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# An Efficient Technique for Finding the Eigenvalues of Sixth-Order Sturm-Liouville Problems

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**Abstract**

In this paper, we will develop a numerical technique for finding the eigenvalues of sixth order nonsingular Sturm–Liouville problems. We used the variational iteration methods as a basis for this technique. Numerical results and conclusions will be presented. Comparison results with others will be presented.

**Mathematics Subject Classifications:** 65

**Keywords:** Eigenvalues, Sixth-Order Sturm-Liouville problems, Variational iteration method

# 1. Introduction

In this paper, we will develop a numerical technique for finding the eigenvalues of the following sixth-order non-singular Sturm-Liouville problem

$$\begin{aligned}(q_0(x)y'''(x))''' &= (q_1(x)y''(x))'' - (q_2(x)y'(x))' + (q_3(x) - \mu w(x))y(x) \quad ((1))\\ &= F(y(x), y'(x), y''(x), y'''(x), y^{(4)}(x), \mu), \quad x \in (a, b)\end{aligned}$$

subject to some six point specified conditions at the boundary of the domain on $y, y', y'', q_0y''', q_1y'' - (q_0y''')'$and/or $(q_0y''')'' - (q_1y'')' + q_2y'$ where $q_3(x), q_2(x), q_1(x), q_0(x),$

and $w(x)$ are piecewise continuous functions with $q_0(x), w(x) \geq 0$. The boundary value problems for ordinary differential equations play a very important role in both theory and applications. They are used to describe a large number of physical, biological and chemical phenomena. Equation (1) is often to as the ring structures. When considering the instability setting in an infinite horizontal layer of fluid, which is heated from below and is subject to the action of rotation, we model the instability as ordinary convection by a sixth-order eigenvalue problem similar to Problem (1). For more details, see [3], [4], [12], [17], [20], [22].

Equation (1) is often referred to as the circular ring structure with constraints which has rectangular cross-sections of constant width and parabolic variable thickness. For more details, see [18] and [23].

Greenberg [13] and [14], showed that Problem (1) has an infinite sequence of eigenvalues $\{\mu_0, \mu_1, \mu_2, ...\}$ such that

$$\eta < \mu_0 \leq \mu_1 \leq \mu_2 \leq ...$$

with

$$\underset{n\to\infty}{Lim}\mu_n = \infty$$

where $\eta$ is a constant and each eigenvalue has multiplicity at most 3.

The numerical treatment of such problems has always far from trivial. For this reason several authors have been extensively involved in the solution of such problems and numerous innovative methods and approaches have enriched the scientific literature; each with its particular merits and advantages. Lesnic [19] used the Adomian decomposition method (ADM) to solve sixth order eigenvalue problems. Also, the only code available in this regard is the Sturm-Liouville Eigenvalues Using Theta Matrices (SLEUTH) [15]. In this paper, we will treat Problem (1) using the iterated variation method (IVM). Comparisons with the ADM and the code SLEUTH will be presented.

We will organize our paper as follows. In Section 2, We will develop the IVM for sixth -order differential equation (1). Several numerical examples are discussed in Section 3. Some conclusions are drawn in Section 4.

# 2. Variational Iteration Method

For the purpose of illustration of the variational iteration method, we rewrite Equation (1) in the following form

$$\begin{aligned} y^{(6)} &= G(y(x), y'(x), y''(x), y'''(x), y^{(4)}(x), y^{(5)}(x)\mu) \\ &= (F(y(x), y'(x), y''(x), y'''(x), \mu) - q_0'''(x)y'''(x) - 3q_0''(x)y^{(4)}(x) \\ &\quad -3q_0'(x)y^{(5)}(x))/q_0(x). \end{aligned}$$

which can be written in the operator form as

$$L(y(x)) + N(y(x)) = 0, \tag{(2)}$$

where

$$\begin{aligned} L(y(x)) &= y^{(6)} \\ N(y(x) &= -G(y(x), y'(x), y''(x), y'''(x), y^{(4)}(x), y^{(5)}(x)\mu). \end{aligned}$$

Following the variational iteration method [1], [2], [5]−[11], [21] we can construct a correct functional as follows:

$$y_{n+1}(x) = y_n(x) + \int_0^x \lambda \left[ L(y_n(\tau)) + N(\widetilde{y_n}(\tau)) \right] d\tau, \tag{(3)}$$

where $\lambda$ is a general Lagrange multiplier which can be found using the variational theory, the subscript $n$ means the $n^{th}$ order approximation, and $\widetilde{y_n}$ is a restricted variation with the property $\delta\widetilde{y_n} = 0$, for more details see [5] and [21]. Simple calculations gives us:

$$\lambda(\tau)y_n^{(6)}(\tau) = \frac{d\left(\lambda(\tau)y_n^{(5)}(\tau)\right)}{d\tau} - \lambda'(\tau)y_n^{(5)}(\tau), \tag{(4)}$$

$$\lambda'(\tau)y_n^{(5)}(\tau) = \frac{d\left(\lambda'(\tau)y_n^{(4)}(\tau)\right)}{d\tau} - \lambda''(\tau)y_n^{(4)}(\tau), \tag{(5)}$$

$$\lambda''(\tau)y_n^{(4)}(\tau) = \frac{d\left(\lambda''(\tau)y_n^{(3)}(\tau)\right)}{d\tau} - \lambda'''(\tau)y_n^{(3)}(\tau), \tag{(6)}$$

$$\lambda'''(\tau)y_n^{(3)}(\tau) = \frac{d\left(\lambda'''(\tau)y_n^{(2)}(\tau)\right)}{d\tau} - \lambda^{(4)}(\tau)y_n^{(2)}(\tau), \tag{(7)}$$

$$\lambda^{(4)}(\tau)y_n^{(2)}(\tau) = \frac{d\left(\lambda^{(4)}(\tau)y_n'(\tau)\right)}{d\tau} - \lambda^{(5)}(\tau)y_n'(\tau), \tag{(8)}$$

$$\lambda^{(5)}(\tau)y_n'(\tau) = \frac{d\left(\lambda^{(5)}(\tau)y_n(\tau)\right)}{d\tau} - \lambda^{(6)}(\tau)y_n(\tau) \qquad ((9))$$

Using equations $(4)-(9)$, one can see that

$$\begin{aligned} y_{n+1}(x) &= y_n(x) + \left[ \begin{array}{c} \lambda(\tau)y_n^{(5)}(\tau) - \lambda'(\tau)y_n^{(4)}(\tau) + \lambda''(\tau)y_n^{(3)}(\tau) \\ -\lambda'''(\tau)y_n^{(2)}(\tau) + \lambda^{(4)}(\tau)y_n'(\tau) - \lambda^{(5)}(\tau)y_n(\tau) \end{array} \right]_{\tau=0}^{\tau=x} \qquad ((10)) \\ &\quad + \int_0^x \left[\lambda^{(6)}(\tau)y_n(\tau) + \lambda N(\widetilde{y_n}(\tau))\right] d\tau. \end{aligned}$$

In order to determine the Lagrange multiplier, we make the functional in (10) stationary, and by noticing that $\delta y_n(0) = 0$, we obtain

$$\begin{aligned} \delta y_{n+1}(x) &= (1-\lambda^{(5)}(x))\delta y_n(x) + \lambda(x)\delta y_n^{(5)}(x) - \lambda'(x)\delta y_n^{(4)}(x) + \lambda''(x)\delta y_n^{(3)}(x) \\ &\quad -\lambda'''(x)\delta y_n^{(2)}(x) + \lambda^{(4)}(x)\delta y_n'(x) + \int_0^x \left[\lambda^{(6)}(\tau)\delta y_n(\tau) + \lambda N(\widetilde{\delta y_n}(\tau))\right] d\tau. \end{aligned}$$

Consequently, we obtain the following stationary conditions

$$\begin{array}{rc} \delta y_n : & \lambda^{(6)}(\tau) = 0 \\ \delta y_n : & 1 - \lambda^{(5)}(\tau) \mid_{\tau=x} = 0 \\ \delta y_n' : & \lambda^{(4)}(\tau) \mid_{\tau=x} = 0 \\ \delta y_n'' : & \lambda'''(\tau) \mid_{\tau=x} = 0 \\ \delta y_n''' : & \lambda''(\tau) \mid_{\tau=x} = 0 \\ \delta y_n^{(4)} : & \lambda'(\tau) \mid_{\tau=x} = 0 \\ \delta y_n^{(5)} : & \lambda(\tau) \mid_{\tau=x} = 0 \end{array}$$

Solving the last initial value problem, it can be seen that the lagrange multiplier is

$$\lambda(\tau) = \frac{(\tau - x)^5}{120}.$$

Therefore, we get the following iteration formulae

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(\tau - x)^5}{120} \left[L(y_n(\tau)) + N(y_n(\tau))\right] d\tau, \qquad ((11))$$

where $y_0(x)$ can be chosen to be the solution of the equation $L(y(x)) = 0$. This implies that

$$y_0(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4 + a_5x^5. \qquad ((12))$$

The convergence of the iteration variational methods, as applied to differential equations has been investigated in $[5]-[8]$. When carrying out the numerical details and if three conditions are specified initially at $x = a$, the approximate solution will be in the form

$$y_n(x,\mu) = c\ f_n(x,\mu) + d\ g_n(x,\mu) + eh_n(x,\mu), n > 0.$$

To satisfy the other conditions, for example

$$y_n(a,\mu) = y_n'(a,\mu) = y_n''(a,\mu) = 0,$$

we should solve the following system

$$\begin{aligned} c\ f_n(a,\mu) + d\ g_n(a,\mu) + eh_n(a,\mu) &= 0 \\ c\ f_n'(a,\mu) + d\ g_n'(a,\mu) + eh_n'(a,\mu) &= 0 \\ c\ f_n''(a,\mu) + d\ g_n''(a,\mu) + eh_n''(a,\mu) &= 0 \end{aligned}$$

for $c, d$ and $e$. Using Crammer's rule, we will get a nonzero solution for the last system if

$$W_n(\mu) = \det \begin{pmatrix} f_n(a,\mu) & g_n(a,\mu) & h_n(a,\mu) \\ f_n'(a,\mu) & g_n'(a,\mu) & h_n'(a,\mu) \\ f_n''(a,\mu) & g_n''(a,\mu) & h_n''(a,\mu) \end{pmatrix} = 0.$$

Therefore, the eigenvalues of problem (1) are the roots of $W_n(\mu)$.We summarize the whole procedure in the following algorithm.

**Algorithm (2.1):**

**Step 1:** Use formula (12) and the initial conditions to find $y_0$.

**Step 2:** Use iteration formula (11) to generate the sequence $\{y_n\}_0^N$ for some positive integer $N$.

**Step 3:** Compute $W_N(\mu)$.

**Step 4:** Find the roots of the equation $W_N(\mu) = 0$.

The roots in Step (4) in Algorithm (2.1) are the eigenvalues of Problem (1).

# 3. Numerical Results

In this section we will present two of our numerical results of sixth-order Sturm-Liouville problems using the method outlined in the previous section. We are interesting in approximating an eigen pair solution $(\mu, y(x))$ to such boundary value problems.

**Example (3.1):** Consider the following sixth order eigenvalue problem

$$y^{(6)}(x) = \mu y(x), x \in (0,1) \qquad ((13))$$

*subject to*

$$\begin{aligned} y(0) &= y''(0) = y^{(4)}(0) = 0, \\ y(1) &= y''(1) = y^{(4)}(1) = 0. \end{aligned}$$

Let $L(y) = y^{(6)}$ and $N(y) = -\mu y(x)$. Choose $y_0$ so that

$$L(y_0) = 0, y(0) = y''(0) = y^{(4)}(0) = 0.$$

Simple calculations implies that $y_0(x) = a\ x + b\ x^3 + c\ x^5$ where $a, b$ and $c$ are constants. The iteration formula (11) becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(\tau - x)^5}{120} \left[ y_n^{(6)}(\tau)) - \mu y_n(\tau) \right] d\tau$$

which implies that

$$\begin{aligned} y_1(x) &= a\left(x + \frac{x^7\mu}{5040}\right) + b\left(x^3 + \frac{x^9\mu}{60480}\right) + c\left(x^5 + \frac{x^{11}\mu}{332640}\right), \\ y_2(x) &= a\left(x + \frac{x^7\mu}{5040} + \frac{x^{13}\mu^2}{6227020800}\right) + b\left(x^3 + \frac{x^9\mu}{60480} + \frac{x^{15}\mu^2}{217945728000}\right) \\ &\quad + c\left(x^5 + \frac{x^{11}\mu}{332640} + \frac{x^{17}\mu^2}{2964061900800}\right) \\ &\quad \vdots \end{aligned}$$

It is easy to see that

$$y_n(x,\mu) = a\ f_n(x,\mu) + b\ g_n(x,\mu) + c\ h_n(x,\mu), n > 0.$$

Using the boundary conditions at 1, we get three equations of the form

$$\begin{aligned} y_n(1,\mu) &= a\ f_n(1,\mu) + b\ g_n(1,\mu) + c\ h_n(1,\mu) = 0, \\ y_n''(1,\mu) &= a\ f_n''(1,\mu) + b\ g_n''(1,\mu) + c\ h_n''(1,\mu) = 0, \\ y_n^{(4)}(1,\mu) &= a\ f_n^{(4)}(1,\mu) + b\ g_n^{(4)}(1,\mu) + c\ h_n^{(4)}(1,\mu) = 0. \end{aligned}$$

To have nonzero values for $a, b$ and $c$ and using Crammer's rule,

$$W_n(\mu) = \det \begin{pmatrix} f_n(1,\mu) & g_n(1,\mu) & h_n(1,\mu) \\ f_n''(1,\mu) & g_n''(1,\mu) & h_n''(1,\mu) \\ f_n^{(4)}(1,\mu) & g_n^{(4)}(1,\mu) & h_n^{(4)}(1,\mu) \end{pmatrix} = 0.$$

If we compute the first elven terms, then $W_n(\mu)$ has 11 zeros, six of them are the first six eigenvalues of Problem (12). the ratios $\frac{\sqrt[6]{-\mu_k}}{\pi}$ are given in **Table** (1). Normalize the first three eigenfunction as

$$\bar{y}_j = \frac{y_{10}(x,\mu_j)}{\int_0^1 \left|y_{10}(x,\mu_j)\right| dx}, j = 1,2,3.$$

We will sketch the graph of $\bar{y}_1, \bar{y}_2, \bar{y}_3$with the exact corresponding eigenfunctions in **Figure** 1. The solid line represents the exact solutions and the dots represents $\bar{y}_1, \bar{y}_2, \bar{y}_3$.One can easily check that the eigenvalue are $\mu_k = -(k\pi)^6$ and the corresponding normailized eigenfunction is $\sin(k\pi x)$.Thus, the exact value for the ratio $\frac{\sqrt[6]{-\mu_k}}{\pi}$ is $k$.

| $k$ | $\frac{\sqrt[6]{-\mu_k}}{\pi}$ | $k$ | $\frac{\sqrt[6]{-\mu_k}}{\pi}$ |
|---|---|---|---|
| 1 | 1.0000000000000 | 4 | 4.0000000000005 |
| 2 | 1.9999999999999 | 5 | 4.9999999999987 |
| 3 | 2.9999999999999 | 6 | 5.9999999999214 |

**Table (1): The first six ratios for Example (3.1)**

**Example (3.2):** Consider the following sixth-order eigenvalue problem [19]

$$y^{(6)}(x) = (q_1(x)y''(x))'' - (q_2(x)y'(x))' + (q_3(x) - \mu)\, y(x),\ x \in (0,5)$$

*subject to*

$$\begin{aligned} y(0) &= y''(0) = y^{(4)}(0) = 0, \\ y(5) &= y''(5) = y^{(4)}(5) = 0, \end{aligned}$$

where

$$\begin{aligned} q_1(x) &= 3\alpha x^2, q_2(x) = 3\alpha^2 x^4 - 8\alpha, \\ q_3(x) &= \alpha^3 x^6 - 14\alpha^2 x^2, \end{aligned}$$

and $\alpha$ is a constant. To compare our results with [19] and [16], we will choose $\alpha = 0.01$.

Let $L(y) = y^{(6)}$ and

$$N(y) = -(q_1(x)y''(x))'' + (q_2(x)y'(x))' - \ (q_3(x) - \mu w(x))y(x)$$

Choose $y_0$ so that

$$L(y_0) = 0, y(0) = y''(0) = y^{(4)}(0) = 0.$$

Simple calculations implies that

$$y_0(x) = a\ x + bx^3 + cx^5$$

where $a$ and $b$ are constants. The iteration formula (11) becomes

$$y_{n+1}(x) = y_n(x) + \int_0^x \frac{(\tau - x)^5}{120} \left[L(y_n(\tau)) + N(y_n(\tau))\right] d\tau, \qquad ((14))$$

which implies that

$$y_i(x) = a\ f_i(x, \mu) + b\ g_i(x, \mu) + c\ h_i(x, \mu), i = 1, 2, ...$$

where

$$\begin{aligned}
f_1(x,\mu) &= x - \frac{13x^9}{302400000} + \frac{x^{13}}{1235520000000} - \frac{\mu x^7}{5040} \\
g_1(x,\mu) &= x^3 + \frac{13x^7}{42000} - \frac{17x^{11}}{831600000} + \frac{x^{15}}{3603600000000} - \frac{\mu x^9}{60480} \\
h_1(x,\mu) &= x^5 + \frac{17x^9}{75600} - \frac{67x^{13}}{6177600000} + \frac{x^{17}}{8910720000000} - \frac{\mu x^{11}}{332640} \\
&\vdots
\end{aligned}$$

We use the same procedure as we did in Example 3.1 and we generate the first ten terms in the sequence (14) . The first few eigenvalues and eigenfunctions are given in **Table** 2.

| $k$ | $\mu_k$ (our results) | $k$ | $\mu_k$ (Lesnic'results) |
|---|---|---|---|
| 1 | 0.099726772771261 | 1 | 0.099726772771272 |
| 2 | 4.572321409243512 | 2 | 4.572321409243511 |
| 3 | 48.04166837750243 | 3 | 48.04166837750251 |
| 4 | 262.0590748451667 | 4 | 262.0590748451697 |
| 5 | 985.8390701194121 | 5 | 985.8390701194187 |
| 6 | 2921.330653932955 | 6 | 2921.330653932924 |
| 7 | 7332.492153601054 | 7 | 7332.492153601076 |
| 8 | 16466.17014262612 | 8 | 16466.17014262633 |
| 9 | 21015.61609499579 | 9 | 21015.61609499584 |

**Table (2): The first nine eigenvalues for Example (3.2)**

# 1 Conclusions

In this paper, we developed a numerical technique for finding the eigenvalues of sixth-order non-singular Sturm-Liouville problems. To verify the accuracy of the presented technique, we have considered two numerical examples. In the first example, we computed the first six eigenvalues. Also, we sketched the first three eigenfunctions. Comparing our results with the exact values of the eigenvalues, we see that our results are very accurate and the error is of order $10^{-13}$. Similarly, from **Table** 1, we see that the approximated eigenfunctions are very close from the exact functions. In Example 3.2, we compare our results with Lesnic' results [19] and the code SLEUTH [16]. From these comparisons, we see that our results are very close from the results of Lesnic' results [19]. In addition, comparing the first two eigenvalues with the computed eigenvalues using the code SLEUTH, we see that the difference between them is less than $10^{-11}$. These results show that VIM is accurate and very efficient. Moreover, the calculations here is more simpler than the ADM.

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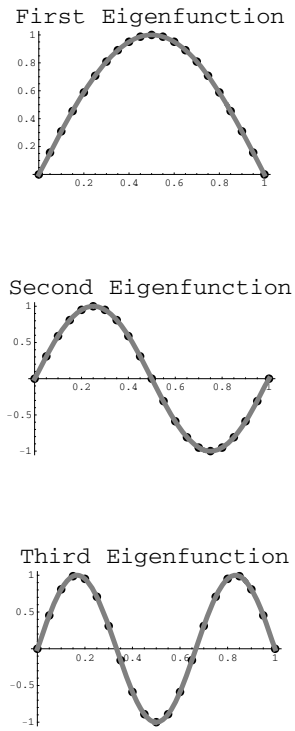
First Eigenfunction
Second Eigenfunction
Third Eigenfunction

Figure 1: