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Legendre Wavelets Method for Fractional Integro-Differential Equations

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Abstract

Legendre wavelets methods are commonly used for the numerical solution of integral equations. In this paper, we apply the Legendre wavelets method to approximate the solution of fractional integro-differential equations. Numerical examples are also presented to demonstrate the validity of the method.

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1 Introduction.

We extend the iterative algorithms due to S. Yousefi and M. Razzaghi [11] for computing the sequence $\{y_{k,M}(t)\}$ of the truncated wavelets series

$$y_{k,M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(t), \quad t \in [0,1]$$

interpolating the exact solution $y(t)$ of the initial-value problem for fractional integro-differential equation

$$D_*^q y(t) = f(t) + p(t)y(t) + \int_0^t k(t,s)y(s)ds, \quad t \in I = [0,1], \quad y(0) = \alpha \qquad (1)$$

where $0 < q < 1$ and $\alpha \in \mathbb{R}$. Here, the given functions f, p, k are assumed to be sufficiently smooth on their respective domains I and S ($S = \{(t, s) : 0 \le s \le t \le 1\}$). In equation (1), $\psi_{n,m}(t)$ is the Legendre wavelets polynomial and D_*^q denotes Caputo fractional derivative of order q. Amongst a variety of definition for fractional order derivatives, Caputo fractional derivative has been used because it allows physically interpretable initial conditions; see for example [5]. Recently some attentions have been paid to the numerical solution of equation (1). Rawashdeh [8] applied the collocation method to find a spline approximation, Shaer and algarellh [6] used the decomposition method to find an analytic solution, and A. Arikoglu and I. Ozkol [1] extended the fractional transform method which is a semi analytical numerical technique to approximate the solution of equation (1).

Equation (1) occurs in a wide variety of applied sciences. It is encountered in various areas of physics and engineering [7, 9]. Damping laws, diffusion processes [2] and fractals [9] are better formulated with the use of fractional derivatives.

The outline of this paper is as follows. In Section 2, we present some definitions and preliminaries. Section 3 contains the application of the Legendre wavelets method to equation (1). Some numerical examples are provided in Section 4.

2 Definitions and Preliminaries

Definition 2.1. *A real function $f(x), x > 0$, is said to be in the space C_α, $\alpha \in \mathbb{R}$ if there exist a real number $p > \alpha$, such that $f(x) = x^p f_1(x)$, where $f_1(x) \in C[0, \infty)$.*

Definition 2.2. *A real function $f(x), x > 0$, is said to be in the space C_α^k, $k \in \mathbb{N} \cup \{0\}$, if $f^{(k)} \in C_\alpha$.*

Definition 2.3. *The Riemann-Liouville fractional integral of order $q \ge 0$ of a function $f \in C_\alpha$, $\alpha \ge -1$, is defined by*

$$J^q f(x) = \frac{1}{\Gamma(q)} \int_0^x (x - s)^{q-1} f(s) ds.$$

Definition 2.4. *Let $f \in C_{-1}^k, k \in \mathbb{N}$. Then the Caputo fractional derivative of f is defined by*

$$D_*^q f(x) = \begin{cases} J^{k-q} f^{(k)}(x), & \text{if } k - 1 < q \le k, \\ f^{(k)}(t), & \text{if } q = k. \end{cases}$$

Some important properties of the Caputo operator can be found in [7].

To obtain a numerical scheme for the approximation of Caputo derivative, we can use a representation that has been introduced by Elliots [4];

$$D_*^{(q)} f(x) = \frac{1}{\Gamma(-q)} \int_0^x \frac{f(s) - f(0)}{(x-s)^{1+q}} ds, \tag{2}$$

where the integral in equation (2) is a Hadamard finite-part integral.

Definition 2.5. *The following functions*

$$\psi_{k,n}(t) = |a_0|^{k/2} \psi(a_0^k t - n b_0),$$

form a family of discrete wavelets, where $a_0 > 1, b_0 > 0$ *and* n, k *are positive integers and* ψ *is a given function called mother wavelet. Moreover, the functions*

$$\psi_{n,m}(t) = \begin{cases} \sqrt{m + \frac{1}{2}} 2^{k/2} p_m(2^k t - \widehat{n}), & \frac{\widehat{n}-1}{2^k} \leq t < \frac{\widehat{n}+1}{2^k}, \\ 0, & otherwise \end{cases} \tag{3}$$

are called Legendre wavelets polynomials where $\widehat{n} = 2n - 1, n = 1, \ldots, 2^{k-1}$, $k \in \mathbb{N}$, $t \in [0, 1]$ *and* m *is the order of the Legendre polynomials* p_m.

3 Legendre wavelets method

In the present paper, we consider the following fractional integro-differential equation

$$D_*^q y(t) = f(t) + p(t) y(t) + \int_0^t k(t,s) y(s) ds, \quad t \in I = [0, 1], \quad y(0) = \alpha. \tag{4}$$

The exact solution of equation (4) can be expanded as a Legendre wavelets series as follows:

$$y(t) = \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{n,m}(t).$$

where $\psi_{n,m}(t)$ is given by equation (3). We approximate the solution $y(t)$ by the truncated series

$$y_{k,M}(t) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \psi_{n,m}(t). \tag{5}$$

Then a total number of $2^{k-1}M$ conditions should exist for determination of $2^{k-1}M$ coefficients

$$c_{10}, c_{11}, \ldots, c_{1M-1}, c_{20}, \ldots, c_{2M-1}, \ldots, c_{2^{k-1}0}, \ldots, c_{2^{k-1}M-1}.$$

Since one condition is furnished by the initial condition, namely

$$y_{k,M}(0) = \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm}\psi_{n,m}(0) = \alpha. \tag{6}$$

We see that there should be $2^{k-1}M - 1$ extra conditions to recover the unknown coefficients c_{nm}. These conditions can be obtained by substituting equation (5) in equation (4);

$$\begin{aligned} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} D_*^q \psi_{n,m}(t) &= f(t) + \textstyle\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} p(t)\psi_{n,m}(t) && (7) \\ &+ \textstyle\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \int_0^t k(t,s)\psi_{n,m}(s)ds. && (8) \end{aligned}$$

We now assume equation (7) is exact at $2^{k-1}M - 1$ points x_i as follows:

$$\begin{aligned} \sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} D_*^q \psi_{n,m}(x_i) &= f(x_i) + \textstyle\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} p(x_i)\psi_{n,m}(x_i) && (9) \\ &+ \textstyle\sum_{n=1}^{2^{k-1}} \sum_{m=0}^{M-1} c_{nm} \int_0^{x_i} k(x_i,s)\psi_{n,m}(s)ds. && (10) \end{aligned}$$

The best choice of the x_i points are the zeros of the shifted Chebyshev polynomials of degree $2^{k-1}M-1$ in the interval [0,1] that is $x_i = \frac{s_i+1}{2}$, where $s_i = \cos\left(\frac{(2i-1)\pi}{2^{k-1}M-1}\right), i = 1, \ldots, 2^{k-1}M - 1$.

Approximating $D_*^q \psi_{n,m}$ using Diethelm method [3] on the representation that has been given by equation (2), we get

$$\begin{aligned} D_*^{(q)}\psi_{n,m}(x_i) &= \int_0^{x_i} \frac{\psi_{n,m}(s)-\psi_{n,m}(0)}{(x_i-s)^{1+q}} ds = \frac{x_i^{-q}}{\Gamma(-q)} \int_0^1 \frac{\psi_{n,m}(x_i - x_i w)-\psi_{n,m}(0)}{w^{1+q}} dw \\ &\approx \textstyle\sum_{r=0}^{L} \alpha_r (\psi_{n,m}(x_i - x_i r/L) - \psi_{n,m}(0)). \end{aligned}$$

where $L \in \mathbb{N}$ and the weights α_r are given by

$$q(1-q)L^{-q}\frac{\Gamma(-q)}{x_i^{-q}}\alpha_r = \begin{cases} -1, & \text{if } r = 0, \\ 2r^{1-q} - (r-1)^{1-q} - (r+1)^{1-q}, & \text{if } r = 1, 2, \ldots, L-1, \\ (q-1)r^{-q} - (r-1)^{1-q} + r^{1-q}, & \text{if } r = L. \end{cases}$$

Then equation (8) becomes

$$\sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1}\sum_{r=0}^{L}\alpha_r(\psi_{n,m}(x_i - x_i r/L) - \psi_{n,m}(0))c_{nm} = f(x_i) + \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}p(x_i)\psi_{n,m}(x_i) \quad (11)$$

$$+ \sum_{n=1}^{2^{k-1}}\sum_{m=0}^{M-1} c_{nm}\int_0^{x_i} k(x_i,s)\psi_{n,m}(s)ds \quad (12)$$

Combine equation (6) and (9) to obtain $2^{k-1}M$ linear equations from which we can compute values for the unknowns coefficients, c_{nm}.

4 Numerical Examples

To demonstrate the effectiveness of the proposed method we consider here some fractional integro-differential equations. We use the software **Maple** to get the numerical results.

Example 4.1. *Consider the following fractional integro-differential equation*

$$D_*^{0.75}y(t) = \frac{t^{0.25}}{\Gamma(1.25)} - t^2 - t - \frac{1}{3}t^4 - \frac{1}{2}t^3 + ty(t) + \int_0^t tsy(s)ds, \quad (13)$$

with the initial condition

$$y(0) = 1, \quad (14)$$

and the exact solution of (10) and (11) is $y(t) = t + 1$.

We solved the linear system that was obtained by equations (6) and (9) with $k = 2$ and $M = 2$ to get

$$c_{10} = 0.88388, c_{11} = 0.10206, c_{20} = 1.2374, c_{21} = 0.10206.$$

Then the approximate solution will be

$$\sum_{n=1}^{2}\sum_{m=0}^{1} c_{nm}\psi_{n,m}(t) = \begin{cases} 0.88388(\sqrt{2}) + 0.10206(\sqrt{6})(4t-1) = 0.99997t + 1.00005, & 0 \le t < 0.5, \\ 1.2374(\sqrt{2}) + 0.10206(\sqrt{6})(t-3) = 0.99997t + 0.99996, & 0.5 \le t \le 1. \end{cases}$$

It is clear that the approximate solution almost coincides with the analytic solution.

Example 4.2. *Consider the following fractional integro-differential equation*

$$D_*^{0.25}y(t) = \frac{6t^{2.75}}{\Gamma(3.75)} - \frac{3}{10}t^2e^t - \frac{1}{5}t^2e^ty(t) + \int_0^t e^tsy(s)ds, \quad (15)$$

with the initial condition

$$y(0) = 1, \tag{16}$$

and the exact solution of (12) and (13) is $y(t) = t^3 + 1$.

We use the Legendre wavelets method with $k = 1$ and $M = 6$ to obtain

$$c_{10} = 1.2507, c_{11} = 0.26054, c_{12} = 0.11211$$

$$c_{13} = 0.01895, c_{14} = 8.129 \times 10^{-10}, c_{15} = -1.8037 \times 10^{-10}.$$

Then the approximate solution will be

$$\begin{aligned} \sum_{m=0}^{5} c_{1m}\psi_{1,m}(t) \qquad &= -0.15075 \times 10^{-6}t^5 + 0.5476 \times 10^{-6}t^4 \\ &+1.00282t^3 + 0.8307 \times 10^{-6}t^2 - 0.312 \times 10^{-7}t + 1. \end{aligned}$$

We see that the approximate solution is almost overlapping the exact solution.

Example 4.3. *Consider the following fractional integro-differential equation*

$$D_*^{0.5}y(t) = \frac{2e^{-t}}{\sqrt{\pi}} \int_0^t \frac{e^s}{\sqrt{s}}ds - e^t \sin ty(t) + \int_0^t e^s \cos sy(s)ds, \tag{17}$$

with the initial condition

$$y(0) = 2, \tag{18}$$

and the exact solution of (14) and (15) is $y(t) = 2e^{-t}$.

Solving the linear system that was obtained by equations (6) and (9) with $k = 1$ and $M = 8$, we get the approximate solution

$$\begin{aligned} \sum_{m=0}^{7} c_{1m}\psi_{n,m}(t) \qquad &= 2 - 2t + 1.0003t^2 - 0.3336t^3 + 0.8310 \times 10^{-1}t^4 \\ &-0.1667 \times 10^{-1}t^5 + 0.2778 \times 10^{-2}t^6 - 0.3968 \times 10^{-3}t^7. \end{aligned}$$

This approximate solution shows excellent agreement with the Taylor polynomial of degree 7 of the function $2e^{-t}$.

5 Conclusion

In this work we illustrated a numerical algorithm for solving fractional integro-differential equations using Legendre wavelets method. We derived a system of equations that characterizing the numerical solution. It has been numerically demonstrated that the proposed method is effective and easy to use.

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