Applied Mathematical Sciences, Vol. 5, 2011, no. 52, 2571 - 2579

On the Modification of the p-RF Method for Inclusion of a Zero of a Function

N. A. Bakar [1], M. Monsi, M. A. Hassan and W. J. Leong

Department of Mathematics, Faculty of Science
University Putra Malaysia
Serdang, Selangor, Malaysia

Abstract

An improved parameter regula falsi method p-RFM1 which is based on the interval parameter regula falsi method p-RF is presented in this paper. This method is generally using the midpoint of the current interval in the algorithm of p-RFM1 method. It is verified on several number of test examples that the CPU times of the algorithm are lesser than does p-RF method.

Mathematics Subject Classification: 65B99, 65G40

Keywords: CPU time, interval analysis, parameter regula falsi, zero of a function

1 Introduction

This paper is about finding a zero of a function $f(x)$ of one real variable. The work that had been carried out in [7], [3], [6] and [5] are basically on iterative procedures for finding a zero of a function.

Iteration procedures are often used in making successive approximation where the current calculated value is closer to the actual solution. Besides iterative procedures, the main tool to be used in this paper is interval analysis [3] upon which the very simple idea of enclosing the solution of the function f in an interval. Gargantini [4] and Hansen [2] have shown that the iterative procedures which involve interval computation approach gives better accuracy and also the inclusion of a zero is always guaranteed.

The p-RF method which was discussed in [3] where the order of convergence of the method is at least $\frac{1}{2}(p + \sqrt{p^2 + 4})$. In this paper, we will focus on a

[1] n_izaruffedge@yahoo.com

modification of p-RF method called p-RFM1 method. The numerical results and the order of convergence of this method will be presented in this paper.

1.1 Convergence Theorem

The following theorem is the basic concept of convergence of a sequence $\{X^{(k)}\}_{k=0}^{\infty}$ which is generated from an algorithm I.

Theorem 1.1 *Let I be an iteration procedure with the limit x^* and let $\zeta(I, x^*)$ be the set of all sequences generated by I having the properties that $\lim_{k\to\infty} x^{(k)} = x^*$ and $x^* \subseteq x^{(k)}$, $k \geq 0$. If there exist a $p \geq 1$ and a constant γ such that for all $\{x^{(k)}\} \in \zeta(I, x^*)$ and for a norm $\|.\|$ it holds that $\|x^{(k+1)}\| \leq \gamma \|x^{(k)}\|^p$. Then, it follows that the R-order of convergence of I satisfies the inequality*

$$O_R(I, x^*) \geq p$$

or the R-order of convergence of I is at least p.

Theorem 1.2 *Let the function f be twice be continuously differentiable in the interval X and assume that f has a zero ξ in X. Let the conditions*

$$f'(x) \in H\,, \qquad x \in X \quad with \quad 0 \notin H \tag{1}$$

$$f''(x) \in K\,, \qquad x \in X \tag{2}$$

be satisfied. The parameter p is given as an integer number for which $p \geq 2$. Then, the sequence $\{X^{(k)}\}$ calculated according to algorithm p-RF method satisfies for $p \geq 2$:

$$\xi \in X^{(k)}\,, \quad k \geq 0$$

$$X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset ... \; with \lim_{k\to\infty} X^{(k)} = \xi$$

and

$$d\left(X^{(k+1)}\right) \leq \gamma d\left(X^{(k)}\right)^{(p)} d\left(X^{(k-1)}\right) \; ; \; (\gamma \geq 0)$$

where $d\left(X^{(k+1)}\right)$, $d\left(X^{(k)}\right)$, *and* $d\left(X^{(k-1)}\right)$ *are the width of the intervals* $X^{(k+1)}$, $X^{(k)}$ *and* $X^{(k-1)}$ *respectively. Then the R-order of convergence of p-RF satisfies the inequality*

$$O_R\left((p-RF),\ \xi\right) \geq \frac{1}{2}\left(p+\sqrt{p^2+4}\right) \tag{3}$$

The prove of these theorem is available in [3].

2 *p*-RFM1 Method

By using the same hypothesis given in Theorem 1.2, we use the midpoint $z^{(i-1)}\,(i=2,3,...,p)$ of the interval $X^{(k+1,i-2)}$ instead of $x^{(k)}$ in the algorithm p-RF. This new modification is called p-RFM1 method. The midpoint $z^{(i)}$ and $z^{(i-1)}$ are updated for every iteration k. The interval p-RFM1 method is described by:

$x^{(0)} = m\left(X^{(0)}\right)$ (midpoint of $X^{(0)}$)

$$X^{(1)} = \left\{x^{(0)} - \frac{f(x^{(0)})}{H}\right\} \bigcap X^{(0)}$$

For $k \geq 1$;

$x^{(k)} = m\left(X^{(k)}\right)$ (midpoint of $X^{(k)}$)

$$X^{(k+1,0)} = \left\{x^{(k)} - \frac{f\left(x^{(k)}\right)}{H}\right\} \bigcap X^{(k)}$$

$$X^{(k+1,1)} = \begin{cases} \left\{x^{(k)} - \dfrac{x^{(k)} - x^{(k-1)}}{f(x^{(k)}) - f(x^{(k-1)})}\left[f(x^{(k)}) + \right.\right. \\ \left.\left.\dfrac{1}{2}K\left(X^{(k+1,0)} - x^{(k)}\right).\left(X^{(k+1,\,0)} - x^{(k-1)}\right)\right]\right\} \bigcap X^{(k+1,0)} \\ \qquad \text{if } f(x^{(k)}) \neq 0 \\ X^{(k+1,\,0)} \qquad \text{otherwise.} \end{cases}$$

for $i = 2, ..., p$.

$z^{(i)} = m\left(X^{(k+1,i-1)}\right), \quad z^{(i-1)} = m\left(X^{(k+1,i-2)}\right)$

$$X^{(k+1,i)} = \begin{cases} \left\{ z^{(i)} - \dfrac{z^{(i)} - z^{(i-1)}}{f(z^{(i)}) - f(z^{(i-1)})} \left[f(z^{(i)}) + \right. \right. \\ \left. \left. \dfrac{1}{2} K \left(X^{(k+1,i-1)} - z^{(i)} \right) . \left(X^{(k+1,\, i-1)} - z^{(i-1)} \right) \right] \right\} \bigcap X^{(k+1,i-1)} \\ \qquad \text{if } f(z^{(i)}) \neq 0 \\ \\ X^{(k+1,\, i-1)} \qquad \text{otherwise.} \end{cases}$$

$$X^{(k+1)} = X^{(k+1,\, p)}.$$

Theorem 2.1 *Let the function f be twice continuously differentiable in the interval X and assume that f has a zero ξ in X. Let the conditions (1) and (2) be satisfied. The parameter p is an integer number for which $p \geq 2$. Then, the sequence $\left\{X^{(k)}\right\}$ generated from p-RFM1 satisfies:*

$$\xi \in X^{(k)}, \quad k \geq 0$$

$$X^{(0)} \supset X^{(1)} \supset X^{(2)} \supset \ldots \textit{ with } \lim_{k \to \infty} X^{(k)} = \xi$$

and for $\gamma \geq 0$

$$d\left(X^{(k+1)}\right) \leq \gamma d\left(X^{(k)}\right)^{(2p-1)} d\left(X^{(k-1)}\right). \tag{4}$$

Then the R-order of convergence of p-RFM1 satisfies the inequality,

$$O_R\left((p - RFM1),\, \xi\right) \geq \frac{1}{2}\left((2p-1) + \sqrt{(2p-1)^2 + 4}\right). \tag{5}$$

3 Algorithm *p*-RFM1

Given that: $X^{(0)} = \left[x_1^{(0)}, x_2^{(0)}\right]$; $\xi \in X^{(0)}$; $\epsilon = 10^{-15}$; $d\left(X^{(k+1)}\right) = \left|x_2^{(k+1)} - x_1^{(k+1)}\right|$ with conditions (1) and (2).

Step 1 : Set $k = 0$, $i = 2$

Step 2 : $x^{(k)} = m\left(X^{(k)}\right)$ (midpoint of $X^{(k)}$)

Step 3 : $X^{(k+1)} = \left\{ x^{(k)} - \dfrac{f\left(x^{(k)}\right)}{H} \right\} \bigcap X^{(k)}$

Step 4 : $k := k + 1$

Step 5 : $x^{(k)} = m\left(X^{(k)}\right)$

Step 6 : $X^{(k+1,i-2)} = \left\{x^{(k)} - \dfrac{f\left(x^{(k)}\right)}{H}\right\} \bigcap X^{(k)}$

Step 7 : Compute

$$X^{(k+1,i-1)} = \left\{x^{(k)} - \frac{x^{(k)} - x^{(k-1)}}{f(x^{(k)}) - f(x^{(k-1)})}\left[f(x^{(k)}) + \frac{1}{2}K\left(X^{(k+1,i-2)} - x^{(k)}\right)\left(X^{(k+1,i-2)} - x^{(k-1)}\right)\right]\right\} \bigcap X^{(k+1,i-2)}$$

if $f\left(x^{(k)}\right) \neq 0$. Otherwise, $X^{(k+1,i-1)} = X^{(k+1,i-2)}$

Step 8 : $z^{(i)} = m\left(X^{(k+1,i-1)}\right)$ and $z^{(i-1)} = m\left(X^{(k+1,i-2)}\right)$

Step 9 : Compute

$$X^{(k+1,i)} = \left\{z^{(i)} - \frac{z^{(i)} - z^{(i-1)}}{f(z^{(i)}) - f(z^{(i-1)})}\left[f(z^{(i)}) + \frac{1}{2}K\left(X^{(k+1,i-1)} - z^{(i)}\right)\left(X^{(k+1,i-1)} - z^{(i-1)}\right)\right]\right\} \bigcap X^{(k+1,i-1)}$$

if $f\left(z^{(i)}\right) \neq 0$. Otherwise, $X^{(k+1,i)} = X^{(k+1,i-1)}$

Step 10 : If $d\left(X^{(k+1,i)}\right) < \epsilon$, go to 14

Step 11 : If $i < p$ then $i := i + 1$ and go to 8

Step 12 : $X^{(k+1)} = X^{(k+1,p)}$

Step 13 : If the width of the interval $X^{(k+1)}$ greater than ϵ, that is $d\left(X^{(k+1)}\right) > \epsilon$, then go to 4

Step 14: Stop.

4 Numerical Results

We present complete numerical results using two test examples where the stopping criterion is $d\left(X^{(k+1)}\right) < 10^{-15}$ and $p = 5$. The following results are obtained by using Matlab R2007a [1] in associate with Intlab [8].

Example 4.1

The function $f(x) = (x-1)(x^4+1)$ has a zero $\xi = 1$ in $X^{(0)} = [0.2,\ 2.0]$. The results obtained by using p-RF method and p-RFM1 method are presented in Table 1 and Table 2 respectively. For p-RF method, the CPU time taken is 0.0312500 second and ends at iteration $k = 3$, $i = 5$ compared to p-RFM1 method where the CPU time taken is 0.01562500 second and ends at iteration $k = 2$, $i = 2$.

Example 4.2

k	i	$X^{(k+1,i)}$	$d\left(X^{(k+1,i)}\right)$
0		[0.85358999999999, 1.09498146639512]	0.241391466395113
1	0	[0.97528133236619, 1.02316965237787]	0.047888320011663
	1	[0.99513872890716, 1.02316965237787]	0.028030923470691
	2	[0.99513872890716, 1.01935332764408]	0.024214598736904
	3	[0.99513872890716, 1.01547680514940]	0.020338076242224
	4	[0.99513872890716, 1.01194208444040]	0.016803355533230
	5	[0.99513872890716, 1.00907728269631]	0.013938553789132
2	0	[0.99787416316147, 1.00202177682740]	0.004147613665922
	1	[0.99989175268693, 1.00202177682740]	0.002130024140458
	2	[0.99993828354072, 1.00006977151653]	1.314879757978726e-4
	3	[0.99999603617088, 1.00000399775332]	7.961582433879677e-6
	4	[0.99999976544888, 1.00000023469400]	4.692451113230689e-7
	5	[0.99999998619637, 1.00000001380423]	2.760785011712841e-8
3	0	[0.99999999999969, 1.00000000000029]	5.900835375882707e-13
	1	[0.99999999999996, 1.00000000000001]	3.552713678800501e-14
	2	[0.99999999999999, 1.00000000000001]	3.330669073875470e-16
	3	[0.99999999999999, 1.00000000000001]	3.330669073875470e-16
	4	[0.99999999999999, 1.00000000000001]	3.330669073875470e-16
	5	[0.99999999999999, 1.00000000000001]	3.330669073875470e-16

Table 1: The results of p-RF method with Example 4.1

k	i	$X^{(k+1,i)}$	$d\left(X^{(k+1,i)}\right)$
0		[0.85358999999999, 1.09498146639512]	0.241391466395113
1	0	[0.97528133236619, 1.02316965237787]	0.047888320011663
	1	[0.99513872890716, 1.02316965237787]	0.028030923470691
	2	[0.99513872890716, 1.00923360327808]	0.014094874370908
	3	[0.99733393674489, 1.00274565275650]	0.005411716011595
	4	[0.99963385731345, 1.00036649028933]	7.326329758698957e-4
	5	[0.99999583298527, 1.00000416704239]	8.334057113312809e-6
2	0	[0.99999999998616, 1.00000000001327]	2.710176527642716e-11
	1	[0.99999999998616, 1.00000000000258]	1.640854119244750e-11
	2	[0.99999999999999, 1.00000000000001]	3.330669073875470e-16

Table 2: The results of p-RFM1 method with Example 4.1

The function $f(x) = x\,(x^9 - 1) - 1$ has zero $\xi \approx 1.07576...$ in $X^{(0)} = [0.1, 2.15]$. The results obtained by using p-RF method and p-RFM1 method are presented in Table 3 and Table 4 respectively. The CPU time of p-RF method is 0.03125 second and ends at iteration $k = 3$, $i = 5$. While the CPU

time of p-RFM1 method is 0.015625 second and ends at iteration $k = 2$, $i = 2$.

k	i	$X^{(k+1,i)}$	$d\left(X^{(k+1,i)}\right)$
0		[1.00029766383684, 1.12207347842121]	0.121775814584360
1	0	[1.06183800081867, 1.08898632512707]	0.027148324308388
	1	[1.07286786740548, 1.08898632512707]	0.016118457721578
	2	[1.07286786740548, 1.08898632512707]	0.016118457721578
	3	[1.07286786740548, 1.08898632512707]	0.016118457721578
	4	[1.07286786740548, 1.08898632512707]	0.016118457721578
	5	[1.07286786740548, 1.08898632512707]	0.016118457721578
2	0	[1.07286786740548, 1.08067519964994]	0.007807332244443
	1	[1.07543570827449, 1.08067519964994]	0.005239491375439
	2	[1.07543570827449, 1.07669506897587]	0.001259360701370
	3	[1.07556002323090, 1.07598559639626]	4.255731653519668e-4
	4	[1.07569584953018, 1.07583658668433]	1.407371541206626e-4
	5	[1.07574338687893, 1.07578875214866]	4.536526972076516e-5
3	0	[1.07576606254727, 1.07576606935031]	6.803029695134910e-9
	1	[1.07576606394863, 1.07576606616207]	2.213424110664164e-9
	2	[1.07576606608683, 1.07576606608684]	2.220446049250313e-16
	3	[1.07576606608683, 1.07576606608684]	2.220446049250313e-16
	4	[1.07576606608683, 1.07576606608684]	2.220446049250313e-16
	5	[1.07576606608683, 1.07576606608684]	2.220446049250313e-16

Table 3: The results of p-RF method with Example 4.2

k	i	$X^{(k+1,i)}$	$d\left(X^{(k+1,i)}\right)$
0		[1.00029766383684, 1.12207347842121]	0.121775814584360
1	0	[1.06183800081867, 1.08898632512707]	0.027148324308388
	1	[1.07286786740548, 1.08898632512707]	0.016118457721578
	2	[1.07286786740548, 1.08250938586611]	0.009641518460619
	3	[1.07343530774921, 1.07818318802548]	0.004747880276261
	4	[1.07513541284123, 1.07639744906999]	0.001262036228751
	5	[1.07573926681963, 1.07579286549305]	5.359867340382074e-5
2	0	[1.07576606601504, 1.07576606615303]	1.379751868313406e-10
	1	[1.07576606601504, 1.07576606610024]	8.518541427804394e-11
	2	[1.07576606608683, 1.07576606608684]	2.220446049250313e-16

Table 4: The results of p-RFM1 method with Example 4.2

5 Conclusion

From Table 1 - Table 4, the numerical results clearly show that the p-RFM1 procedure gives better results in terms of processing time and the width of the last interval that satisfies the stopping criterion. Furthermore, the computation of p-RFM1 method is terminated earlier than does the computation of p-RF method whenever the stooping criterion is satisfied. In fact, the R-order of convergence of p-RFM1 is at least

$$\frac{1}{2}\left((2p-1)+\sqrt{(2p-1)^2+4}\right)\ ;p\geq 2$$

On the other hand, the R-order of convergence of p-RF method [3] is at least

$$\frac{1}{2}\left(p+\sqrt{p^2+4}\right)\ ;p\geq 2.$$

where

$$O_R\left((p-\mathrm{RFM1})\,,\ \xi\right) > O_R\left((p-\mathrm{RF})\,,\ \xi\right)\ ;\ p\geq 2.$$

References

[1] A. Gilat, MATLAB: An Introduction with Applications, Hobuken, NJ: Wiley, 2008.

[2] E. Hansen, Interval forms of Newton's method. Computing, 20 (1978), 153-163.

[3] G. Alefeld and J. Herzberger, Introduction to Interval Computation, Academic Press, New York, 1983.

[4] I. Gargantini, Further applications circular arithmetic: Shoroeder-like algorithms with error bounds for finding zeroes of polynomial. SIAM Journal on Numerical Analysis, 15 (1978), 149-154.

[5] L.W. Ehrlic, A modified Newton's method for polynomials, Communications of the ACM, 10 (1967), 107-107.

[6] N. Anderson and $\mathring{A}$. Björck, A new high order method of regula falsi type for computing a root of an equation, BIT, 13 (1973), 253-264.

[7] O. Aberth, Iteration methods for finding all zeros of polynomial simultaneously, Mathematics of Computation, 27 (1973), 339-344.

[8] S.M. Rump, INTLAB - INTerval LABoratory: In Tibor Csendes, Developments in Reliable Computing, Kluwer Academic Publishers, Dordrecht, (1999), 77-104. **http://www.ti3.tu-harburg.de/rump/**.

Received: February, 2011