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M_i-Edge Colorings of Graphs

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Abstract

An edge coloring of a graph G is called M_i-edge coloring if at most i colors appear at any vertex of G. Let $K_i(G)$ denote the maximum number of colors used in an M_i-edge coloring of G. In this paper we determine the exact value of $K_2(G)$ for any graph G of maximum degree at most 3.

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1 Introduction

Let G be a graph with vertex set $V(G)$ and edge set $E(G)$. An edge k-coloring ψ of a graph G is an assignment of k colors, $\{1,\dots,k\}$, to the edges of G. The coloring ψ is proper if no two adjacent edges have the same color. Unless otherwise stated, the edge colorings of graphs in this paper are not necessarily proper.

Given two vertices $u, v \in V(G)$, let $E(uv)$ be the set of edges joining u and v in G. The multiplicity $\mu(uv)$ of an edge uv is the size of $E(uv)$. Set $\mu(v) = \max\{\mu(uv) : u \in V(G)\}$, which is called the multiplicity of a vertex v. Let $\deg(v)$, $\delta(G)$ denote the degree of a vertex v and the minimum degree of a graph G, respectively.

An edge coloring of G is called edge cover coloring if each color appears at least once at each vertex $v \in V(G)$. The maximum integer k such that G has an edge cover coloring with k colors is called the edge cover chromatic index of G and is denoted by $\chi'_c(G)$. Gupta [1] proved that $\min\{\deg(v) - \mu(v) : v \in V(G)\} \leq \chi'_c(G) \leq \delta(G)$. So for simple graphs we have the following bounds $\delta(G) - 1 \leq \chi'_c(G) \leq \delta(G)$. Xu and Liu [5] improved the lower bound of $\chi'_c(G)$ for multigraphs. They proved that $\chi'_c(G) \geq \delta(G) - 1$ for $\delta(G) \in \{2, 3, 4, 5\}$.

In general, the problem of determining the edge covering chromatic index of graphs is NP-hard because deciding whether a 3-connected 3-regular graph G is proper edge 3-colorable is NP-complete, see [3].

Song and Liu [4] generalized the edge cover coloring to a g-edge cover coloring and gave a lower bound for the g-edge cover chromatic index. Let g be a function which assigns a positive integer $g(v)$ to each vertex $v \in V(G)$. A g-edge cover coloring is an edge coloring of G such that each color appears at each vertex v at least $g(v)$ times. The g-edge cover chromatic index of G, denoted by $\chi'_{gc}(G)$, is the maximum number of colors in a g-edge cover coloring of G. Observe that an edge cover coloring is a g-edge cover coloring where $g(v) = 1$ for each vertex $v \in V(G)$. Set $\delta_g(G) = \min\{\lfloor \deg(v)/g(v) \rfloor : v \in V(G)\}$. It is known that $\chi'_{gc}(G) = \delta_g(G)$ if G is bipartite or $g(v)$ is positive and even for all $v \in V(G)$, see [4]. In the same paper is demonstrated that $\min\limits_{v\in V(G)} \{\lfloor \deg(v)/g(v) \rfloor\} \geq \chi'_{gc}(G) \geq \min\limits_{v\in V(G)} \{\lfloor (\deg(v) - \mu(v))/g(v) \rfloor\}$.

Let f be a function which assigns a positive integer $f(v)$ to each vertex $v \in V(G)$. An f-coloring of G is an edge coloring of G such that each vertex v is incident with at most $f(v)$ edges colored with the same color. The minimum number of colors needed to f-color G is called the f-chromatic index of G and is denoted by $\chi'_f(G)$. If $f(v) = 1$ for all $v \in V(G)$ then the f-coloring problem is reduced to the proper edge coloring problem. Since the ordinary edge coloring is NP-complete, the f-coloring problem is also NP-complete in general. Hakimi and Kariv [2] obtained the following result: $\max\limits_{v\in V(G)} \{\lceil \deg(v)/f(v) \rceil\} \leq \chi'_f(G) \leq \max\limits_{v\in V(G)} \{\lceil (\deg(v) + \mu(v))/f(v) \rceil\}$ for any multigraph G. When G does not contain multiple edges we have $\mu(v) \leq 1$ for each $v \in V(G)$. Therefore for simple graphs the following claim holds: $\max\limits_{v\in V(G)} \{\lceil \deg(v)/f(v) \rceil\} \leq \chi'_f(G) \leq \max\limits_{v\in V(G)} \{\lceil \deg(v)/f(v) \rceil\} + 1$.

Note that g and f colorings arise in many applications such as the network design, the file transfer problem on computer networks and so on.

Now we introduce two new types of edge colorings similar to f and g colorings. We say that an edge coloring of a graph G is M_i-edge coloring if at most i colors appear at any vertex of G. The problem is to determine the maximum number of colors $K_i(G)$ used in an M_i-edge coloring of G.

An edge coloring of a graph G is called m_i-edge coloring if at least i colors appear at any vertex of G. The problem is to determine the minimum number of colors $k_i(G)$ used in an m_i-edge coloring of G.

In this paper we determine the exact value of $K_2(G)$ for any graph G of maximum degree at most 3.

2 Results

Lemma 2.1 *Let T be a tree on n vertices such that each vertex of T has degree 1 or 3. Then $K_2(T) = \frac{n}{2}$.*

Proof Every graph has an even number of vertices of odd degree. So n is even, hence $\frac{n}{2}$ is an integer.

Assume that the claim holds for any such tree on m vertices, $m \geq 6$. Let T_1 be a tree on $m+2$ vertices such that the maximum degree of T_1 is 3 and T_1 does not contain a vertex of degree 2. Let T_2 be a tree obtained from T_1 by deleting two leaves incident with the same vertex. Let e_1, e_2 be the edges incident with the removed vertices. The tree T_2 has m vertices and every vertex of T_2 has degree 1 or 3, therefore $K_2(T_2) = \frac{m}{2}$.

We can extend the M_2-edge coloring of T_2 to the M_2-edge coloring of T_1 by coloring the edges e_1, e_2 with color $\frac{m}{2} + 1$. Hence, $K_2(T_1) \geq \frac{m}{2} + 1$.

Now we assume that $K_2(T_1) \geq \frac{m}{2} + 2$. If we remove two leaves incident with the same vertex we get a tree T_2 on m vertices with vertices of degree 1 or 3. Moreover, any M_2-edge coloring of T_1 with $K_2(T_1)$ colors induces a coloring of T_2 with at least $K_2(T_1) - 1 \geq \frac{m}{2} + 1$ colors, a contradiction since T_2 has m vertices. So $K_2(T_1) \leq \frac{m}{2} + 1$. □

Corollary 2.1 *Let F be a forest on n vertices such that each vertex of F has degree 1 or 3. Then $K_2(F) = \frac{n}{2}$.*

Proof Assume that $T_1, \ldots, T_m$ are the components of F. From Lemma 2.1 it follows that $K_2(T_i) = \frac{|V(T_i)|}{2}$, $i = 1, \ldots, m$. Clearly, $K_2(F) = K_2(T_1) + \ldots K_2(T_m) = \frac{|V(T_1)|}{2} + \ldots \frac{|V(T_m)|}{2} = \frac{|V(F)|}{2} = \frac{n}{2}$. □

Corollary 2.2 *Let F be a forest on n vertices with maximum degree at most 3. Let m denote the number of vertices of degree 2. Then $K_2(F) = \frac{m+n}{2}$.*

Proof Assume that a vertex v of degree 2 is incident with vertices v_1 and v_2. We replace each such vertex v by a pair of vertices v_x, v_y and the edges v_1v, vv_2 by edges v_1v_x, v_yv_2, see Figure 1 for illustration.

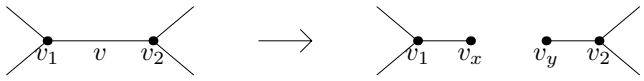

Figure 1: Splitting of a vertex v of degree 2.

In this way we obtain a forest on $n + m$ vertices such that each vertex has degree 1 or 3. Hence, the claim follows from Corollary 2.1. □

Lemma 2.2 *Let G be a graph different from a forest (tree) and let each vertex of G have degree 1 or 3. Let φ be an M_2-edge coloring of G with $K_2(G)$ colors. Then at least one cycle is monochromatic.*

Proof We can assume that φ is such an M_2-edge coloring of G that no cycle is monochromatic. First we prove several properties of the color classes under the coloring φ.

Claim 2.1 *Each color class induces a connected subgraph.*

Proof Assume that there is a color, say j, such that the graph induced by the edges of color j is not connected. Then we recolor one component with a new color and we obtain an M_2-edge coloring of G with $K_2(G)+1$ colors, a contradiction. □

Claim 2.2 *Each color class induces a path.*

Proof Each color class induces a tree since each color class induces a connected subgraph (see Claim 2.1) and there is no monochromatic cycle in G. Moreover, each tree must be a path, otherwise there is an M_2-edge coloring of G with $K_2(G)+1$ colors (we recolor the longest path in a tree with a new color). □

Let C be a shortest cycle in G. Let $v_1, \ldots, v_n$ be the vertices incident with C in a clockwise order. Clearly, each vertex v_i has degree 3 in G. Let e_i be the edge incident with v_i which does not lie on the cycle C.

Claim 2.3 *If $i \neq j$ then the edges e_i and e_j are different.*

Proof Otherwise C is not a shortest cycle in G. □

Claim 2.4 *There is a color which appears only on the cycle C in G.*

Proof Otherwise we recolor the edges of C with a new color and we get an M_2-edge coloring of G with $K_2(G)+1$ colors. □

Let $O_C = \{c_1, \ldots, c_m\}$ be a set of colors which appear only on the cycle C in G.

Claim 2.5 *If $i \neq j$ then two edges of color c_i, c_j cannot be incident.*

Proof It follows from the fact that each vertex of C has degree 3 and the colors c_i, c_j appear only on the cycle C. □

Claim 2.6 *Let $v_{i-1}v_i \ldots v_{j+1}v_{j+2}$, $j \geq i$, be a path in C such that the edges $v_{i-1}v_i, v_{j+1}v_{j+2}$ have color from O_C and no other edge in this path has color from O_C. Then $\varphi(e_i) = \varphi(v_iv_{i+1})$ and $\varphi(e_{j+1}) = \varphi(v_jv_{j+1})$.*

Proof It follows from the fact that the colors $\varphi(v_{i-1}v_i), \varphi(v_{j+1}v_{j+2})$ appear only on the cycle C. □

Claim 2.7 *Let $e_i, v_iv_{i+1}, v_jv_{j+1}, e_{j+1}$, $j \geq i$, be edges such that $\varphi(e_i) = \varphi(v_iv_{i+1})$ and $\varphi(e_{j+1}) = \varphi(v_jv_{j+1})$. If there is no $k \in \{i+1, \ldots, j\}$ such that $\varphi(e_k) = \varphi(v_kv_{k+1})$ or $\varphi(e_k) = \varphi(v_{k-1}v_k)$, then $\varphi(e_i) = \varphi(v_rv_{r+1}) = \varphi(e_{j+1})$ for $r = i, \ldots, j$.*

Proof If $i = j$ then the claim trivially holds. If $j > i$ then necessarily $\varphi(v_iv_{i+1}) = \varphi(v_{i+1}v_{i+2}) = \ldots = \varphi(v_jv_{j+1})$. □

Let $P_{i,j+1}$ denote a path $v_iv_{i+1}, v_{i+1}v_{i+2}, \ldots, v_jv_{j+1}$ in C, $i \leq j$. If a subgraph $e_i, P_{i,j+1}, e_{j+1}$, $i \leq j$, is monochromatic then it is called U-structure. From Claims 2.6 and 2.7 it follows that the number of U-structures in G is at least m.

Now we show that if G does not contain a monochromatic cycle under the coloring φ then there is an M_2-edge coloring of G with $K_2(G) + 1$ colors.

Assume that k is such a color that there is a U-structure of color k. Let U_k denote the number of U-structures of color k. Remove the cycle C from G. We obtain a graph G' with an M_2-edge coloring which uses $K_2(G) - m$ colors. Since each color class induces a path in G (see Claim 2.2) and each U-structure contains an edge of C there are at least $U_k + 1$ components of color k in G'. For each such k we recolor U_k components from these $U_k + 1$ ones successively with colors $c_1, \ldots, c_m$ (there are at least m U-structures, hence at least m such components). Finally, we return C to G and color the edges of C with color $K_2(G) + 1$. It is easy to see that this coloring is an M_2-edge coloring of G, a contradiction. □

Lemma 2.3 *Let G be a graph on n vertices such that each vertex has degree* 1 *or* 3. *Then $K_2(G) = t + \frac{n}{2}$, where t is the maximum number of disjoint cycles in G.*

Proof

First we show that $K_2(G) \geq t + \frac{n}{2}$. Color the disjoint cycles $C_1, \ldots, C_t$ with t colors. Let G' be a graph obtained from G be deleting the edges of cycles $C_1, \ldots, C_t$. The graph G contains exactly t disjoint cycles, hence, G' is a forest on n vertices. Corollary 2.1 implies that G' has an M_2-edge coloring with $\frac{n}{2}$ colors. The M_2-edge coloring of G' with $\frac{n}{2}$ colors and a coloring of disjoint cycles with t colors induces an M_2-edge coloring of G with $t + \frac{n}{2}$ colors.

Now we assume that φ is an M_2-edge coloring of G with $K_2(G)$ colors. We say that a color c is cyclic if a subgraph induced by the edges of color c contains a cycle. Let t_0 denote the number of cyclic colors. Let G_1 be a graph obtained from G by deleting the edges of cyclic color. The coloring φ

of G induces an M_2-edge coloring of G_1 with $K_2(G) - t_0$ colors. Observe that $K_2(G_1) = K_2(G) - t_0$. Lemma 2.2 and the fact that φ uses $K_2(G)$ colors imply that G_1 is a forest with vertices of degree 1 or 3. From Corollary 2.1 it follows that $K_2(G_1) = \frac{n}{2}$. On the other hand the number of disjoint cycles t in G is not smaller than t_0. So $K_2(G) = t_0 + \frac{n}{2} \leq t + \frac{n}{2}$. □

Splitting of a vertex v of degree 2 is defined in the following way. Assume that v is incident with vertices v_1 and v_2. We replace each such vertex v by a pair of vertices v_x, v_y and the edges v_1v, vv_2 by edges v_1v_x, v_yv_2.

Theorem 2.1 *Let G be a graph on n vertices with maximum degree at most 3. Let m denote the number of vertices of degree 2 in G. Let G' be a graph obtained from G by splitting the vertices of degree 2. Let t denote the maximum number of disjoint cycles in G'. Then $K_2(G) = \frac{n+m}{2} + t$.*

Proof Clearly, the graph G' has $n + m$ vertices and each vertex has degree 1 or 3. From Lemma 2.3 it follows that $K_2(G') = \frac{n+m}{2} + t$. Since any M_2-edge coloring of G (G') induces an M_2-edge coloring of G' (G) it follows $K_2(G) = K_2(G')$. □

References

[1] R.P. Gupta, On decompositions of a multigraph into spanning subgraphs, Bull. Amer. Math. Soc. 80 (1974), 500–502.

[2] S.L. Hakimi and O. Kariv, A generalization of edge-coloring in graphs, J. Graph Theory 10 (1986), 139–154.

[3] I. Holyer, The NP-completeness of edge-coloring, SIAM J. Comput. 10 (4) (1981), 718–720.

[4] H. Song and G. Liu, On f-edge cover-coloring in multigraphs, Acta Mathematica Sinica 48 (5) (2005), 910–919.

[5] C. Xu and G. Liu, A note on the edge cover chromatic index of multigraphs, Discrete Math. 308 (2008), 6564–6568.

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