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A New Operational Matrix of Derivative for Chebyshev Wavelets and its Applications in Solving Ordinary Differential Equations with Non analytic Solution

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Abstract

In this letter a new operational matrix of derivative for Chebyshev wavelets is presented. Shifted Chebyshev polynomials and their properties are employed for deriving a general procedure for forming this matrix. Then Chebyshev wavelets expansions along with these operational matrices are applied for solving ordinary differential equations with non analytic solution. Although spectral methods such as Galerkin, tau and pseudospectral methods by using orthogonal polynomials do not work well for solving ordinary differential equations in which, at least, one of the coefficient functions or solution function is not analytic, but it is shown that the Chebyshev wavelet Galerkin method is very efficient and suitable for solving this kind of problem. Several numerical examples are given to illustrate the efficiency and performance of the presented method.

Keywords: Chebyshev wavelets, Shifted Chebyshev polynomials, Operational

matrix, Galerkin method, Singular ordinary differential equations.

1. Introduction

Orthogonal functions and polynomial series have received considerable attention in dealing with various problems of dynamic systems. The main characteristic of this technique is that it reduces these problems to those of solving a system of algebraic equations, thus greatly simplifying the problem. In this method, a truncated orthogonal series is used for numerical integration of differential equations, with the goal of obtaining efficient computational solutions. For this purpose there are two approaches.

One approach is based on converting the underlying differential equations into integral equations through integration, approximating various signals involved in the equation by truncated orthogonal series and using the operational matrix P of integration, to eliminate the integral operations. The operational matrix of integration P is given by

$$\int_o^t \Psi(s)ds = P\,\Psi(t) \tag{1}$$

Where $\Psi(t) = [\psi_1, \psi_2, ..., \psi_N]$ and $\psi_i, i = 1,2,...N$ are orthogonal basis functions, orthogonal on a certain interval $[a,b]$. Special attentions have been given to applications of block Walsh functions [6], Legendre polynomials [13], Chebyshev polynomials [8], Haar wavelets [7], Legendre wavelets [11] and Chebyshev wavelets [1].
Another approach is based on using operational matrix of derivatives in order to reducing the underlying problem into solving a system of algebraic equations. The operational matrix of derivatives D is given by

$$\frac{d\Psi(t)}{dt} = D\,\Psi(t) \tag{2}$$

Where $\Psi(t) = [\psi_1, \psi_2, ..., \psi_N]$ and $\psi_i, i = 1,2,...N$ are orthogonal basis functions. Several papers have appeared in the literature concerned with the application of operational matrix of derivatives [12, 13].

It is well known that for a given ordinary differential equation which is defined on interval $[a,b]$, if solution function and coefficient functions are analytic on $[a,b]$, spectral methods are very efficient and suitable. The cases in which, at least, one of the coefficient functions or solution function is not analytic on $[a,b]$, spectral methods do not work well. For more details refer to [2, 4].

Here, the cases in which, at least, one of the coefficient functions or solution function is not analytic will be studied. As described before, spectral (Galerkin,

tau and pseudospectral) methods do not work well for this kind of problems. To overcome this difficulty Chebyshev wavelet Galerkin method is proposed, in this paper. An important property of the proposed method is that spectral accuracy can be found for solving specific singular ordinary differential equations. It is clearly demonstrated in examples.

In this paper The Chebyshev wavelets are first introduced, then by using shifted Chebyshev polynomial and their properties the operational matrix of derivative for Chebyshev wavelets is derived. Then, applications of this matrix for solving ordinary differential equations with non analytic solution are described. Illustrative examples are given to demonstrate the efficiency and capability of the proposed method.

The article is organized as follows: We begin by introducing some necessary definitions and mathematical preliminaries Chebyshev and shifted Chebyshev polynomials which are required for establishing our results. In Section 3 the Chebyshev operational matrix of derivative is obtained. Section 4 is devoted to applying the Chebyshev operational matrix of fractional derivative for solving ordinary differential equations. In Section 5 the proposed method is applied to several examples. Also a conclusion is given in Section 6.

2. Preliminaries and notations

In this section we give some necessary definitions and mathematical preliminaries which are used further in this paper.

2.1 Shifted Chebyshev polynomials and their properties

The Chebyshev polynomials which are orthogonal with respect to the weight function $w(t)=\frac{1}{\sqrt{1-t^2}}$ defined on the interval $[-1,1]$ and they obey the following recurrence relation [4]

$$T_0(t)=1,\ T_1(t)=2t-1,\quad T_{m+1}(t)=2tT_m(t)-T_{m-1}(t),\quad m=1,\,2,\,3,\ldots$$

In order to use Chebyshev polynomials on the interval $[0,1]$ we define the so-called shifted Chebyshev polynomials by introducing the change of variable $t=2x-1$. Let the shifted Chebyshev polynomials $T_m(2x-1)$ be denoted by $\tilde{T}_m(x)$, Then $\tilde{T}_m(x)$ can be obtained as follows:

$$\tilde{T}_0(x)=1,\ \tilde{T}_1(x)=2x-1,\quad \tilde{T}_{m+1}(x)=2(2x-1)\tilde{T}_m(x)-\tilde{T}_{m-1}(x),\quad m=1,\,2,\,3,\ldots \qquad (3)$$

The orthogonality condition for these shifted polynomials is

$$\int_0^1 \frac{\tilde{T}_m(x)\tilde{T}_n(x)}{\sqrt{1-(2x-1)^2}}\,dx=\begin{cases}\dfrac{\pi\gamma_m}{4} & m=n\\ 0 & m\neq n\end{cases} \qquad (4)$$

In which

$$\gamma_m = \begin{cases} 2 & m = 0, \\ 1 & m \geq 1. \end{cases} \tag{5}$$

Theorem 1: Let $\tilde{T}_m(x)$ be the shifted Chebyshev polynomials into $[0,1]$ and $\tilde{T}_m'(x)$ be derivative of $\tilde{T}_m(x)$ with respect to x, then we have

$$\tilde{T}_m'(x) = \sum_{\substack{k=0 \\ k+m\ odd}}^{m-1} \frac{4m}{\gamma_k} \tilde{T}_k(x). \tag{6}$$

Proof: See [12].■

2.2 wavelets and Chebyshev wavelets

In recent years, wavelets [1, 3, 7, 10, 11, 12] have found their way into many different fields of science and engineering. Wavelets constitute a family of functions constructed from dilation and translation of single function called the mother wavelet. When the dilation parameter a and the translation parameter b vary continuously, we have the following family of continuous wavelets:

$$\psi_{a,b}(t) = |a|^{-\frac{1}{2}} \psi\left(\frac{t-b}{a}\right) \; a,b \in R, a \neq 0 \tag{7}$$

Chebyshev wavelets $\psi_{nm}(t) = \psi(k,n,m,t)$ have four arguments; n argument k can assume any positive integer, m is the order for Chebyshev polynomials and t is the normalized time. They are defined on the interval $[0,1)$ by

$$\psi_{n,m}(t) = \begin{cases} 2^{\frac{k+1}{2}} P_m(2^{k+1}t - 2n - 1) & \frac{n}{2^k} \leq t < \frac{n+1}{2^k} \\ 0 & otherwise \end{cases} \tag{8}$$

where

$$P_m(t) = \begin{cases} \frac{1}{\sqrt{\pi}} & m = 0 \\ \sqrt{\frac{2}{\pi}} T_m(t) & m \geq 1 \end{cases}$$

and $m = 0,1,...,M$ and $n = 0,1,...,2^k - 1$. We should note that in dealing with Chebyshev wavelets the weight function $w(x)$ have to be dilated and translated as

$w_n(t) = w(2^k t - 2n + 1)$.

A function $f(t)$ defined over $L^2[0,1]$ can be expanded in the terms of Chebyshev wavelets as

$$f(t) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} c_{nm} \psi_{nm}(t), \tag{9}$$

where $c_{nm} = (f(t), \psi_{nm}(t))$, in which $(.,.)$ denotes the inner product. If the infinite series in (9) is truncated, then it can be written as

$$f(t)=\sum_{n=0}^{2^k-1}\sum_{m=0}^{M-1}c_{nm}\psi_{nm}(t)=C^T\Psi(t), \tag{10}$$

where C and $\Psi(t)$ are $2^k(M+1)\times 1$ matrices given by

$$C=[c_{0,0},c_{0,1},...,c_{0,M},...,c_{2,M},...,c_{(2^k-1),0},c_{(2^k-1),1},...,c_{(2^k-1),M}]^T$$

$$\Psi(t)=[\psi_{0,0},\psi_{0,1},...,\psi_{0,M},...,\psi_{(2^k-1),M},...,\psi_{(2^k-1),0},\psi_{(2^k-1),1},...,\psi_{(2^k-1),M}]^T \tag{11}$$

3. Operational matrix of derivative

In the following section we introduce a new method for deriving operational matrix of derivative for Chebyshev wavelet, and finally we defined the operational matrices of product for Chebyshev wavelets.

Theorem 2: Let $\Psi(t)$ be the Chebyshev wavelets vector defined in (11). The derivative of this vector $\Psi(t)$ can be expressed by

$$\frac{d\Psi(t)}{dt}=D\Psi(t) \tag{12}$$

where D is the $2^k(M+1)$ operational matrix of derivative defined as follow

$$D=\begin{pmatrix} F & 0 & \cdots & 0\\ 0 & F & \cdots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & 0 & F\end{pmatrix}, \tag{13}$$

In which F is $(M+1)\times(M+1)$ matrix and its (r,s)-th element is defined as follow

$$F_{r,s}=\begin{cases} 2^{k+2}m\sqrt{\dfrac{c_{r-1}}{c_{s-1}}} & r=2,...,(M+1),\ s=1,...,r-1\ \ and\ (r+s)\ odd\\ 0 & otherwise\end{cases} \tag{14}$$

Proof: By using shifted Chebyshev polynomial into $[0,1]$ the rth element of vector $\Psi(t)$ in (11) can be written as

$$\Psi_r(t)=\psi_{n,m}(t)=2^{\frac{k+1}{2}}\sqrt{\frac{2}{\pi c_m}}\tilde{T}_m(2^kt-n)\ \chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]},\quad r=1,2,...,2^k(M+1) \tag{15}$$

Where $r=n(M+1)+(m+1)$, $m=0,1,...,M$, $n=0,1,...,(2^k-1)$ and $\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}$ is the characteristic function defined as

$$\chi_{[\frac{n}{2^k},\frac{n+1}{2^k}]}(t)=\begin{cases}1 & t\in[\dfrac{n}{2^k},\dfrac{n+1}{2^k}]\\ 0 & otherwise\end{cases}.$$

By differentiation with respect to t in (15) we have

$$\frac{d\Psi_r(t)}{dt} = 2^{\frac{k+1}{2}} \sqrt{\frac{2}{\pi c_m}}\, 2^k \tilde{T}'_m(2^k t - n)\, \chi_{[\frac{n}{2^k}, \frac{n+1}{2^k}]} \tag{16}$$

This function is zero outside the interval $[\frac{n}{2^k}, \frac{n+1}{2^k}]$, hence its Chebyshev wavelets expansion only have those elements of basis Chebyshev wavelets in $\Psi(t)$ that are nonzero in the interval $[\frac{n}{2^k}, \frac{n+1}{2^k}]$ i.e. $\Psi_i(t),\ i = n(M+1)+1, n(M+1)+2, ..., (n+1)(M+1)$, so its Chebyshev wavelet expansion has the following form

$$\frac{d\Psi_r(t)}{dt} = \sum_{i=n(M+1)+1}^{(n+1)(M+1)} a_i \Psi_i(t)$$

This results that operational matrix D is a block matrix as defined in (13). Moreover we have

$$\frac{d}{dt} P_0(t) = 0$$

This results that $\frac{d\Psi_r(t)}{dt} = 0$ for $r = 1, (M+1)+1, ..., (2^k - 1)(M+1)+1$. Consequently, the first row of matrix F defined in (13) is zero. Now by substituting $\tilde{T}'_m(2^k t - n)$ from Eq. (6) into (16) we have

$$\frac{d\Psi_r(t)}{dt} = 2^{\frac{k+1}{2}} \sqrt{\frac{2}{\pi c_m}}\, 2^k \sum_{\substack{j=0 \\ j+m\ odd}}^{m-1} \frac{4m}{c_j} \tilde{T}_j(2^k t - n)\, \chi_{[\frac{n}{2^k}, \frac{n+1}{2^k}]}, \tag{17}$$

expanding this equation in Chebyshev wavelets basis we have

$$\frac{d\Psi_r(t)}{dt} = 2^{\frac{k+1}{2}} \sqrt{\frac{2}{\pi c_m}}\, 2^k \sum_{\substack{j=0 \\ j+m\ odd}}^{m-1} \frac{4m}{c_j} \tilde{T}_j(2^k t - n)\, \chi_{[\frac{n}{2^k}, \frac{n+1}{2^k}]} = 2^k \sum_{\substack{s=1 \\ s+r\ odd}}^{r-1} 4m \sqrt{\frac{c_{r-1}}{c_{s-1}}}\, \Psi_{n(M+1)+s}(t),$$

So if we choose $F_{r,s}$ as

$$F_{r,s} = \begin{cases} 2^{k+2} m \sqrt{\frac{c_{r-1}}{c_{s-1}}} & r = 2, ..., (M+1),\ s = 1, ..., r-1 \ \textit{and}\ (r+s)\textit{odd} \\ 0 & \textit{othewise} \end{cases}$$

Equation (12) is hold and this leads to desired results.■

Corollary: By using Eq. (12) the operational matrix for n-th derivative can be derived as

$$\frac{d^n \Psi(x)}{dx^n} = D^n \Psi(x)\ , \tag{19}$$

where D^n is the n-th power of matrix D .■

4. Applications of the operational matrix of derivative

In this section, we demonstrate the effectiveness of the proposed Chebyshev wavelet Galerkin method with four illustrative examples. For all examples, spectral methods by using orthogonal polynomials do not work well and cannot present spectral accuracy. It will be shown that the Chebyshev wavelet Galerkin method is very efficient for solving these singular problems and spectral accuracy will be obtained. For two examples, by using Chebyshev wavelet Galerkin methods the exact solution is obtained. The algorithms are performed by Maple 12 with 20 digits precision.

4.1. Linear differential equation

Consider the linear second order differential equation

$$y''(x)+f_1(x)y'(x)+f_2(x)y(x)=g(x) \tag{20}$$

With the initial conditions

$$y(0)=A, y'(0)=B, \tag{21}$$

Or boundary conditions

$$y(0)=A, y(1)=B, \tag{22}$$

To solve problem (20) we approximate $y(x), f_1(x), f_2(x)$ and $g(x)$ by the Chebyshev wavelets as

$$y(x) \simeq C^T\psi(x), \quad f_1(x) \simeq F_1^T\psi(x), \quad f_2(x) \simeq F_2^T\psi(x), \quad g(x) \simeq G^T\psi(x), \tag{23}$$

By using Eq. (19) we have

$$y'(x) \simeq C^T D\psi(x), y''(x) \simeq C^T D^2\psi(x) \tag{24}$$

Employing Eqs. (23) and (24) the residual $R(x)$ for Eq. (22) can be written as

$$R(x)=\left(C^T D^2\Psi(x)+F_1^T\Psi(x)\Psi^T(x)D^T C+F_2^T\Psi(x)\Psi^T(x)C-G^T\Psi(x)\right) \tag{25}$$

By using the product operation matrix of Chebyshev wavelets (25) can be written as

$$R(x)=\left(\Psi^T(x)\left(D^2\right)^T C+\Psi^T(x)\tilde{F}_1 D^T C+\Psi^T(x)\tilde{F}_2 C-\Psi^T(x)G\right) \tag{26}$$

As in a typical tau method [4] we generate $(m+1)-2$ linear equations by applying

$$\int_0^1 \Psi_j(x)R(x)dx=0 \qquad j=1,...,(m+1)-2 \tag{27}$$

Also, by substituting initial conditions form Eq. (21) in Eq. (22) we have

$$y(0)=C^T\Psi(0)=A,\ y'(0)=C^T D\Psi(0)=B \tag{28}$$

And for boundary value problems we have

$$y(0)=C^T\Psi(0)=A, \qquad y(1)=C^T\psi(1)=B \tag{29}$$

Eqs. (27), (28) and (29) generate $(m+1)$ set of linear equations. These linear equations can be solved for unknown coefficients of the vector C. Consequently, $y(x)$ given in Eq. (23) can be calculated.

4.2. Nonlinear differential equation

Consider the nonlinear equation

$$y''(x) = F\left(x, y'(x), y(x)\right) \tag{30}$$

With the initial conditions

$$y(0) = A, y'(0) = B, \tag{31}$$

Or boundary conditions

$$y(0) = A, y(1) = B, \tag{32}$$

In order to use shifted Chebyshev polynomials for this problem, we first approximate $y(x)$ by the Chebyshev wavelets

$$y(x) \simeq C^T \psi(x) \tag{33}$$

By using Eq. (19) we have

$$y'(x) \simeq C^T D \psi(x), \quad y''(x) \simeq C^T D^2 \psi(x) \tag{34}$$

By substituting equations in Eq. (33-34) we get

$$C^T D^2 \Psi(x) = F\left(x, C^T D \Psi(x), C^T \Psi(x)\right) \tag{35}$$

Also, by substituting Eqs. (31) and (32) in Eq. (33) for initial value problems we have

$$y(0) = C^T \Psi(0) = A, \quad y(1) = C^T \psi(1) = B \tag{36}$$

And for boundary value problems we have

$$y(0) = C^T \Psi(0) = A, \quad y(1) = C^T \psi(1) = B \tag{37}$$

To find the solution $y(x)$, we first collocate Eq. (35) at $(m+1)-2$ points. For a better result we use the first $(m+1)-2$ roots of shifted Chebyshev $P_{(m+1)}(x)$. These equations collectively with Eqs. (36) or (37) generate $(m+1)$ nonlinear equations which can be solved using Newton's iterative method. Consequently $y(x)$ given in Eq. (33) can be calculated.

The above-mentioned applications for linear and nonlinear differential equations can be easily execute for higher order differential equations.

5. Illustrative Examples

In this section, we demonstrate the effectiveness of the proposed Chebyshev wavelet method with four illustrative examples. It will be shown that the Chebyshev wavelet method is very efficient for solving both linear and nonlinear differential equations. The algorithms are performed by Maple 12 with 20 digits precision.

Example 1:

Consider the boundary value problem [2]

$$y'' + |2t-1|y' + y = 24|2t-1| + 3(2t-1)^3 + |2t-1|^3, \qquad t \in [0,1],$$
$$y(0) = 1, \quad y(1) = 1.$$

The exact solution $y(t) = |2t-1|^3$ has singular point at $t = \frac{1}{2}$. We solve this boundary value problem using Chebyshev wavelets with M=3 and k=1. The vector C is computed by solving system of linear equation (20). For the obtained vector C, we get

$$y(t) \simeq C^T \Psi(t) = \begin{cases} -8t^3 + 12t^2 - 6t + 1 & t \le \frac{1}{2} \\ 8t^3 - 12t^2 + 6t - 1 & t > \frac{1}{2} \end{cases},$$

which is the exact solution.

Example 2. Consider the boundary value problem [2]

$$y'' + \left|x - \frac{3}{4}\right| y' + \left|x - \frac{1}{4}\right| y = h(x), \qquad x \in [0,1],$$
$$y(0) = \left(\frac{3}{16}\right)^3, \; y(1) = \left(\frac{3}{16}\right)^3.$$

with the exact solution $y(t) = \left|x^2 - x + \frac{3}{16}\right|^3$. In this example exact solution and coefficient functions has singular points at $x = \frac{1}{4}$ and $x = \frac{3}{4}$. Here we solve this problem using Chebyshev wavelets with M=6, k=2. By applying the technique described in section 4.1 we have 28 linear systems for vector C. Solving this linear system we get a 28×1 vector C which result

$$y(x) \simeq C^T \Psi(x) = |x^2 - x + \frac{3}{16}|$$

and this is the exact solution.

Example 3. In this example we consider a boundary value problem in which one of its coefficient function is non analytical but its solution function is infinite differentiable on [0, 1]. Consider the boundary value problem

$$y'' + f(x)y' + y = g(x), \qquad x \in [0, 1],$$
$$y(0) = 0, \quad y(1) = \frac{e\sin(1)}{2}.$$

in which

$$f(x) = \begin{cases} -4x + \frac{4}{3} & 0 \le x < \frac{1}{4} \\ x - \frac{1}{3} & \frac{1}{4} \le x < 1 \end{cases}$$

and the exact solution is $y(x)=\frac{e^x \sin(x)}{1+x^2}$. In this example coefficient functions has singular point at $x=\frac{1}{4}$. Here we solve this problem using Chebyshev wavelets with different values of M and k. Figure 1 show the absolute error produced from Chebyshev wavelets method for M=6, and k=2. The absolute error for this example is shown in Table 1.

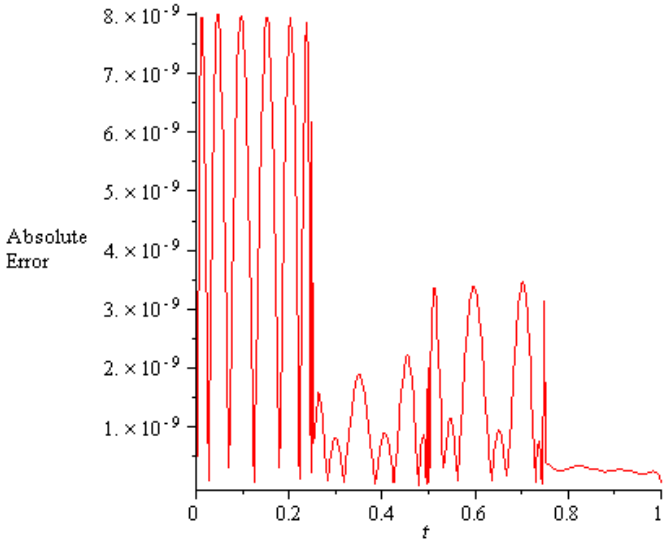

Fig. 1: Absolute error for example 3 with M=6 and k=2.

From Table 1, we see that we can achieve a good approximation with the exact solution by using a few terms of Chebyshev wavelets.
As it can be seen in Fig. 1, the Chebyshev wavelet Galerkin method is very efficient for solving these singular problems and spectral accuracy has been obtained.

Table 1.

x	M=6, k=1	M=6, k=2	M=8, k=1	M=8, k=2
0.2	5.0×10^{-7}	8.0×10^{-9}	2.3×10^{-10}	4.3×10^{-12}
0.4	4.2×10^{-7}	1.0×10^{-9}	6.5×10^{-10}	1.2×10^{-11}
0.6	1.5×10^{-7}	3.2×10^{-9}	3.5×10^{-10}	2.9×10^{-12}
0.8	1.5×10^{-7}	8.5×10^{-10}	3.3×10^{-11}	4.3×10^{-12}

Conclusion

The spectral methods such as Galerkin, tau and pseudospectral methods by using orthogonal polynomials have been successful for solving many application problems and spectral accuracy has been obtained. However, difficulties arise in

dealing with ordinary differential equations with non analytic solution such that cannot produce accurate results. To overcome these difficulties, in this article the Chebyshev wavelet Galerkin method is proposed and is applied for solving ordinary differential equations in which, at least, one of the coefficient functions or solution function is not analytic. The proposed method is shown to be highly accurate. This was clearly demonstrated in the examples.

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