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Strong Convergence Theorem by Hybrid Method for Non-Lipschitzian Mapping

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Abstract

We introduce the hybrid method of modified Mann's iteration for an asymptotically k-strict pseudo-contractive mapping T in the intermediate sense which is necessarily lipschitzian. We establish strong convergence theorem for such method. The result extend and improve the recent ones announced by Inchan and Nammanee, Inchan and concern result of Takahashi, Takeuchi and Kubota [Strong convergence theorems by hybrid methods for families of nonexpansive mappings in Hilbert space, J. Math. Anal. Appl. 341 (2008), 276–286], and many others.

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1 Introduction

Let C be a nonempty subset of a Hilbert space H and $T : C \to C$ a mapping. Recall the following concepts.

(i) T is nonexpansive if $\|Tx - Ty\| \leq \|x - y\|$ for all $x, y \in C$.

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ii) T is asymptotically nonexpansive (cf. [4]) if there exists a sequence $\{k_n\}$ of positive numbers satisfying $\lim_{n\to\infty} k_n = 1$ and $\|T^n x - T^n y\| \leq k_n \|x - y\|$ for all integers $n \geq 1$ and $x, y \in C$.

iii) T is uniformly Lipschitzian if there exists a constant $L > 0$ such that $\|T^n x - T^n y\| \leq L\|x - y\|$ for all integers $n \geq 1$ and all $x, y \in C$.

(iv) T is asymptotically nonexpansive in the intermediate sense [2] provided T is uniformly continuous and

$$\limsup_{n\to\infty} \sup_{x,y\in C} (\|T^n x - T^n y\| - \|x - y\|) \leq 0.$$

It is clear that every nonexpansive mapping is asymptotically nonexpansive and every asymptotically nonexpansive mapping is uniformly Lipschitzian.

The class of asymptotically nonexpansive mappings in the intermediate sense was introduced by Bruck, Kuczumow and Reich [2] and iterative methods for the approximation of fixed points of such types of non-Lipschitzian mappings have been studied by Agarwal, O'Regan and Sahu [1], Bruck, Kuczumow and Reich [2], Chidume, Shahzad and Zegeye [3], Kim and Kim [9] and many others.

In 2008, Kim and Xu [11] introduced the concept of asymptotically k-strict pseudo-contractive mappings in Hilbert space as below:

Definition 1.1. *Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ is said to be an asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ if there exist a constant $k \in [0, 1)$ and a sequence $\{\gamma_n\}$ in $[0, 1)$ with $\lim_{n\to\infty} \gamma_n = 0$ such that*

$$\|T^n x - T^n y\|^2 \leq (1 + \gamma_n)\|x - y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 \qquad (1)$$

for all $x, y \in C$ and $n \in N$.

They studied weak and strong convergence theorems for this class of mappings. It is important to note that every asymptotically k-strict pseudo-contractive mapping with sequence $\{\gamma_n\}$ is a uniformly L-Lipschitzian mapping with $L = \sup\{\frac{k+\sqrt{1+(1-k)\gamma_n}}{1+k} : n \in N\}$.

Recently, Sahu et al.[16] introduced the concept of asymptotically k-strict pseudo-contractive mappings in the intermediate sense which are not necessarily Lipschitzian (see Lemma 2.6 [16]) as below:

Definition 1.2. *Let C be a nonempty subset of a Hilbert space H. A mapping $T : C \to C$ will be called an asymptotically k-strict pseudo-contractive*

mapping in the intermediate sense with sequence $\{\gamma_n\}$ *if there exist a constant* $k \in [0,1)$ *and a sequence* $\{\gamma_n\}$ *in* $[0,1)$ *with* $\lim_{n\to\infty}\gamma_n = 0$ *such that*

$$\limsup_{n\to\infty}\sup_{x,y\in C}(\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - T^n x - (y - T^n y)\|^2) \le 0. \tag{2}$$

Throughout this paper we assume that

$$c_n := \max\{0, \sup_{x,y\in C}(\|T^n x - T^n y\|^2 - (1+\gamma_n)\|x-y\|^2 - k\|x - T^n x - (y - T^n y)\|^2)\}.$$

Then $c_n \ge 0$ *for all* $n \in N$, $c_n \to 0$ *as* $n \to \infty$ *and (2) reduces to the relation*

$$\|T^n x - T^n y\|^2 \le (1+\gamma_n)\|x-y\|^2 + k\|x - T^n x - (y - T^n y)\|^2 + c_n \tag{3}$$

for all $x, y \in C$ *and* $n \in N$

Remark 1.3. *If* $c_n = 0$ *for all* $n \in N$ *in (3) then* T *is an asymptotically* k*-strict pseudocontractive mapping with sequence* $\{\gamma_n\}$.

Fixed point iteration processes for nonexpansive mappings and asymptotically nonexpansive mappings in Hilbert spaces and Banach spaces including Mann and Ishikawa iteration processes have been studied extensively by many authors to solve nonlinear operator equations as well as variational inequalities: see [5, 13, 17, 20]. However, Mann and Ishikawa iterations processes have only weak convergence even in Hilbert space: see [10, 20].

Iteration method for finding a fixed point of an asymptotically k-strict pseudo-contractive mapping T is the modified Mann's iteration method studied in [12, 18, 19, 21] which generates a sequence $\{x_n\}$ via

$$x_{n+1} = \alpha_n x_n + (1-\alpha_n)T^n x_n, \quad n \ge 0, \tag{4}$$

where the initial guess $x_0 \in C$ is arbitrary and the sequence $\{\alpha_n\}_{n=0}^{\infty}$ line in the interval $(0,1)$.

In 2007, Takahashi, Takeuchi and Kubota [20] introduced the modification Mann iteration method for a family of nonexpansive mappings $\{T_n\}$. Let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:

$$\begin{cases} y_n = \alpha_n u_n + (1-\alpha_n)T_n u_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \le \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{5}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then they prove that the sequence $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$. In 2008, Kumam [8], introduce an iterative scheme by a new hybrid method for finding a common element of the set of fixed points of a nonexpansive mapping, the set of solutions of an equilibrium problem and the set of solutions of the variational inequality for α-inverse-strongly monotone mappings in a real Hilbert space.

In 2008, Inchan [6], introduce the modified Mann iteration processes for an asymptotically nonexpansive mapping. Let C be a nonempty closed bounded convex subset of a Hilbert space H, T be an asymptotically nonexpansive mapping of C into itself and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{6}$$

where $\theta_n = (1-\alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then him prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Recently, Inchan and Nammanee [7], introduce the modified Mann iteration processes for an asymptotically k-strict pseudo-contractive mapping. Let C be a nonempty closed convex subset of a Hilbert space H, T be an asymptotically k-strict pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0$ as $n \to \infty$ and $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then they prove that $\{x_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

Inspired and motivated by these fact, it is the purpose of this paper to introduce the modified Mann iteration processes for an asymptotically k-strict pseudo-contractive mapping in the intermediate sense by idear in (7). Let C be a closed convex subset of a Hilbert space H, $T : C \to C$ be an asymptotically k-strictly pseudo-contractive mapping in the intermediate sense and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}(x_0)$, define $\{x_n\}$ as follows way:

$$\begin{cases} y_n = \alpha_n x_n + (1-\alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{8}$$

where $\theta_n = (diamC)^2(1-\alpha_n)\gamma_n \to 0,\ (n \to \infty)$.

We shall prove that the iteration generated by (8) converges strongly to $z_0 = P_{F(T)}x_0$.

2 Preliminary

A point $x \in C$ is a fixed point of T provided $Tx = x$. Denote by $F(T)$ the set of *fixed points* of T; that is, $F(T) = \{x \in C : Tx = x\}$. Let H be a real Hilbert space with norm $\|\cdot\|$ and inner product $\langle\cdot,\cdot\rangle$ and let C be a closed convex subset of H. For every point $x \in H$, there exists a unique nearest point in C, denote by $P_C x$, such that

$$\|x - P_C x\| \leq \|x - y\|, \quad \text{for all } y \in C.$$

P_C is called the metric projection of H onto C. It is well known that P_C is a nonexpansive mapping of H onto C.

We collect some lemmas which will be used in the proof for the main result.

Lemma 2.1. *[14] There holds the identity in a Hilbert space H:*

(i) $\|x+y\|^2 = \|x\|^2 + \|y\|^2 + 2\langle x, y\rangle, \forall x, y \in H$.

(ii) $\|\lambda x + (1-\lambda)y\|^2 = \lambda\|x\|^2 + (1-\lambda)\|y\|^2 - \lambda(1-\lambda)\|x-y\|^2$ *for all* $x, y \in H$ *and* $\lambda \in [0,1]$.

Lemma 2.2. *[15] Let C be a closed convex subset of a real Hilbert space H. Given $x \in H$ and $y \in C$. Then $y = P_C x$ if and only if there holds the inequality*

$$\langle x - y, y - z\rangle \geq 0, \quad \forall z \in C.$$

Lemma 2.3. *[16] Let C be a nonempty subset of a Hilbert space H and let $T : C \to C$ a uniformly continuous asymptotically k-strict pseudo-contractive in the intermediate sense with sequence $\{\gamma_n\}$. Let $\{x_n\}$ be a sequence in C such that $\|x_n - x_{n+1}\| \to 0$ and $\|x_n - T^n x_n\| \to 0$ as $n \to \infty$. Then $\|x_n - Tx_n\| \to 0$ as $n \to \infty$.*

Lemma 2.4. *[16] Let C be a nonempty closed convex subset of Hilbert space H and $T : C \to C$ a continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense. Then $I - T$ is demiclosed at zero in the sense that $\{x_n\}$ is sequence in C such that $x_n \rightharpoonup x \in C$ and $\limsup_{n\to\infty} \|x_n - T^n x_n\| = 0$, then $(I-T)x = 0$.*

3 Main Results

In this section, we prove strong convergence theorem by hybrid method for asymptotically k-strict pseudo-contractive mapping in the intermediate sense in Hilbert spaces.

Theorem 3.1. *Let H be a Hilbert space and let C be a nonempty closed convex bounded subset of H. Let T be a uniformly continuous asymptotically k-strict pseudo-contractive mapping in the intermediate sense of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (8) converges strongly to $z_0 = P_{F(T)}x_0$.*

Proof. We first show that $F(T) \subset C_n$ for all $n \in \mathbb{N}$, by induction. For any $z \in F(T)$ we have $z \in C = C_1$ hence $F(T) \subset C_1$. Let $F(T) \subset C_m$ for each $m \in \mathbb{N}$. For $u \in F(T) \subset C_m$. By lemma 2.1, we have,

$$\begin{aligned}
\|y_m - u\|^2 &= \|\alpha_m x_m + (1-\alpha_m)T^m x_m - u\|^2 \\
&= \|\alpha_m(x_m - u) + (1-\alpha_m)(T^m x_m - u)\|^2 \\
&= \alpha_m\|x_m - u\|^2 + (1-\alpha_m)\|T^m x_m - u\|^2 - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&\leq \alpha_m\|x_m - u\|^2 + (1-\alpha_m)[(1+\gamma_m)\|x_m - u\|^2 \\
&\quad + k\|x_m - T^m x_m\|^2 + c_m] - \alpha_m(1-\alpha_m)\|x_m - T^m x_m\|^2 \\
&= (1 + (1-\alpha_m)\gamma_m)\|x_m - u\|^2 + (k - \alpha_m)(1-\alpha_m)\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + (1-\alpha_m)\gamma_m\|x_m - u\|^2 \\
&\quad + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + c_m \\
&\leq \|x_m - u\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \qquad (1)
\end{aligned}$$

It follows that $u \in C_{m+1}$ and $F(T) \subset C_{m+1}$, hence $F(T) \subset C_n$ for all $n \in \mathbb{N}$. Next, we show that C_n is closed and convex for all $n \in \mathbb{N}$. It follows obvious that $C_1 = C$ is closed and convex. Suppose that C_m is closed and convex for each $m \in \mathbb{N}$. Let $z_j \in C_{m+1} \subset C_m$ with $z_j \to z$. Since C_m is closed, $z \in C_m$ and $\|y_m - z_j\|^2 \leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$. Then

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - z_j + z_j - z\|^2 \\
&= \|y_m - z_j\|^2 + \|z_j - z\|^2 + 2\langle y_m - z_j, z_j - z\rangle \\
&\leq \|z_j - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&\quad + \|z_j - z\|^2 + 2\|y_m - z_j\|\|z_j - z\|.
\end{aligned}$$

Taking $j \to \infty$,

$$\|y_m - z\|^2 \leq \|z - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.$$

Hence $z \in C_{m+1}$. Let $x, y \in C_{m+1} \subset C_m$ with $z = \alpha x + (1-\alpha)y$ where $\alpha \in [0,1]$. Since C_m is convex, $z \in C_m$ and $\|y_m - x\|^2 \leq \|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, $\|y_m - y\|^2 \leq \|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m$, we have

$$\begin{aligned}
\|y_m - z\|^2 &= \|y_m - (\alpha x + (1-\alpha)y)\|^2 \\
&= \|\alpha(y_m - x) + (1-\alpha)(y_m - y)\|^2 \\
&= \alpha\|y_m - x\|^2 + (1-\alpha)\|y_m - y\|^2 - \alpha(1-\alpha)\|(y_m - x) - (y_m - y)\|^2 \\
&\leq \alpha(\|x - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad +(1-\alpha)(\|y - x_m\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m) \\
&\quad -\alpha(1-\alpha)\|y - x\|^2 \\
&= \alpha\|x - x_m\|^2 + (1-\alpha)\|y - x_m\|^2 - \alpha(1-\alpha)\|(x_m - x) - (x_m - y)\|^2 \\
&\quad +[k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m \\
&= \|\alpha(x_m - x) + (1-\alpha)(x_m - y)\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 \\
&\quad +\theta_m + c_m \\
&= \|x_m - z\|^2 + [k - \alpha_m(1-\alpha_m)]\|x_m - T^m x_m\|^2 + \theta_m + c_m.
\end{aligned}$$

Then $z \in C_{m+1}$, it follows that C_{m+1} is closed and convex. Hence C_n is closed and convex for all $n \in \mathbb{N}$. This implies that $\{x_n\}$ is well-defined. From $x_n = P_{C_n} x_0$. By Lemma 2.2, we have

$$\langle x_0 - x_n, x_n - y\rangle \geq 0, \text{ for all } y \in C_n.$$

Moreover, by the same proof of Theorem 3.1 of [7], we have that

$$\lim_{n\to\infty} \|x_n - x_{n+1}\| = 0. \tag{2}$$

On the other hand, $x_{n+1} \in C_{n+1} \subset C_n$ implies that

$$\|y_n - x_{n+1}\|^2 \leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 + \theta_n + c_n, \tag{3}$$

By the definition of y_n, we have

$$\begin{aligned}
\|y_n - x_n\| &= \|\alpha_n x_n + (1-\alpha_n)T^n x_n - x_n\| \\
&= (1-\alpha_n)\|T^n x_n - x_n\|.
\end{aligned}$$

From (3), we have

$$\begin{aligned}(1-\alpha_n)^2\|T^n x_n - x_n\|^2 &= \|y_n - x_n\|^2 \\ &= \|y_n - x_{n+1} + x_{n+1} - x_n\|^2 \\ &\leq \|y_n - x_{n+1}\|^2 + \|x_{n+1} - x_n\|^2 \\ &\quad +2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &\leq \|x_n - x_{n+1}\|^2 + [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 \\ &\quad +\theta_n + c_n + \|x_{n+1} - x_n\|^2 + 2\|y_n - x_{n+1}\|\|x_{n+1} - x_n\| \\ &= [k - \alpha_n(1-\alpha_n)]\|x_n - T^n x_n\|^2 \\ &\quad +2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) \\ &\quad +\theta_n + c_n.\end{aligned}$$

It follows that

$$((1-\alpha_n)^2 - (k - \alpha_n(1-\alpha_n)))\|x_n - T^n x_n\|^2 \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n.$$

Hence

$$(1 - k - \alpha_n)\|T^n x_n - x_n\| \leq 2\|x_{n+1} - x_n\|(\|x_{n+1} - x_n\| + \|y_n - x_{n+1}\|) + \theta_n + c_n. \tag{4}$$

From $\limsup_{n\to\infty} \alpha_n < 1 - k$, we can chosen $\epsilon > 0$ such that $\alpha_n \leq 1 - k - \epsilon$ for large enough n. From (2) and (4), we have

$$\lim_{n\to\infty} \|T^n x_n - x_n\| = 0. \tag{5}$$

From (2), (5) and Lemma 2.3, we have

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0. \tag{6}$$

Since H is reflexive and $\{x_n\}$ is bounded we get that $\emptyset \neq \omega_w(x_n)$. From Lemma 2.4, we have $\omega_w(x_n) \subset F(T)$. By the fact that $\|x_n - x_0\| \leq \|z_0 - x_0\|$ for all $n \geq 0$ where $z_0 = P_{F(T)}(x_0)$ and the weak lower semi-continuity of the norm, we have

$$\begin{aligned}\|x_0 - z_0\| \leq \|x_0 - w\| &\leq \liminf_{n\to\infty} \|x_0 - x_n\| \\ &\leq \limsup_{n\to\infty} \|x_0 - x_n\| \leq \|x_0 - z_0\|,\end{aligned}$$

for all $w \in \omega_w(x_n)$. However, since $\omega_w(x_n) \subset F(T)$, we must have $w = z_0$ for all $w \in \omega_w(x_n)$. Thus $\omega_w(x_n) = \{z_0\}$ and then $x_n \rightharpoonup z_0$. Hence, $x_n \to z_0 = P_{F(T)}(x_0)$ by

$$\|x_n - z_0\|^2 = \|x_n - x_0\|^2 + 2\langle x_n - x_0, x_0 - z_0\rangle + \|x_0 - z_0\|^2$$

$$\leq 2(\|z_0 - x_0\|^2 + \langle x_n - x_0, x_0 - z_0\rangle) \to 0 \text{ as } n \to \infty.$$

This complete the proof. □

Using this Theorem 3.1, we have the following corollaries.

Corollary 3.2. *[7] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically k-strict pseudo-contractive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, assume that the control sequence $\{\alpha_n\}_{n=1}^{\infty}$ is chosen so that $\limsup_{n\to\infty} \alpha_n < 1 - k$. Then $\{x_n\}$ generated by (7) converges strongly to $z_0 = P_{F(T)}x_0$.*

Corollary 3.3. *[6] Let H be a Hilbert space and let C be a nonempty closed convex subset of H. Let T be an asymptotically nonexpansive mapping of C into itself such that $F(T) \neq \emptyset$ and let $x_0 \in C$. For $C_1 = C$ and $x_1 = P_{C_1}x_0$, defined $\{x_n\}$ as follows;*

$$\begin{cases} y_n = \alpha_n x_n + (1 - \alpha_n)T^n x_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\|^2 \leq \|x_n - z\|^2 + \theta_n\}, \\ x_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{7}$$

where $\theta_n = (1 - \alpha_n)(k_n^2 - 1)(diamC)^2 \to 0$ as $n \to \infty$ and $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{x_n\}$ generated by (7) converges strongly to $z_0 = P_{F(T)}x_0$.

Corollary 3.4. *([20] Theorem 4.1) Let H be a Hilbert space and C be a nonempty closed convex subset of H. Let T be a nonexpansive mapping of C into H such that $F(T) \neq \emptyset$ and let $x_0 \in H$. For $C_1 = C$ and $u_1 = P_{C_1}x_0$, define a sequence $\{u_n\}$ of C as follows:*

$$\begin{cases} y_n = \alpha_n u_n + (1 - \alpha_n)Tu_n, \\ C_{n+1} = \{z \in C_n : \|y_n - z\| \leq \|u_n - z\|\}, \\ u_{n+1} = P_{C_{n+1}}x_0, \ n \in \mathbb{N}, \end{cases} \tag{8}$$

where $0 \leq \alpha_n \leq a < 1$ for all $n \in \mathbb{N}$. Then $\{u_n\}$ converges strongly to $z_0 = P_{F(T)}x_0$.

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