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Exponential Stability of Switched Linear Systems with Time-Varying Delay

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Abstract. This paper investigates the exponential stability of uncertain switched linear system with time varying delay. Based on Lyapunov functional method, new sufficient exponential stability robustness criteria are presented. The conditions are delay dependent and can be easily checked. Numerical examples are provided to demonstrate efficiency and reduced conservatism of the results in this paper.

Keywords: Exponential stability; time varying delay; switched systems; linear matrix inequalities

1. Introduction

In the past decades, the researches focus on the stability of switched systems and designing stabilizing controllers in both continuous time domain and discrete-time domain [13, 20, 16-17]. In fact switching systems arise in many practical processes that cannot be described by exclusively continuous or exclusively discrete models in manufacturing, communication networks automotive engine control, chemical processes and so on see e.g. [12, 14].
The stability analysis of switched time delay systems has attracted a lot of attention [6, 9]. The main approach for stability analysis relies on the use of Lyapunov Krasovskii functionals and LMI approach for constructing common Lyapunov function and switching rules [1, 3, 19]. In [18], the asymptotic stability of switched linear time delay symmetric systems has been studied. Switching systems composed by a finite number of linear point time delay differential equations has been considered in [3], it has been shown that the asymptotic stability may be achieved by using a common Lyapunov function method switching rule. In [9], sufficient stability conditions have been obtained by using a generalisation of Halanay's inequality. The exponential stability problem was considered in [21] for switching linear systems with impulsive effects by using the

matrix measure concept. The exponential stability of time delay systems has not received much attention [11, 1].

In this paper, we study the exponential stability of a class of switched linear systems with time varying delay. Our objective is to derive delay dependent conditions for the exponential asymptotic stability. The approach is based on Lyapunov Krasovskii functional. The results allow easily to design switching rules and to compute the exponential stability rate of the systems solution. Some numerical examples are given for illustration.

2. Preliminaries

Consider a class of switched uncertain linear systems with time varying delay of the form:

$$\begin{cases} \dot{x}(t) = \left[A_\sigma + \Delta A_\sigma\right]x(t) + \left[D_\sigma + \Delta D_\sigma\right]x(t-h(t)), & t > 0 \\ x(t) = \phi(t), & t \in [-\bar{h},0] \end{cases} \tag{1}$$

Where $x \in R^n$ is the state, $\sigma(x)$: $R^n \rightarrow I:=\{1,2,\dots,N\}$ is the switching rule which is piece-wise constant function depending on the state in each time, $\phi \in C([-h,0],R^n)$ is the initial function, with norm $\|\phi\| = \sup\limits_{\theta \in [-\bar{h},0]} \|\phi(\theta)\|$,

$A_\sigma, D_\sigma \in \{[A_i, D_i], i = 1,2,...,N\}$, A_i and D_i are given matrices. Moreover $\sigma(x)=i$ implies that the ith subsystem is activated and we have the following subsystem,

$$\dot{x}(t) = \left[A_i + \Delta A_i\right]x(t) + \left[D_i + \Delta D_i\right]x(t-h(t)), \quad t > 0 \tag{2}$$

h(t), is the unknown time varying delay term, but bounded by:

$$0 \le h(t) \le \bar{h}, \quad \dot{h}(t) \le \mu \tag{3}$$

Where $\bar{h}$ and μ are given non negative constants. The uncertainty matrices $\Delta A_i(t)$ and $\Delta D_i(t)$ satisfy the following conditions,

$$\Delta A_i(t) = E_{0i}F_{0i}(t)H_{0i}; \quad \Delta D_i(t) = E_{1i}F_{1i}(t)H_{1i} \tag{4}$$

Where $E_{ki}, H_{ki}, k = 0,1, i = 1,2,...,N$ are given constant matrices with appropriate dimensions, $F_{ki}(t)$ are unknown real matrices satisfying,

$$F_{ki}^T F_{ki} \le I, \quad k = 0,1, \; i = 1,2,...,N \tag{5}$$

Definition 1. Given $\alpha>0$, the system (1) is robustly α-exponentially stable if there exist a switching rule σ and a constant $\beta \ge 1$ such that every solution $x(t,\phi)$ of the system satisfies the following inequality:

$$\|x(t,\phi)\| \le \beta e^{-\alpha t}\|\phi\|, \quad t \ge 0$$

Lemma 1. For any vectors $X, Y \in R^n$, and any matrix $M>0$, one has

$$2X^T Y \le X^T M^{-1} X + Y^T M Y$$

Lemma 2. For any constant matrix $M = M^T \in R^{n\times n}$, M>0, scalar $\gamma \ge \eta(t) > 0$, vector function $\omega : [0,\gamma] \to R^n$ such that the integrations in the following are well defined, then: $\eta(t)\int_0^{\eta(t)} \omega^T(\beta)M\omega(\beta)d\beta \ge \left[\int_0^{\eta(t)} \omega(\beta)d\beta\right]^T M\left[\int_0^{\eta(t)} \omega(\beta)d\beta\right]$

3. Main Results

Let the matrices R, Q and $\tilde{Q}$ be given by:

$$R = \begin{pmatrix} R_{11} & R_{12} \\ R_{12}^T & R_{22} \end{pmatrix},\ Q = \begin{bmatrix} Q_{11} & Q_{12} \\ Q_{12}^T & Q_{22} \end{bmatrix};\ \tilde{Q} = \begin{pmatrix} I & A_i^T \\ 0 & D_i^T \end{pmatrix} Q \begin{pmatrix} I & 0 \\ A_i & D_i \end{pmatrix} \tag{6}$$

And let,

$$\beta_1 = \lambda_{\min}(R_{11}) \tag{7}$$

$$\beta_2 = \left(1+\bar{h}^2\right)\lambda_{\max}(R) + \frac{1-e^{-2\alpha\bar{h}}}{2\alpha}\lambda_{\max}(P) + \frac{e^{-2\alpha\bar{h}}+2\alpha\bar{h}-1}{4\alpha^2}\max_{1\le i\le N}\lambda_{\max}\left(\tilde{Q}_i\right) \tag{8}$$

$$L_i = R_{12} + R_{12}^T + P - \frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22} + F_0A_i + A_i^TF_0^T + M \tag{9}$$

We will first give exponential stability conditions for the nominal systems of (1) i.e. the system described by:

$$\dot{x}(t) = A_ix(t) + D_ix(t-h(t)),\quad i \in I,\quad t > 0 \tag{10}$$

Theorem 1. For given $\alpha \ge 0$, switched linear system (12) is α-exponentially stable if there exist symmetric positive definite matrices M, P, Q_{11}, Q_{22}, R_{11}, R_{22}, matrices R_{12}, Q_{12}, F_0, F_1, F_2 and F_3 satisfying,

There exist $\rho_i \ge 0$, i=1,2,…,N such that $\sum_{i=1}^N \rho_i > 0$ and $\sum_{i=1}^N \rho_i L_i < 0$. (11)

$$G_i = \begin{pmatrix} G_{11}^i & G_{12}^i & G_{13}^i & \frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T - R_{22} - 2\alpha R_{12} - A_i^TF_3^T \\ * & G_{22}^i & G_{23}^i & (1-\mu)R_{22} - \frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T - D_i^TF_3^T \\ * & * & G_{33}^i & F_3^T - R_{12} \\ * & * & * & \frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{11} - 2\alpha R_{22} \end{pmatrix} > 0 \tag{12}$$

The switching rule is chosen as $\sigma(x) = \arg\min\left\{x^T(t)L_ix(t)\right\}$. Moreover the solution $x(t,\phi)$ of the system satisfies:

$$\|x(t,\phi)\| \le \beta e^{-\alpha t}\|\phi\|,\ \beta = \sqrt{\frac{\beta_2}{\beta_1}},\ t \ge 0 \tag{13}$$

Where '*' denotes the symmetric part, $G_{33}^{i} = F_2 + F_2^T - \bar{h}Q_{22}$; $G_{23}^{i} = F_1 - D_i^T F_2^T$;
$G_{13}^{i} = F_0 - R_{11} - \bar{h}Q_{12} - A_i^T F_2^T$; $G_{12}^{i} = (1-\mu)R_{12} - \frac{1-\mu}{\bar{h}} e^{-2\alpha\bar{h}} Q_{22} - F_0 D_i - A_i^T F_1^T$;
$G_{22}^{i} = (1-\mu)e^{-2\alpha\bar{h}} P + \frac{1-\mu}{\bar{h}} e^{-2\alpha\bar{h}} Q_{22} - F_1 D_i - D_i^T F_1^T$; $G_{11}^{i} = M - 2\alpha R_{11} - \bar{h}Q_{11}$

Proof. Let $x_t := \{x(t+s), s \in [-\bar{h}, 0]\}$ and consider the following Lyapunov Krasovskii functional:

$$V(x_t) = V_1(x_t) + V_2(x_t) + V_3(x_t)$$

With

$$V_1(x_t) = \begin{pmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \end{pmatrix}^T R \begin{pmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \end{pmatrix};$$

$$V_2(x_t) = \int_{-h(t)}^{0} e^{2\alpha s} x^T(t+s) P x(t+s) ds$$

$$V_3(x_t) = \int_{-h(t)}^{0} \int_{s}^{0} e^{2\alpha\theta} \begin{pmatrix} x(t+\theta) \\ \dot{x}(t+\theta) \end{pmatrix}^T Q \begin{pmatrix} x(t+\theta) \\ \dot{x}(t+\theta) \end{pmatrix} d\theta ds$$

It is easy to verify that,

$$\beta_1 \|x(t)\|^2 \le V(x_t) \le \beta_2 \|x_t\|^2 \tag{14}$$

where β_1 and β_2 are defined by (7) and (8). Computing the first time derivative of $V(x_t)$, we obtain:

$$\dot{V}_1(x_t) = \begin{pmatrix} x(t) \\ \int_{t-h(t)}^{t} x(s)ds \end{pmatrix}^T R \begin{pmatrix} \dot{x}(t) \\ x(t) - (1-\dot{h}(t))x(t-h(t)) \end{pmatrix}$$

$$\dot{V}_2(x_t) = x^T(t)Px(t) - (1-\dot{h}(t))e^{-2\alpha h(t)} x^T(t-h(t))Px(t-h(t)) - 2\alpha V_2(x_t)$$

$$\dot{V}_3(x_t) = h(t)\begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix}^T Q \begin{pmatrix} x(t) \\ \dot{x}(t) \end{pmatrix} - (1-\dot{h}(t)\int_{t-h(t)}^{t} e^{-2\alpha h(t)} \begin{pmatrix} x(s) \\ \dot{x}(s) \end{pmatrix}^T Q \begin{pmatrix} x(s) \\ \dot{x}(s) \end{pmatrix} ds - 2\alpha V_3(x_t)$$

Let $\xi(t) = \begin{bmatrix} x^T(t) & x^T(t-h(t) & \dot{x}^T(t) & \int_{t-h(t)}^{t} x^T(s)ds \end{bmatrix}^T$. Taking account of (3) and applying lemma 2 we obtain,

$$\dot{V}_1(x_t) \le x^T(t)\left[R_{12} + R_{12}^T\right]x(t) - \xi^T(t)T_1\xi(t) - 2\alpha V_1(x_t)$$

$$\dot{V}_2(x_t) \le x^T(t)Px(t) - \xi^T(t)T_2\xi(t) - 2\alpha V_2(x_t)$$

$$\dot{V}_3(x_t) \le -\frac{1-\mu}{\bar{h}} e^{-2\alpha\bar{h}} x^T(t)Q_{22}x(t) - \xi^T(t)T_3\xi(t) - 2\alpha V_3(x_t)$$

where,

$$T_1 = \begin{pmatrix} -2\alpha R_{11} & (1-\mu)R_{12} & -R_{11} & -R_{22}-2\alpha R_{12} \\ (1-\mu)R_{12}^T & 0 & 0 & (1-\mu)R_{22} \\ -R_{11} & 0 & 0 & -R_{12} \\ -R_{22}-2\alpha R_{12}^T & (1-\mu)R_{22} & -R_{12}^T & -2\alpha R_{22} \end{pmatrix}$$

$$T_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (1-\mu)e^{-2\alpha\bar{h}}P & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$T_3 = \begin{pmatrix} -hQ_{11} & -\frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22} & -\bar{h}Q_{12} & \frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T \\ -\frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22} & \frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22} & 0 & -\frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T \\ -\bar{h}Q_{12}^T & 0 & -\bar{h}Q_{22} & 0 \\ \frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12} & -\frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12} & 0 & \frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{11} \end{pmatrix}$$

Letting $T = T_1 + T_2 + T_3$ we have,

$$\dot{V}(x_t) + 2\alpha V(x_t) \le x^T(t)\left[R_{12} + R_{12}^T + P - \frac{(1-\mu)}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22}\right]x(t) - \xi^T(t)T\xi(t) \tag{15}$$

Now let $B_i = [A_i \quad D_i \quad -I \quad 0]$ and $F = \left[F_0^T \quad F_1^T \quad F_2^T \quad F_3^T\right]^T$. We can easily verify that $B_i\xi = 0,\ \forall\xi \neq 0$, and,

$$\xi^T(t)\left[FB_i + B_i^T F^T\right]\xi(t) = 0,\ i=1,2\ldots,N \tag{16}$$

Adding (16) to the right hand side of (15), and adding and subtracting the term $x^T(t)Mx(t)$ with M a positive definite matrix we get, $\dot{V}(x_t) + 2\alpha V(x_t) \le x^T(t)L_i x(t) - \xi^T(t)G_i\xi(t)$. Since the condition (12) holds, it follows that

$$\dot{V}(x_t) + 2\alpha V(x_t) \le x^T(t)L_i x(t) \tag{17}$$

From condition (11) we have $\sum_{i=1}^N \rho_i L_i < 0$, where $\rho_i \ge 0$, i=1,2,…,N and $\sum_{i=1}^N \rho_i > 0$. Since $\rho_i \ge 0$, and $\sum_{i=1}^N \rho_i > 0$, so $\sum_{i=1}^N \rho_i \min_{i=1,\ldots,N} x^T(t)L_i x(t) \le \sum_{i=1}^N \rho_i x^T(t)L_i x(t) < 0$. By choosing the switching rule as:

$$\sigma(x) = \arg\min_{i=1,\ldots,N} x^T(t)L_i x(t)$$

We have $\dot{V}(x_t) + 2\alpha V(x_t) \le x^T(t)L_i x(t) \le \eta\sum_{i=1}^N \rho_i x^T(t)L_i x(t) < 0$. Where $\eta = \left(\sum_{i=1}^N \rho_i\right)^{-1}$. This implies that $V(x_t) \le V(\phi)e^{-2\alpha t},\ t \ge 0$. Taking account of (14) we obtain $\beta_1\|x(t)\|^2 \le V(x_t) \le V(\phi)e^{-2\alpha t} \le \beta_2 e^{-2\alpha t}\|\phi\|^2$. And then, $\|x(t)\| \le \beta e^{-\alpha t}\|\phi\|,\ \ t \ge 0$ which concludes the proof. □

Remark 2. In [11] the results are given in terms of a set of generalized Lyapunov equations type and in [1] the results are expressed in terms of generalized algebraic Riccati equation type while in this paper the results are expressed in terms of linear matrix inequalities.

Theorem 2. For given α≥0, switched linear system (1) is robustly α-exponentially stable if there exist symmetric positive definite matrices M, P, Q_{11}, Q_{22}, R_{11}, R_{22}, S_1, S_2, matrices R_{12}, Q_{12}, F_0, F_1, F_2 and F_3 such that condition (11) holds and

$$H_i = \begin{pmatrix} H_{11}^i & H_{12}^i & H_{13}^i & H_{14}^i & -F_0E_{0i} & -F_0E_{1i} \\ * & H_{22}^i & H_{23}^i & H_{24}^i & -F_1E_{0i} & -F_1E_{1i} \\ * & * & H_{33}^i & H_{34}^i & -F_2E_{0i} & -F_2E_{1i} \\ * & * & * & H_{44}^i & -F_3E_{0i} & -F_3E_{1i} \\ * & * & * & * & S_1 & 0 \\ * & * & * & * & * & S_2 \end{pmatrix} > 0,\ i=1,2\ldots,N \tag{18}$$

The switching rule is chosen as $\sigma(x) = \arg\min\{x^T(t)L_i x(t)\}$. Moreover the solution $x(t,\phi)$ of the system satisfies: $\|x(t,\phi)\| \le \sqrt{\dfrac{\beta_2}{\beta_1}}e^{-\alpha t}\|\phi\|,\quad t \ge 0$

Where '*' denotes the symmetric part, β_1 given by (11), $\beta_2 = \left(1+\bar{h}^2\right)\lambda_{max}(R) + \dfrac{1-e^{-2\alpha\bar{h}}}{2\alpha}\lambda_{max}(P) + \dfrac{e^{-2\alpha\bar{h}}+2\alpha\bar{h}-1}{4\alpha^2}\max_{1\le i\le N}\left[\lambda_{max}\left(\tilde{Q}_i\right)+\lambda_{max}\left(\overline{Q}_i\right)\lambda_{max}\left(\overline{H}_i\right)\right]$

And where $\overline{Q}_i = [E_{0i}\ \ E_{1i}]^T Q_{22}[E_{0i}\ \ E_{1i}]$; $\overline{H}_i = \mathrm{diag}\{H_{0i}^T H_{0i}, H_{1i}^T H_{1i}\}$;

$H_{11}^i = M - 2\alpha R_{11} - hQ_{11} - H_{0i}^T S_1 H_{0i}$; $H_{12}^i = (1-\mu)R_{12} - \dfrac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22} - F_0D_i - A_i^T F_1^T$

$H_{13}^i = F_0 - R_{11} - \bar{h}Q_{12} - A_i^T F_2^T$; $H_{14}^i = \dfrac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T - R_{22} - 2\alpha R_{12} - A_i^T F_3^T$

$H_{22}^i = (1-\mu)e^{-2\alpha\bar{h}}P + \dfrac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22} - F_1D_i - D_i^T F_1^T - H_{1i}^T S_2 H_{1i}$

$H_{23}^i = F_1 - D_i^T F_2^T$; $H_{24}^i = (1-\mu)R_{22} - \dfrac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T - D_i^T F_3^T$

$H_{33}^i = F_2 + F_2^T - \bar{h}Q_{22}$; $H_{34}^i = F_3^T - R_{12}$; $H_{44}^i = \dfrac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{11} - 2\alpha R_{22}$

Proof. By letting $\tilde{B}_i = [A_i + \Delta A_i\ \ \ D_i + \Delta D_i\ \ -I\ \ 0]$, we can verify that $\tilde{B}_i\xi = 0,\ \forall\xi \ne 0$. We can write $\tilde{B}_i = B_i + \Delta B_i$ with $\Delta B_i = [\Delta A_i\ \ \Delta D_i\ \ 0\ \ 0]$.

Rewriting ΔB_i as: $\Delta B_i = \begin{bmatrix} E_{0i} & E_{1i} \end{bmatrix} \begin{pmatrix} F_{0i}(t) & 0 \\ 0 & F_{1i}(t) \end{pmatrix} \begin{bmatrix} H_{0i} & 0 & 0 & 0 \\ 0 & H_{1i} & 0 & 0 \end{bmatrix}$, using (5) and Lemma 1, the proof follows as in theorem 1 with the use of Schur complement. The computation of β_2 may be easily carried when taking account of (5) and the following equality where $\chi(\theta) = \begin{bmatrix} x^T(\theta) & x^T(\theta - h) \end{bmatrix}^T$,

$$\begin{pmatrix} x(\theta) \\ \dot{x}(\theta) \end{pmatrix}^T Q \begin{pmatrix} x(\theta) \\ \dot{x}(\theta) \end{pmatrix} = \chi^T(\theta)\tilde{Q}_i\chi(\theta) + \chi^T(\theta)\begin{pmatrix} 0 & 0 \\ \Delta A_i & \Delta D_i \end{pmatrix}^T Q \begin{pmatrix} 0 & 0 \\ \Delta A_i & \Delta D_i \end{pmatrix}\chi(\theta) \qquad \square$$

For the case when N=1 (without switching), theorem 2 gives an exponential estimate for the robust stability of linear time delay uncertain system described by,

$$\dot{x}(t) = \left(A + \Delta A(t)\right)x(t) + \left(D + \Delta D(t)\right)x(t - h(t)), \qquad t > 0 \tag{19}$$

where

$$\Delta A(t) = E_0 F_0(t) H_0; \;\; \Delta D(t) = E_1 F_1(t) H_1; \;\; F_k^T F_k \le I, \;\; k = 0,1 \tag{20}$$

Corollary 1. For given $\alpha \ge 0$, linear time delay system (19) is α-exponentially stable if there exist symmetric positive definite matrices P, Q_{11}, Q_{22}, R_{11}, R_{22}, S_1, S_2 matrices R_{12}, Q_{12}, F_0, F_1, F_2 and F_3 satisfying the following LMI:

$$\tilde{G} = \begin{pmatrix} G_{11} + H_0^T S_1 H_0 & G_{12} & G_{13} & G_{14} & F_0 E_0 & F_0 E_1 \\ * & G_{22} + H_1^T S_2 H_1 & G_{23} & G_{24} & F_1 E_0 & F_1 E_1 \\ * & * & G_{33} & G_{34} & F_2 E_0 & F_2 E_1 \\ * & * & * & G_{44} & F_3 E_0 & F_3 E_1 \\ * & * & * & * & -S_1 & 0 \\ * & * & * & * & * & -S_2 \end{pmatrix} < 0 \tag{21}$$

Moreover the solution $x(t,\phi)$ of the system satisfies:

$$\|x(t,\phi)\| \le \sqrt{\frac{\beta_2}{\beta_1}} e^{-\alpha t} \|\phi\|, \quad t \ge 0 \tag{22}$$

Where '*' denotes the symmetric part, $\beta_1 = \lambda_{min}(R)$,

$\beta_2 = \left(1 + h^2\right)\lambda_{max}(R) + h\lambda_{max}(P) + h^2 \max\limits_{1 \le i \le N} \left[\lambda_{max}\left(\tilde{Q}\right) + \lambda_{max}\left(\overline{Q}\right)\lambda_{max}\left(\overline{H}\right)\right]$ with

$\tilde{Q} = \begin{pmatrix} I & A^T \\ 0 & D^T \end{pmatrix} Q \begin{pmatrix} I & 0 \\ A & D \end{pmatrix}$, $\overline{Q} = \begin{bmatrix} E_0 & E_1 \end{bmatrix}^T Q_{22} \begin{bmatrix} E_0 & E_1 \end{bmatrix}$; $\overline{H} = \mathrm{diag}\left\{H_0^T H_0, H_1^T H_1\right\}$

And where $G_{11} = R_{12} + R_{12}^T + 2\alpha R_{11} + P + hQ_{11} - \frac{1-\mu}{\bar{h}} e^{-2\alpha\bar{h}} Q_{22} + F_0 A + A^T F_0^T$;

$G_{22} = -(1-\mu)e^{-2\alpha\bar{h}} P - \frac{1-\mu}{\bar{h}} e^{-2\alpha\bar{h}} Q_{22} + F_1 D + D^T F_1^T$; $G_{23} = -F_1 + D^T F_2^T$;

$$G_{33}=\bar{h}Q_{22}-F_2-F_2^T;G_{34}=R_{12}-F_3^T;G_{44}=2\alpha R_{22}-\frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{11};$$

$$G_{13}=R_{11}+\bar{h}Q_{12}-F_0+A^TF_2^T;G_{14}=R_{22}+2\alpha R_{12}-\frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T+A^TF_3^T;$$

$$G_{12}=-(1-\mu)R_{12}+\frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{22}+F_0D+A^TF_1^T;$$

$$G_{24}=-(1-\mu)R_{22}+\frac{1-\mu}{\bar{h}}e^{-2\alpha\bar{h}}Q_{12}^T+D^TF_3^T$$

4. Examples

In this section we will give three examples to illustrate the feasibility of the approach.

Example 1 [1]: Consider the uncertain switched linear systems (1) where N=2, $h(t)=0.5\sin^2 t$ and

$$A_1=\begin{pmatrix}-20 & 1\\ -4 & 6\end{pmatrix},\ A_2=\begin{pmatrix}5 & -1\\ 1 & -30\end{pmatrix},\ D_1==\begin{pmatrix}1 & -1\\ 1 & -1\end{pmatrix},\ D_2=\begin{pmatrix}1 & -1\\ 3 & -4\end{pmatrix}$$

$$H_{0i}=H_{1i}=\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix},\ E_{0i}=E_{1i}=\begin{pmatrix}0.2 & 0\\ 0 & 0.2\end{pmatrix}$$

In this case we have $\bar{h}=0.5$ and $\mu=0.5$. Applying theorem 3.1 of [1] it is found that the system is robustly stable with decay rate α=1. Applying our results, by theorem 2 it is found that the solutions

$$P=\begin{pmatrix}0.4503 & -0.3576\\ -0.3576 & 0.4542\end{pmatrix},\ R=\begin{pmatrix}0.1441 & -0.0583 & 0.0014 & -0.0004\\ -0.0583 & 0.0476 & 0.0002 & -0.0001\\ 0.0014 & 0.0002 & 0.0015 & -0.0003\\ -0.0004 & -0.0001 & -0.0003 & 0.0001\end{pmatrix}$$

$$M=\begin{pmatrix}1.5703 & -0.6601\\ -0.6601 & 0.4725\end{pmatrix},\ Q=\begin{pmatrix}0.2065 & -0.0472 & 0.0160 & 0.0054\\ -0.0472 & 0.0163 & -0.0037 & -0.0011\\ 0.0460 & -0.0037 & 0.0014 & 0.0005\\ 0.0054 & -0.0011 & 0.0005 & 0.0002\end{pmatrix}$$

$$G_1=\begin{pmatrix}0.0372 & -0.0135\\ -0.0135 & 0.0098\end{pmatrix},\ G_2=\begin{pmatrix}0.0078 & -0.0031\\ -0.0031 & 0.0022\end{pmatrix},\ F_0=\begin{pmatrix}0.1457 & -0.0511\\ -0.0619 & 0.0386\end{pmatrix},$$

$$F_1=\begin{pmatrix}0.0023 & 0.0017\\ -0.0011 & -0.0022\end{pmatrix},\ F_2=10^{-3}\begin{pmatrix}0.8562 & 0.3579\\ 0.3640 & 0.4922\end{pmatrix},\ F_3=10^{-4}\begin{pmatrix}-0.5149 & -0.1154\\ 0.3018 & 0.0459\end{pmatrix}$$

Satisfy $H_1>0$, $H_2>0$, and the matrices

$$L_1=\begin{pmatrix}-3.3942 & -0.0947\\ -0.0947 & 1.2664\end{pmatrix},\ L_2=\begin{pmatrix}3.3777 & 0.0987\\ 0.0987 & -1.2686\end{pmatrix}$$

Satisfy $0.5\,L_1+0.5\,L_2<-0.006\,I$. And the system is robustly stable with decay rate 2.15 which is greater that that in [1]. Moreover the solution of the system satisfies

$$\|x(t,\phi)\|\le 4.6108e^{-2.15t}\|\phi\|,\quad t\ge 0$$

Example 2 [11]: consider switched linear delay system with N=2, $h(t) = \sin^2 0.5t$, and

$$A_1 = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}, \; A_2 = \begin{pmatrix} a_3 & 0 \\ 0 & a_4 \end{pmatrix}, \; D_1 = \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}, \; D_2 = \begin{pmatrix} e^{-2} & e^2 \\ e^2 & e^{-2} \end{pmatrix}$$

Where $a_1 = -1.875e^4 - 3, \; a_2 = -3.75e^4 - 3, \; a_3 = -0.125e^8 - 0.3125e^4 - 3.5$ and $a_4 = -2e^8 - 5e^4 - 3.5$. In this case we have $\bar{h} = 1$ and μ=0.5. In [11] it is found that the systems is exponentially stable with decay rate α=2. Applying theorem 1 with the decay rate α=3.2, we found that both the condition (11) and (12) are satisfied with the system matrices L_i given by:.

$$\left\{ L_1 = \begin{pmatrix} -3.7084 & -2.4387 \\ -2.4387 & -7.5107 \end{pmatrix}, \; L_2 = 10^4 \begin{pmatrix} -0.0632 & -0.0594 \\ -0.0594 & -1.8317 \end{pmatrix} \right\}$$

Consequently the system is robustly stable with decay rate α=3.2.

Example 3 [2]: consider the uncertain linear time delay systems (21) with,

$$A = \begin{pmatrix} -4 & 0 \\ 1 & -4 \end{pmatrix}, \; D = \begin{pmatrix} 0.1 & 0 \\ 1 & 0.1 \end{pmatrix}, \; h(t)=0.2\cos^2(2.5t)$$

$$E_0 = E_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \; H_0 = H_1 = \begin{pmatrix} 0.2 & 0 \\ 0 & 0.2 \end{pmatrix},$$

Then we have $\bar{h} = 0.2$, μ=0.5. the criteria for exponential stability obtained in [4, 7] are not applicable. Applying corollary 1, we obtain a decay rate α=3.4039. For the comparison with other approaches we give table 1.

approach	Decay rate α
Corollary 2	3.4039
Hien and Phat [2]	2.03
Mondie and Kharitonov [8]	1.786
Niculescu et al. [10]	1.625

Table 1: decay rate α

From the numerical examples it is clear that our approach may give improved decay rate for both switched or non switched systems.

5. Conclusion

In this paper, some new delay dependent robust exponential stability criteria for switched linear systems have been presented. These criteria are expressed in terms of linear matrix inequality. The approach uses Lyapunov Krasovskii functional combined with slack variable approach. Some numerical examples are given to illustrate these results presented in this paper have significant improvement over the existing ones.

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