001	8610984
1987	6372803		2002	8857375
1988	6597216		2003	9098466
1989	6816864		2004	9361254
1990	6999318		2005	9611065
1991	7143743		2006	9867219
1992	7298571		2007	10141078
1993	7596681		2008	10440771
1994	6505533			

Figure 1, below, shows the actual population data from 1980 to 2008 (inclusive) plotted against time in years.

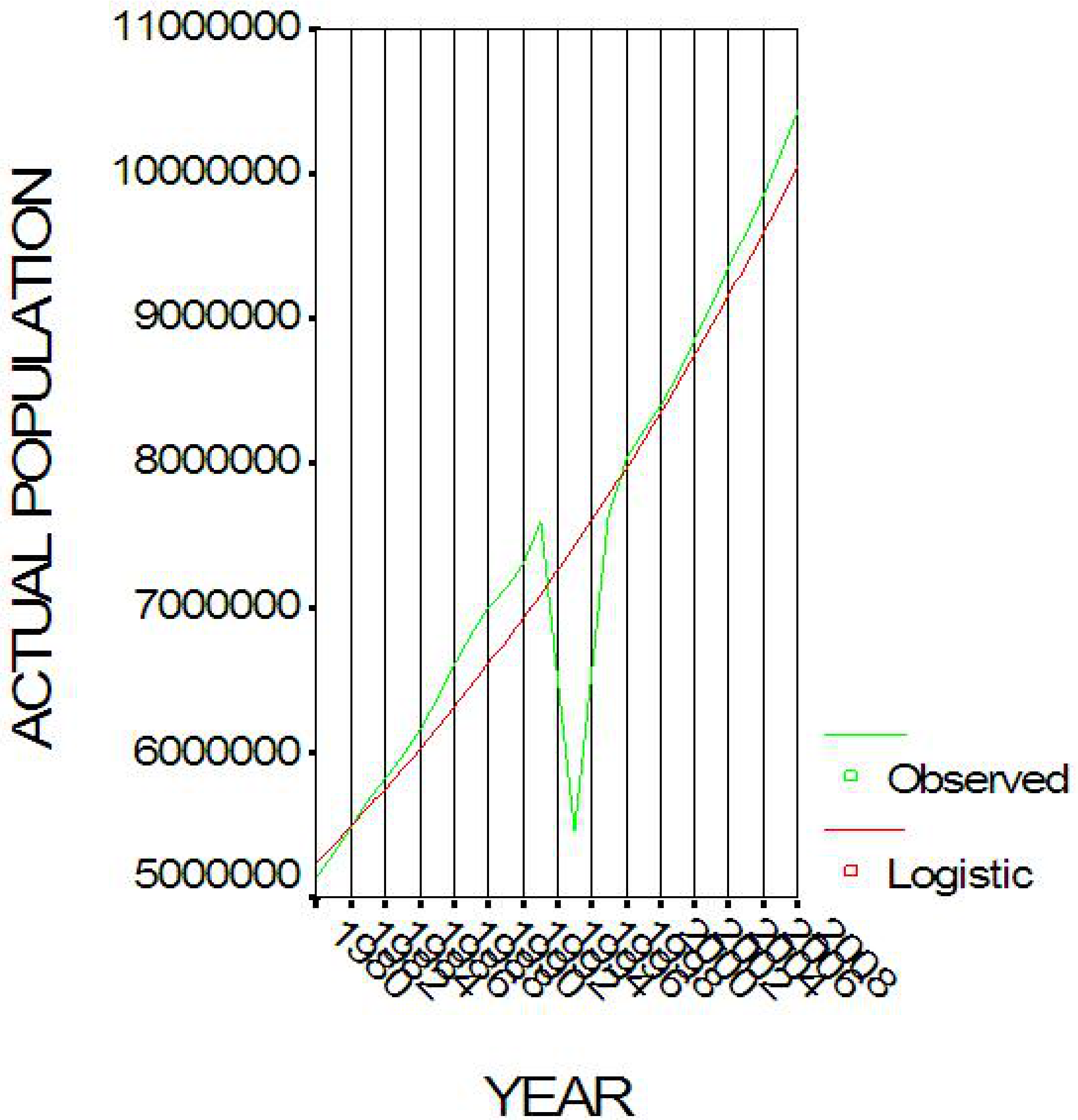

Figure 1: Graph of actual population values from 1980 to 2008 (inclusive) against time in years

Substituting the values of N_0, N_1 and N_2 into equation (3.13), we get $N_{max} = \frac{a}{b} = 77208025.64$.
This is the predicted carrying capacity or limiting value of the population of Rwanda. From equation (3.11), we obtain $e^{-a} \approx 0.97$ so that $a \approx -ln(0.97)$. Solving this for the value of a we get

$$a \approx 0.03 \tag{14}$$

This implies that the predicted rate of Rwanda population growth is approximately 3% per year. From $\frac{a}{b} = 77208025.64$ and equation (4.14), we have $b = 3.885606419 \times 10^{-10}$. Substituting the values of N_0, e^{-a} and $\frac{a}{b}$ into equation (3.7) gives

$$N = \frac{77208025.64}{1 + (14.0214901) \times (0.97)^t} \tag{15}$$

Table 2 shows the predicted population values and the corresponding actual population values within a given year. The predicted population values were computed using equation (4.15).

Table 2: Actual and predicted values of population

Year	Actual pop.	Predicted pop.		Year	Actual pop.	Predicted pop.
1980	5139838	5139838		1995	5472977	7815269
1981	5308986	5287915		1996	6546456	8031834
1982	5483277	5439936		1997	7647419	8253686
1983	5663174	5595987		1998	8041839	8480915
1984	5834293	5756155		1999	8222350	8713609
1985	5986589	5920529		2000	8398413	8951855
1986	6161714	6089197		2001	8610984	9195742
1987	6372803	6262248		2002	8857375	9445353
1988	6597216	6439771		2003	9098466	9700774
1989	6816864	6621857		2004	9361254	9962086
1990	6999318	6808597		2005	9611065	10229370
1991	7143743	7000080		2006	9867219	10502706
1992	7298571	7196398		2007	10141078	10782170
1993	7596681	7397642		2008	10440771	11067836
1994	6505533	7603902				

Figure 2 shows the predicted population values plotted against time.

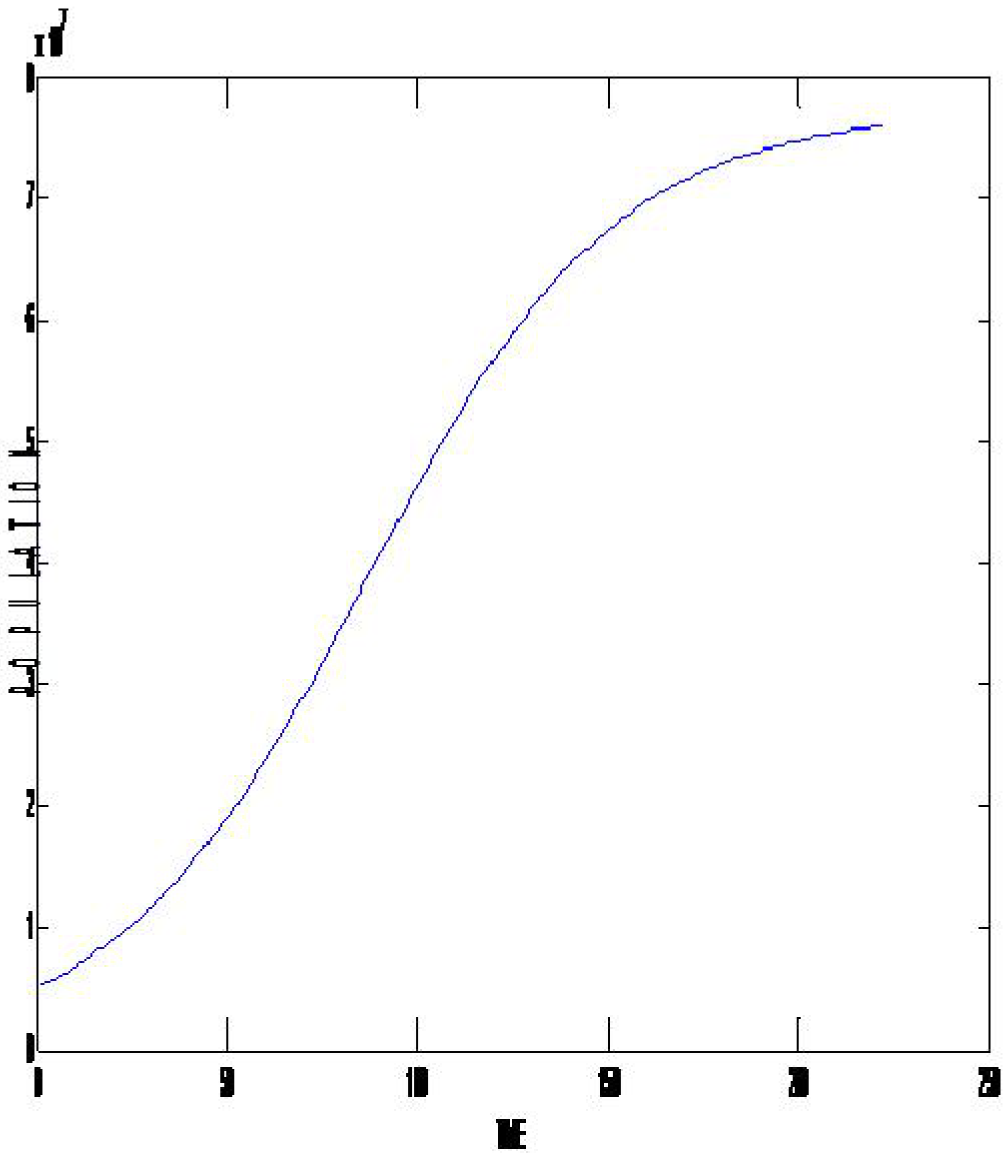

Figure 2: Graph of predicted population values against time.

Next we compute the expected time for the population of Rwanda to be $\frac{a}{2b} = 38604012.82$. From equation (4.15) and putting 38604012.82 as the value for N, we get $t = \frac{\log(0.071319)}{\log(0.97)}$ and solving for t we obtain $t \approx 87$. Thus the population of Rwanda is predicted to be 38604012.82 in the year 2067.

5 Discussion

From figure 1, it can be seen that in 1980 population of Rwanda was 5139838 and continued to increase till 1993 when it decreased considerably from 7596681 to 6505533 in 1994. This decrease was probably due to some people leaving the country. Afterwards, the population decreased farther to 5472977 in 1995 after the 1994 genocide in which approximately one million people were killed. The population begun to rise once again from 5472977 in 1995 to 10440771 in 2008. This increase is attributed to the rebuilding and peace prevailing in the country. There has been a lot of improvement in education, agricultural productivity, water and sanitation and health services with an overall effect on population increase. After the genocide of 1994 there was little formal and civic education available especially to women. This meant that women were unaware of various methods of birth control, where to access them and how to use them. There was a belief in Rwanda that the more children one had, one would have a higher social and economic status and receive better care in old age. Since many families lost their members during the genocide of 1994, they wanted to replace them by having many children. There were also early marriages to girls and more people migrated back to the country after peace was installed. This coupled with other factors had an overall effect on the increase in population. In figure 2, we can see that the graph of the predicted population values is an S-shaped curve. This shows that the values fitted the logistic curve. At first, the population starts to grow going through an exponential growth phase at an approximate rate of 3% per annum reaching 38604012.82 in the year 2067 after which the rate of growth is expected to slow down. As it gets closer to the carrying capacity, 77208025.64, the growth drastically slows down and reaches a stable level. This slow down to a carrying capacity is perhaps the result of war, pestilence, and starvation as more and more people contend for the resources that are now at their upper bound. The population growth rate of Rwanda according to information in International Data Base (IDB) was approximately 2.8% per annum in 2006 and 2.9% in the years 2007, 2008 and 2009 which corresponds well with the findings in this research work of 3% per annum.

6 Conclusion

In conclusion we found that the predicted carrying capacity for the population of Rwanda is 77208025.64. Population growth of any country depends also on the vital coefficients. In the case of Rwanda we found out that the vital coefficients a and b are 0.03 and $3.885606419 \times 10^{-10}$ respectively. Thus the population growth rate of Rwanda, according to this model, is 3% per annum. This approximated population growth rate compares well with the statistically predicted values in literature. Based on this model we also found out that the population of Rwanda is expected to be 38604012.82 in the year 2067. The following are some recommendations: Technological developments, pollution and social trends have significant influence on the vital coefficients a and b . Therefore, they must be re-evaluated every few years to enhance the determination of variations in the population growth rate. In order to derive more benefits from models of population growth, one should subdivide populations into different age groups for effective capture, analyses and planning purposes. Other models can be developed by subdividing the population into males and females, since the reproduction rate in a population usually depends more on the number of females than on the number of males. The government should work towards industrialization of the country for the attainment of vision 2020. This will have an effect in improving its absorptive capacity for development through population growth rate measures. The more industrialized a Nation is, the more living space and food it has and the smaller the coefficient b, thus raising the carrying capacity. The government should also step up the dissemination of civic education on birth control methods to enable it to manage its resources allocation through efficient and effective population growth rate management principles. However, present attempts appear to provide acceptable predictions for the Rwanda population growth.

Appendix

Map of Rwanda

Figure 3: Map of Rwanda

Matlab program

```
  t=0:28;
 N=77208025.64 ./(1+(14.0214901)*(0.97).^t);
 Format long
N
plot(t,N)
  % t and N , represent time and population respectively.
```

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