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Efficiency Decomposition for Two-process Production Systems with Shared Inputs in Data Envelopment Analysis

Ali Ashrafi

Institute for Mathematical Research, Universiti Putra Malaysia,
43400 UPM Serdang, Selangor, Malaysia
Faculty of Mathematics, Statistics and Computer Science,
University of Semnan, Semnan, Iran

A. B. Jaafar, L. S. Lee and M. R. Abu Bakar

Institute for Mathematical Research, Universiti Putra Malaysia,
43400 UPM Serdang, Selangor, Malaysia

Abstract

Zha and Liang [10] develop an approach for measuring the efficiency of two-process production systems where the inputs of the system can be freely allocated between two processes. They represent the overall efficiency of the system as a product form, to integrate the efficiencies of the two processes. A major restriction of Zha and Liang's model is its non-linearity. This paper develops a relational DEA model, taking into account the series relationship among two processes, to measure the overall efficiency of two-process production systems with shared inputs. The linearity of DEA models is maintained in the new model. Also, by introducing dummy process, the original production system can be transformed into a two-stage system where the first stage has a parallel structure. we utilize the relational model to propose a model that the overall efficiency score of the system is decomposed into the product of the efficiencies of the stages. This helps the decision makers identify the inefficient stages and make later improvements. A numerical example of US commercial banks is used to clarify the model.

Mathematics Subject Classification: 90C90, 90B99

Keywords: Data envelopment analysis, Decision making unit, Two-process production system, Shared input, Efficiency

1 Introduction

Data Envelopment Analysis (DEA) was first introduced by Charnes et al. [2] as a non-parametric technique for measuring the relative efficiency of peer production systems or decision making units (DMU) that have multiple inputs and outputs. In recent years, a great number of DEA studies have focused on two-process production systems, where all outputs from the first process are intermediate products that make up the inputs to the second process. For example, Seiford and Zhu [8] developed a two-process DEA approach for measuring the efficiency of the profitability and marketability of US commercial banks. Zhu [11] applied the same two-process system to assess the financial efficiency of the best 500 companies. Sexton and Lewis [9] studied the Major League Baseball performance in a two-process system. Chen and Zhu [3] developed a linear DEA type model where each process's efficiency is defined on its own production possibility set. Kao and Hwang [6] developed a different approach where the overall efficiency of the system can be decomposed into the product of the efficiencies of the two processes. Chen et al. [4] presented a model similar to the Kao and Hwang's model, but in an additive form. Actually, in the real world it may not be possible that a DMU generates its final output only by using intermediate products without any other inputs. For example, a bank uses employees and fixed assets to produce intermediate output, such as deposits in the first process. Some of employees and fixed assets may flow into second process and use together with intermediate products as the inputs for the second process to produce the second process's output, such as profit. These situations impose some limitations e.g. efficiency formulation and linearizing transformation for the conventional two-process DEA models. Recently, Zha and liang [10] introduced a non-linear approach, under the assumption of constant returns to scale (CRS), for measuring the efficiency of two-process production systems with shared inputs. They proposed the overall efficiency of the system as a product form, to integrate the efficiencies of the two processes. By taking one of the efficiencies of the processes as a parameter, they converted the non-linear model into a parametric linear model. In this paper, we develop a relational DEA model, taking into account the series relationship among two processes, to measure the overall efficiency of two-process production systems with shared inputs. Unlike the Zha and Liang's model, the linearity of DEA models is maintained in our model. Also, by introducing dummy process, the original production system can be transformed into a two-stage system where the first stage has a parallel structure. we apply the relational model and introduce a DEA model for measuring the overall efficiency of the system as well as the efficiencies of the two stages. As a result, the overall efficiency score is decomposed into the product of the efficiency scores of the two stages.

The rest of this paper is organized as follows. Section 2 presents some preliminary considerations. In Section 3, we first present a general two-process production system with sharing inputs and then develop a relational DEA model for measuring the efficiencies of the whole system as well as the two processes. Section 4 applies the new approach to the 30 firms in the banking industry in US. Finally, conclusions are provided in the last section. .

2 Preliminaries

Let $x_{ij},\ (i = 1, ..., m)$ and $y_{rj},\ (r = 1, ..., s)$ represent the ith input, and rth output of $DMU_j,\ (j = 1, ..., n)$. It is assumed that all inputs and outputs are non-negative, but at least one of the inputs and outputs are positive. If v_i and u_r be the known multipliers or prices associated with input i and output r, then the relative efficiency score of DMU_o, $(o \in \{1, ..., n\})$, E_o, can be expressed as

$$E_o = \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}. \tag{1}$$

In the absence of known multipliers, Charnes et al. [2] proposed a fractional programming problem (CCR model) to derive appropriate multipliers for a given DMU. The CCR model for evaluating the relative efficiency of DMU_o under the assumption of constant returns to scale (CRS), is indicated as follows:

$$\begin{aligned} E_o^* = max\ & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i x_{io}}, \\ \text{s.t}\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{i=1}^{m} v_i x_{ij}}, \quad j = 1, ..., n, \\ & u_r,\ v_i, \geq \varepsilon,\ r = 1, ..., s, i = 1, ..., m, \end{aligned} \tag{2}$$

where ε is non-Archimedean small value designed to impose strict positivity on the multipliers. $E_0^* = 1$ indicates efficiency and $E_0^* < 1$ for inefficiency.

3 A two-process DEA model with shared inputs

Now, consider a production system composed of a two processes with shared inputs as shown in Fig. 1. Suppose we have n DMUs, that each $DMU_j, (j = 1, ..., n)$ has m inputs $x_{ij}, (i = 1, ..., m)$ that can be freely allocated in any of the two processes. Also, there are p intermediate products $z_{dj}, (d = 1, , p)$ that are the outputs of process 1 as well as the inputs of process 2. The outputs

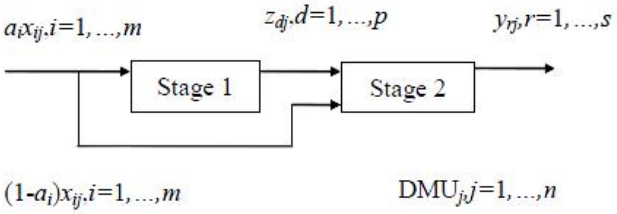

Fig. 1. Two-stage production system with shared inputs

from the second process are $y_{rj}, (r = 1, ..., s)$.
We suppose that $a_i x_{ij}, (i = 1, ..., m)$ and $(1 - a_i)x_{ij}, (i = 1, ..., m)$ are the amounts of shared input i distributed to the process 1 and process 2, respectively. Consider, DMU_o, $(o \in \{1, ..., n\})$ be the DMU under evaluation. Based on the CCR model, the efficiency scores of the whole process and the two individual processes can be calculated as follows:

$$\begin{aligned} E_o &= \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} v_i a_i x_{io} + \sum_{i=1}^{m} \tilde{v}_i (1 - a_i) x_{io}}, \\ E_o^1 &= \frac{\sum_{d=1}^{p} \eta_d z_{do}}{\sum_{i=1}^{m} v_i a_i x_{io}}, \\ E_o^2 &= \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{d=1}^{p} \tilde{\eta}_d z_{do} + \sum_{i=1}^{m} \tilde{v}_i (1 - a_i) x_{io}}, \end{aligned} \tag{3}$$

where $v_i, \tilde{v}_i, u_r, \eta_d$ and $\tilde{\eta}_d$ are unknown non-negative multipliers. Note that η_d can be equal to $\tilde{\eta}_d$.
In an effort to estimate the overall efficiency of the DMU_o, taking into account the series relationship among two processes, we formulate the following fractional program:

$$\begin{aligned} E_o^* = max \ & E_o \\ \text{s.t} \ & E_j^1 \leq 1, \quad j = 1, ..., n, \\ & E_j^2 \leq 1, \quad j = 1, ..., n, \\ & \eta_d = \tilde{\eta}_d, \ d = 1, ..., p. \end{aligned} \tag{4}$$

Note that, similar to Kao and Hwang [6] and Chen et al. [4], we have assumed that every intermediate product has the same multiplier, no matter whether it plays the role of input or output.
Let $v_i a_i = \pi_i$ and $\tilde{v}_i(1 - a_i) = \omega_i$, then model (4) is equivalent to the following

model:

$$\begin{aligned} E_o^* = max\ & \frac{\sum_{r=1}^{s} u_r y_{ro}}{\sum_{i=1}^{m} \pi_i x_{io} + \sum_{i=1}^{m} \omega_i x_{io}}, \\ \text{s.t}\ & \frac{\sum_{d=1}^{p} \eta_d z_{dj}}{\sum_{i=1}^{m} \pi_i x_{ij}} \leq 1, \quad j = 1, ..., n, \\ & \frac{\sum_{r=1}^{s} u_r y_{rj}}{\sum_{d=1}^{p} \eta_d z_{dj} + \sum_{i=1}^{m} \omega_i x_{ij}} \leq 1, \quad j = 1, ..., n, \\ & \pi_i, \omega_i, u_r, \eta_d \geq \varepsilon, \quad i = 1, ..., m,\ r = 1, ..., s,\ d = 1, ..., p. \end{aligned} \tag{5}$$

Note that model (5) is a fractional programming problem that can be converted into a linear programming problem by applying the Charnes-Cooper transformation (Charnes and Cooper [1]) as follows:

$$\begin{aligned} E_o^* = max\ & \sum_{r=1}^{s} u_r y_{ro}, \\ \text{s.t}\ & \sum_{i=1}^{m} \pi_i x_{io} + \sum_{i=1}^{m} \omega_i x_{io} = 1, \\ & \sum_{d=1}^{p} \eta_d z_{dj} - \sum_{i=1}^{m} \pi_i x_{ij} \leq 0, \quad j = 1, ..., n, \\ & \sum_{r=1}^{s} u_r y_{rj} - \sum_{i=1}^{m} \omega_i x_{ij} - \sum_{d=1}^{p} \eta_d z_{dj} \leq 0, \quad j = 1, ..., n, \\ & \pi_i, \omega_i, u_r, \eta_d \geq \varepsilon, \quad i = 1, ..., m,\ r = 1, ..., s,\ d = 1, ..., p. \end{aligned} \tag{6}$$

This model will be solved for n times, once for each DMU, to evaluate the overall efficiency score of the system. On optimality, the efficiency scores of two processes of each DMU_o, $(o \in \{1, ..., n\})$, can be calculated as follows:

$$\begin{aligned} E_o^{1*} =& \frac{\sum_{d=1}^{p} \eta_d^* z_{do}}{\sum_{i=1}^{m} \pi_i^* x_{io}}, \\ E_o^{2*} =& \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{p} \eta_d^* z_{do} + \sum_{i=1}^{m} \omega_i^* x_{io}}, \end{aligned} \tag{7}$$

where $u_i^*, \pi_i^*, \omega_i^*$ and η_d^* are the optimal multipliers calculated from model (6). Using the model (6), we can evaluate the overall efficiency of the DMU_o in a way that takes into account the operations of its two processes. Also, using equation (7), we are able to recognize the inefficient processes and make later improvements.

According to Kao [7], in order to decompose the efficiency of system, we introduce dummy process 3 to present the two-stage production system with shared

inputs in the form of series and parallel structures. The outputs of dummy process 3 are the same as its inputs. With dummy process 3, the system of Fig. 1 can be represented as a new two-stage system connected in series, where stage I has a parallel structure composed of processes 1 and 3. This equivalent two-stage production system is depicted in Fig. 2.

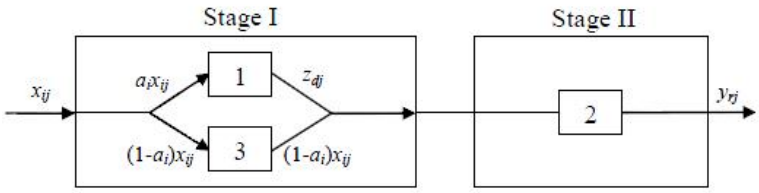

Fig. 2. An equivalent system with series and parallel structures

Note that as demonstrated by Chen and Zhu [3], since the outputs of dummy process 3 are the same its inputs, the dummy process 3 is always efficient. Under these series and parallel structures, according to Kao [7], the overall efficiency of DMU_o will be decomposed into the product of efficiencies of stages I and II as follows:

$$\begin{aligned} E_o^{(I)*} &= \frac{\sum_{d=1}^{p} \eta_d^* z_{do} + \sum_{i=1}^{m} \omega_i^* x_{io}}{\sum_{i=1}^{m} \pi_i^* x_{io} + \sum_{i=1}^{m} \omega_i^* x_{io}}, \\ E_o^{(II)*} &= \frac{\sum_{r=1}^{s} u_r^* y_{ro}}{\sum_{d=1}^{p} \eta_d^* z_{do} + \sum_{i=1}^{m} \omega_i^* x_{io}}, \\ E_o^* &= E_o^{(I)*} \times E_o^{(II)*}. \end{aligned} \tag{8}$$

The performance decomposition helps the decision makers identify the source of inefficiency and make appropriate improvement.

4 Numerical example

In this section, the new approach is applied to the 30 firms in the banking industry in US as studied by Zha and Liang [10]. They divided the production system of the banking industry into two processes. The inputs to the process 1 are the number of employees (x_1), assets (x_2) and equity (x_3). Some of the employees, assets and equity may flow in the process 2 and act as the inputs of the process 2. The outputs from the process 2 are market value (y_1), earning (y_2) and returns (y_3). There are also two intermediate products between the two processes, namely revenue (z_1) and profit (z_2). Table 1 reports the data

set.

By applying the proposed relational model, model (6), the overall efficiencies of the 30 banks are calculated as shown in the second column of Table 2. The efficiency scores of the two processes for each DMU were calculated by using Equation (7) and the results are reported in Table 2 under the heading process efficiency. As we explained in Section 2, by introducing dummy process, we can decompose the system efficiency into the product of the efficiencies of the stages I and II. The remaining columns of Table 2 show the stage efficiencies as calculated from Equation (8). From Table (2), it can be seen that three of

Table 1: Data set

DMU	In.			Int. Product		Out.		
	x_1	x_2	x_3	z_1	z_2	y_1	y_2	y_3
1	85300	256853.0	19581.0	31690.0	3464.0	33221.7	7.21	66.1
2	95288	232446.0	20222.0	20386.0	2664.0	27148.6	6.49	69.4
3	58322	187298.0	12801.0	16298.0	1950.0	20295.9	7.13	59.7
4	39078	182926.0	11912.0	14884.0	1805.0	16971.3	6.73	70.5
5	15600	184879.0	10451.0	13838.0	1296.0	15003.5	6.42	49.4
6	33365	121173.0	9134.0	11336.0	1165.0	12616.4	5.76	82.4
7	35328	122002.0	8450.0	10681.0	1150.0	12351.1	3.45	50.0
8	44536	131879.9	9043.1	10582.9	1430.2	16815.0	5.04	39.9
9	46900	90454.0	8197.5	8970.9	1277.9	14807.4	2.91	54.9
10	14000	104000.0	5000.0	8600.0	215.0	5252.4	2.03	28.3
11	30800	84432.2	6364.8	7919.4	610.0	10428.7	1.57	31.8
12	45404	72134.4	5312.1	7582.3	956.0	12268.6	2.76	45.5
13	26757	73404.0	5768.0	6389.5	408.1	9938.2	1.19	61.4
14	28905	66339.1	5152.5	6054.0	825.0	8671.2	3.45	51.6
15	17881	47397.0	3751.0	5410.6	541.0	5310.1	4.55	84.7
16	19700	50316.0	4055.0	5409.0	1032.0	11342.5	20.37	52.8
17	15850	53685.0	5223.0	5327.0	914.0	10101.5	4.57	69.9
18	27200	58071.0	4154.0	4827.5	885.1	12138.0	11.02	108.5
19	24300	40129.0	4106.0	4514.0	691.0	7476.7	4.50	83.8
20	15996	44981.3	3773.8	3755.4	602.5	7623.6	3.50	46.9
21	19415	46471.5	4269.6	3740.3	565.5	7922.5	4.94	46.9
22	20175	41553.5	3272.2	3680.0	533.3	5774.9	5.30	59.0
23	20767	36199.0	2921.0	3449.9	465.1	4912.2	3.03	33.9
24	13231	33874.0	2725.0	3328.3	568.1	8304.0	4.19	54.3
25	13500	35469.9	2607.7	3112.6	413.4	4537.0	3.54	71.7
26	17023	33703.8	2928.1	2996.1	418.8	4997.0	3.25	57.3
27	14081	31794.3	2617.0	2897.3	329.0	4865.1	2.09	66.8
28	13598	29620.6	2379.4	2868.0	452.2	5788.0	3.22	52.0
29	4900	43881.6	3007.8	2859.6	288.6	3218.0	4.66	41.1
30	11171	13228.9	1265.1	2565.4	353.1	6543.3	1.54	60.7

DMUs, namely DMU 5, DMU 16 and DMU 30, perform efficiently as a whole as well as each processes and stages. Also DMUs 17, 24 and 29 have an overall efficiency scores of greater than 0.9. Since the overall efficiency, E_o^*, is a product of the efficiencies of stages I,$E_o^{(I)*}$, and II, $E_o^{(II)*}$, every E_o^* is always less than or equal to its corresponding $E_o^{(I)*}$ and $E_o^{(II)*}$. Table 2 shows that for most DMUs the efficiency of the stage I is higher than the stage II. This exhibits that the low efficiency score for the whole system is caused by the low efficiency score of the stage II. Because of the stage I has a dummy process connected parallel with a real process, and a dummy process is always efficient, the efficiency score of process 1 is not greater than the efficiency score of stage

Table 2: Results

DMU	System eff.	Process eff.		Stage eff.	
		Process 1	Process 2	Stage I	Stage II
1	0.444	0.726	0.494	0.899	0.494
2	0.361	0.567	0.427	0.846	0.427
3	0.454	0.751	0.498	0.911	0.498
4	0.537	0.581	0.598	0.901	0.598
5	1	1	1	1	1
6	0.541	0.608	0.842	0.642	0.842
7	0.498	0.546	0.540	0.922	0.540
8	0.541	0.698	0.587	0.922	0.587
9	0.498	0.642	0.526	0.946	0.526
10	0.477	0.754	0.499	0.955	0.499
11	0.519	0.580	0.555	0.935	0.555
12	0.476	0.680	0.614	0.775	0.614
13	0.561	0.568	0.614	0.915	0.614
14	0.477	0.695	0.499	0.956	0.499
15	0.793	0.846	0.806	0.985	0.806
16	1	1	1	1	1
17	0.960	1	0.960	1	0.960
18	0.861	0.696	0.898	0.959	0.898
19	0.657	0.745	0.676	0.972	0.676
20	0.739	0.753	0.756	0.977	0.756
21	0.655	0.589	0.675	0.970	0.675
22	0.605	0.656	0.626	0.966	0.626
23	0.401	0.603	0.414	0.969	0.414
24	0.991	0.825	1	0.991	1
25	0.899	0.715	0.918	0.979	0.918
26	0.622	0.634	0.641	0.970	0.641
27	0.805	0.648	0.826	0.975	0.826
28	0.713	0.765	0.725	0.984	0.725
29	0.988	0.818	1	0.988	1
30	1	1	1	1	1

I. Also, because of the process 2 is the same with stage II, the efficiency score of process 2 is equal to efficiency score of stage II.

5 Conclusions

In the DEA literature, a great number of DEA studies have focused on two-process production systems, where all outputs from the first process are intermediate products that make up the inputs to the second process. However, in some real world situations DMUs have a two-process structure that the inputs of the system can be freely allocated among two processes. Efficiency formulation and linearized transformation may be two important issues when we use the conventional two-process DEA models. In this paper, we introduced a relational DEA model for measuring the overall efficiency of the system as well as the efficiencies of two processes. By introducing dummy process, we have decomposed the overall efficiency into the product of the efficiency scores of the stages. This helps the decision makers identify the inefficient stages and make later improvements.

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