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An Optimal Control Problem for Rigid Body Motions in Minkowski Space

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Abstract. In this paper smooth trajectories are computed in the Lie group $SO(2,1)$ as a motion planning problem by assigning a Frenet frame to the rigid body system to optimize the cost function of the elastic energy which is spent to track a spacelike curve in Minkowski space. In this case, the derivative of the tangent vector of the spacelike curve at a point s is taken as timelike. A method is proposed to solve a motion planning problem that minimizes the integral of the Lorentz inner product of Darboux vector of a spacelike curve. This method uses the coordinate free Maximum Principle of Optimal control and results in the theory of integrable Hamiltonian systems. The presence of several conversed quantities inherent in these Hamiltonian systems aids in the explicit computation of the rigid body motions.

Keywords: Optimal control, Hamiltonian vector field, Darboux vector, Maximum Principle, Lie group, Rigid body motion, Lorentzian metric

1. Introduction

An optimal control problem is used for rigid body motions and formulation on Lie group $SO(2,1)$ where the cost function to be minimized is equal to the integral of the Lorentz inner product of Darboux vector of a spacelike curve. Here the derivative of the tangent vector of the spacelike curve at a point s is timelike. This problem is analogous to the elastic problem in [1, 4]. The coordinate free maximum principle [2, 3] is applied to solve this problem. In [4] the author applied an integrable case where the necessary conditions for optimality can be expressed analytically and the corresponding optimal motions are expressed in a coordinate free manner. These optimal motions are showed to trace helical paths. In this study, the optimal control

problem formulation is considered as the general theory of optimal control for the motion planning application, framed curves and left-invariant Hamiltonian systems are applied to this particular setting. Frenet frame of curve is applied to the Lorentz-Minkowski space to solve the problem and a particular set of curves is analyzed that satisfies these necessary conditions and provides analytic solutions for the corresponding optimal motions. An application of the Maximum Principle to this problem results in a system of first order differential equations that yields coordinate free necessary conditions for optimality. This system minimizes the cost function of elastic energy which is spent to track a spacelike curve in Minkowski space.

2. Frenet frame

The Lorentz-Minkowski space is the metric space $E^3 = (R^3, <, >)$ where the metric is given by $< x, y >= -x_1y_1 + x_2y_2 + x_3y_3$. The exterior product is defined as

$$x \times y = (x_2y_3 - y_2x_3)\,\overrightarrow{e_1} + (y_3x_1 - x_3y_1)\,\overrightarrow{e_2} + (x_2y_1 - y_2x_1)\,\overrightarrow{e_3}, \tag{2.1}$$

where $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}$ is the standard orthonormal frame in R^3. The metric $<, >$ is called as Lorentzian metric. Let H^2 denote the hyperboloid $x_1^2 - (x_2^2 + x_3^2) = 1$, $x_1 > 0$. The isometry group for a hyperbolic plane H^2 is denoted by $SO(2, 1)$.

Recall that $SO(2, 1)$ is the group that leaves the bilinear form $<, >$ in E^3 invariant. L is the Lie algebra of $SO(2, 1)$ which is equal to the space of matrices

$$A = \begin{pmatrix} 0 & a_1 & a_2 \\ a_1 & 0 & a_3 \\ a_2 & a_3 & 0 \end{pmatrix}. \tag{2.2}$$

Definition 2.1. A vector $v \in E^3$ is called spacelike if $< v, v > 0$ or $v = 0$, timelike if $< v, v >< 0$, lightlike if $< v, v >= 0$ and $v \neq 0$.

Definition 2.2. For a curve α in E^3, α is spacelike (resp. timelike, lightlike) at t if $\alpha'(t)$ is a spacelike (resp. timelike, lightlike) vector. If it is for any $t \in I$, the curve α is called spacelike (resp. timelike, lightlike).

In this paper it is supposed that α is a spacelike curve, $T(s) = \alpha'(s)$ as the tangent vector at s and supposed T'(s)$\neq$ 0 is the timelike vector independent with $T(s)$. The curvature of α at s is defined as $k(s) = |T'(s)|$. The normal vector $N(s)$ is defined by $N(s) = \frac{T'(s)}{k(s)}$. Moreover $k(s) = \sqrt{- < T'(s), T'(s) >}$ is the curvature of the curve α. The binormal vector $B(s)$ is defined by $B(s) = T(s) \times N(s)$ which is a spacelike vector. $\tau(s) =< N'(s), B(s) >$ is the torsion

of the curve α. For each s, $\{T(s), N(s), B(s)\}$ is an orthonormal base of E^3 which is called the Frenet trihedron of α. By differentiation each one of the vector functions of the Frenet trihedron frame
$R = (T \mid N \mid B) \in L$ about the curve $\alpha : I \to E^3$ described by the following differential equations:

$$\alpha'(t) = T,\ T' = kN,\ N' = kT + \tau B,\ B' = \tau N, \tag{2.3}$$

where k curvature, τ torsion of the curve α. [6]

$$\begin{pmatrix} T' \\ N' \\ B' \end{pmatrix} = \begin{pmatrix} 0 & k & 0 \\ k & 0 & \tau \\ 0 & \tau & 0 \end{pmatrix} \begin{pmatrix} T \\ N \\ B \end{pmatrix}. \tag{2.4}$$

These equations form a rotation motion with Darboux vector [5] $w = \tau T - kB$. Also momentum rotation vector is expressed as follows:

$$T' = w \times T,\ N' = w \times N,\ B' = w \times B.$$

Moreover $< w, w >= \tau^2 < T, T > +k^2 < B, B >= k^2 + \tau^2$. Since T and B are spacelike unitary vectors then $< T, T >=< B, B >= 1$. In this study, this Frenet frame is used to plan rigid body motions by applying the maximum principle to optimal control systems defined on the Lie group [2]. An element $g(t) \in M$ is defined as:

$$g(t) = \begin{pmatrix} 1 & 0 \\ \alpha(t) & R(t) \end{pmatrix}, \tag{2.5}$$

where $R(t) \in L$. There is also associated with (2.3) via the relations

(2.6)
$$[1\ \alpha(t)]^T = g(t)\overrightarrow{e_1},\ \ [0\ T]^T = g(t)\overrightarrow{e_2},\ \ [0\ N]^T = g(t)\overrightarrow{e_3},\ \ [0\ B]^T = g(t)\overrightarrow{e_4},$$

where $\overrightarrow{e_1}, \overrightarrow{e_2}, \overrightarrow{e_3}, \overrightarrow{e_4}$ is the standard orthonormal frame in E^4.

Proposition 2.3. *The left-invariant differential equation:*

$$\frac{dg(t)}{dt} = g(t) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & k & 0 \\ 1 & k & 0 & \tau \\ 0 & 0 & \tau & 0 \end{pmatrix}, \tag{2.7}$$

where $g(t) \in M$ is equivalent to the Frenet frame (2.3).

Proof. It follows from differentiating (2.6) w.r.t to t that

$$
\begin{array}{l}
[1\ \alpha(t)]^T = \frac{dg(t)}{dt}\overrightarrow{e_1} = g(t)\overrightarrow{e_2} = [0\ T]^T, \\
[0\ T]^T = \frac{dg(t)}{dt}\overrightarrow{e_2} = g(t)\,(k\overrightarrow{e_3}) = k\,[0\ N]^T, \\
[0\ N]^T = \frac{dg(t)}{dt}\overrightarrow{e_3} = g(t)\,(k\overrightarrow{e_2} + \tau\overrightarrow{e_4}) = k\,[0\ T]^T + \tau\,[0\ B]^T, \\
[0\ B]^T = \frac{dg(t)}{dt}\overrightarrow{e_4} = g(t)\,(\tau\overrightarrow{e_3}) = \tau\,[0\ N]^T,
\end{array} \tag{2.8}
$$

then equating the L.H.S to the R.H.S yields (2.3). The system (2.7) can be expressed conveniently in coordinate form by defining the following basis for the Lie algebra of M denoted by m

(2.9)

$$
A_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad A_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad A_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
B_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.
$$

Using these notations, it follows that (2.7) can be expressed as $\frac{dg(t)}{dt} = g(t)(B_2 + kA_1 + \tau A_3)$. To minimize the elastic energy of the curve, this is equivalent minimizing the function $J = \frac{1}{2}\int < w, w > dt = \frac{1}{2}\int (k^2 + \tau^2)\,dt$, where the w defined as the Darboux vector with the equation $w = \tau T - kB$. The motion $g(t) \in M$ of the left-invariant differential system (2.8) which minimizes the expression (2.9) is computed on a given interval $[0, T]$ subject to the given boundary conditions $g(0) = g_0$, $g(T) = g_T$ on the next section. □

3. Hamiltonian lift on M

Due to the similarity in between optimal control problem and elastic problem, this optimal control problem is considered as elastic problem and the applicability of Maximum Principle is obvious. The Maximum Principle states that the optimal paths are the projections of the extremal curves onto the base manifold, where the extremal curves are solutions of certain Hamiltonian systems on the cotangent bundle. For the problem, the manifold is M and the cotangent bundle is T^*M. The appropriate pseudo-Hamiltonian on T^*M is defined as:

$$
H\,(p, u, g) = p\,(g(t)\,B_2) + kp\,(g(t)\,A_1) + \tau p\,(g(t)\,A_3) - p_0\frac{1}{2}\left(\tau^2 + k^2\right), \tag{3.1}
$$

where $p(.) : TM \to \mathbb{R}$. In this study, the regular extremals where $p_0 = 1$ (ignoring abnormal extremals where $p_0 = 0$) is carried . The cotangent bundle T^*M can be written as the direct product $M \times m^*$ where m^* is the dual of the Lie algebra m of M. The original Hamiltonian defined on T^*M can be expressed as a reduced Hamiltonian on the dual of the Lie algebra m^*. The linear functions $M_i = \hat{p}(A_i), p_i = \hat{p}(B_i)$ for $i = 1, 2, 3$ where $\hat{p} : m \to \mathbb{R}$ are the Hamiltonian lifts of left-invariant vector fields on M, because $p(g(t) A_i) = \hat{p}(A_i)$ for any $P = \left(g(t), \hat{p}\right)$ and any $A_i \in m$. If M_i, p_i is a collection of linear functions generated by the basis A_i, B_i in m then the vector $(M_1, M_2, M_3, p_1, p_2, p_3)$ is the coordinate vector of $\hat{p}$ relative to the dual basis A_i^*, B_i^*. The Hamiltonian (3.1) becomes

$$H = p_2 + kM_1 + \tau M_3 - \frac{1}{2}\left(\tau^2 + k^2\right). \tag{3.2}$$

It follows from [2] that calculating $\frac{\partial H}{\partial k} = \frac{\partial H}{\partial \tau} = 0$ yields the optimal controls $k = M_1$ and $\tau = M_3$ substituting into (3.2) gives the optimal Hamiltonian

$$H = p_2 + \frac{1}{2}\left(M_3^2 + M_1^2\right). \tag{3.3}$$

In addition the optimal motions are the solutions $g(t) \in M$ of the differential equation:

$$\frac{dg(t)}{dt} = g(t) \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & M_1 & 0 \\ 1 & M_1 & 0 & M_3 \\ 0 & 0 & M_3 & 0 \end{pmatrix}. \tag{3.4}$$

To solve the equation (3.4) for $g(t) \in M$, it is necessary to solve the extremal curves M_1, M_2, M_3 for a special case.

4. Solving the extremal curves

To compute the corresponding Hamiltonian vector fields from the left-invariant Hamiltonian (3.3) the Lie bracket table (a) obtained for the basis (2.9):

$[,]$	A_1	A_2	A_3	B_1	B_2	B_3
A_1	0	A_3	A_2	0	B_3	B_2
A_2	$-A_3$	0	A_1	B_3	0	$-B_1$
A_3	$-A_2$	$-A_1$	0	B_2	B_1	0
B_1	0	$-B_3$	$-B_2$	0	0	0
B_2	$-B_3$	0	$-B_1$	0	0	0
B_3	$-B_2$	B_1	0	0	0	0

table (a)

where the Lie Bracket is defined as $[X,Y]=XY-YX$. The time derivatives of M_i, p_i along the Hamiltonian flow are described by the Poisson bracket given by the equation $\left\{\hat{\wedge}p(.),\hat{\wedge}p(.)\right\}=-\hat{\wedge}p([.,.])$, we can get

$$\begin{gathered}M_1{}'=\{M_1,H\}=\{M_1,p_2\}+M_3\{M_1,M_3\}+M_1\{M_1,M_1\}=-p_3-M_3M_2,\\ M_2{}'=\{M_2,H\}=\{M_2,p_2\}+M_3\{M_2,M_3\}+M_1\{M_2,M_1\}=0,\end{gathered}$$

(4.1)

$$\begin{gathered}M_3{}'=\{M_3,H\}=\{M_3,p_2\}+M_3\{M_3,M_3\}+M_1\{M_3,M_1\}=-p_1+M_1M_2,\\ p_1{}'=\{p_1,H\}=\{p_1,p_2\}+M_1\{p_1,M_1\}+M_3\{p_1,M_3\}=p_2M_3,\\ p_2{}'=\{p_2,H\}=\{p_2,p_2\}+M_1\{p_2,M_1\}+M_3\{p_2,M_3\}=p_1M_3+p_3M_1,\\ p_3{}'=\{p_3,H\}=\{p_3,p_2\}+M_1\{p_3,M_1\}+M_3\{p_3,M_3\}=p_2M_1.\end{gathered}$$

The Hamiltonian vector fields are

$$\begin{gathered}M_1{}'=-p_3-M_3M_2,\ M_2{}'=0,\ M_3{}'=-p_1+M_1M_2,\\ p_1{}'=p_2M_3,\ p_2{}'=p_1M_3+p_3M_1,\ p_3{}'=p_2M_1.\end{gathered}\tag{4.2}$$

A trivial example of an integrable case of vector fields (4.2) occurs when $p_1=p_2=p_3=M_1=M_2=M_3=0$. Moreover, for these values $p_1=p_2=p_3=M_1=M_2=M_3$ are constant for all t and therefore the system is integrable. By substituting these values into (3.4) we can get $\frac{dg(t)}{dt}=g(t)B_2$. This is easily integrated to yield $\alpha(t)=[t,0,0]^T$ with R equal to a 3×3 matrix with zero entries. Therefore, a straight line motion with zero rotation about this line is an optimal rigid body motion. In addition there exists a nontrivial integrable case of the Hamiltonian vector fields (4.2). This case is considered nontrivial as it gives rise to time-dependent extremal curves. It is observed that $p_1=p_2=p_3=0$ is an invariant surface for the Hamiltonian vector fields (4.2). Explicitly, for $p_1=p_2=p_3=0$ the equations (4.2) degenerate to:

$$M_1{}'=-M_3M_2,\ M_2{}'=0,\ M_3{}'=M_1M_2,\ p_1{}'=p_2{}'=p_3{}'=0.\tag{4.3}$$

This implies that M_2 is constant that will be denoted by c. In addition $p_1=p_2=p_3=0$ for all t. It follows that the Hamiltonian vector fields (4.3) are completely integrable. For these particular curves the Hamiltonian (3.3) reduces to $H=M_3^2+M_1^2$. It follows that the differential equations (4.3) are satisfied that the extremal curves are:

$$M_2=c,\ M_1=r\cos ct,\ M_3=r\sin ct.\tag{4.4}$$

To compute the optimal motions corresponding to the extremal curves (4.4) is not trivial as the elements of the Lie algebra are time-dependent.

5. Optimal motions for the rigid body

The Frenet frame (3.4) is split into its translational and rotational part $\frac{d\alpha(t)}{dt} = R\overrightarrow{e_2}$ and

$$\frac{dR}{dt} = R\begin{pmatrix} 0 & M_1 & 0 \\ M_1 & 0 & M_3 \\ 0 & M_3 & 0 \end{pmatrix}, \tag{5.1}$$

where $R^{-1} = R^T$. A basis is described for the Lie algebra m as:

$$E_1 = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},\ E_2 = \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix},\ E_3 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \tag{5.2}$$

The quantities

$$RPR^{-1} = constant,\ RMR^{-1} + \left[X, RPR^{-1}\right] = constant, \tag{5.3}$$

are conversed for all left-invariant Hamiltonian systems on M where

$$M = M_1E_1 + M_2E_2 + M_3E_3,\ P = p_1E_1 + p_2E_2 + p_3E_3,\ X = x_1E_1 + x_2E_2 + x_3E_3, \tag{5.4}$$

where x_1, x_2, x_3 are the position coordinates of the vector $\alpha(t) = [x_1, x_2, x_3]^T$. Using these constants of motion (5.1) is integrated which is stated in the following theorem:

Theorem 5.1. *$R = (T \mid N \mid B) \in L$ is the optimal rotation matrix corresponding to the extremals (4.4) which relates the Frenet frame to a fixed inertial frame where:*

$$\begin{aligned} T &= \begin{pmatrix} \cos Kt\cos ct + \frac{c}{K}\sin Kt\sin ct \\ -\frac{r}{K}\sin ct \\ \sin Kt\cos ct - \frac{c}{K}\cos Kt\sin ct \end{pmatrix}, \\ N &= \begin{pmatrix} -\frac{r}{K}\sin Kt \\ \frac{c}{K} \\ \frac{r}{K}\cos Kt \end{pmatrix}, \\ B &= \begin{pmatrix} \cos Kt\sin ct - \frac{c}{K}\sin Kt\cos ct \\ \frac{r}{K}\cos ct \\ \sin Kt\sin ct + \frac{c}{K}\cos Kt\cos ct \end{pmatrix}, \end{aligned} \tag{5.5}$$

where $K^2 = c^2 + r^2$ and r, c are the constant parameters of the curvatures (4.4).

Proof. For these particular curves $p_1 = p_2 = p_3 = 0$ the conversation laws (5.3) reduce to $RMR^{-1} = constant$ this constant matrix RMR^{-1} is then conjugated for a particular solution R such that $RMR^{-1} = \sqrt{M_1^2 + M_2^2 + M_3^2}E_2$, by using (4.4) gives $RMR^{-1} = \sqrt{c^2 + r^2}E_2$. Therefore $M = KR^{-1}E_2R$ is verified. Expressing R in a convenient coordinate from [2] we have

(5.6)
$$R = \exp(\varphi_1 E_2)\exp(\varphi_2 E_3)\exp(\varphi_3 E_2),\ R^{-1} = \exp(-\varphi_3 E_2)\exp(-\varphi_2 E_3)\exp(-\varphi_1 E_2),$$

therefore we can write:

$$M = K\exp(-\varphi_3 E_2)\exp(-\varphi_2 E_3)E_2\exp(\varphi_2 E_3)\exp(\varphi_3 E_2). \tag{5.7}$$

It is shown that:

(5.8)
$$M = K\begin{pmatrix} 0 & \sinh\varphi_2\cos\varphi_3 & \cosh\varphi_2 \\ \sinh\varphi_2\cos\varphi_3 & 0 & -\sinh\varphi_2\sin\varphi_3 \\ -\cosh\varphi_2 & -\sinh\varphi_2\sin\varphi_3 & 0 \end{pmatrix} = \begin{pmatrix} 0 & M_1 & M_2 \\ M_1 & 0 & M_3 \\ -M_2 & M_3 & 0 \end{pmatrix},$$

equating M in (5.4) to (5.8) gives:

$$M_1 = K\sinh\varphi_2\cos\varphi_3,\ M_2 = K\cosh\varphi_2,\ M_3 = -K\sinh\varphi_2\sin\varphi_3. \tag{5.9}$$

So it is easily shown that:

$$\cosh\varphi_2 = \frac{M_2}{K} = \frac{c}{K},\ \sinh\varphi_2 = \pm\sqrt{\frac{c^2}{K^2} - 1} = \pm\frac{r}{K}, \tag{5.10}$$

in addition form (5.9) $\tan\varphi_3 = -\frac{M_3}{M_1}$ therefore

(5.11)
$$\sin\varphi_3 = \mp\frac{M_3}{\sqrt{M_1^2 + M_3^2}} = \mp\sin ct,\ \cos\varphi_3 = \pm\frac{M_1}{\sqrt{M_1^2 + M_3^2}} = \pm\cos ct,$$

in order to obtain an expression for φ_1, we substitute (5.6) into (5.1) yields:

$$\begin{aligned} \frac{dR}{dt} &= \varphi_1' E_2\exp(\varphi_1 E_2)\exp(\varphi_2 E_3)\exp(\varphi_3 E_2) + \varphi_2'\exp(\varphi_1 E_2)E_3\exp(\varphi_2 E_3)\exp(\varphi_3 E_2) \\ &\quad + \varphi_3'\exp(\varphi_1 E_2)\exp(\varphi_2 E_3)E_2\exp(\varphi_3 E_2), \end{aligned} \tag{5.12}$$

therefore

$$
\begin{aligned}
R^{-1}\frac{dR}{dt} &= \varphi_1{}'\exp(-\varphi_3 E_2)\exp(-\varphi_2 E_3)E_2\exp(\varphi_2 E_3)\exp(\varphi_3 E_2)\\
&\quad + \varphi_2{}'\exp(-\varphi_3 E_2)E_3\exp(\varphi_3 E_2) + \varphi_3{}'E_2\\
&= \varphi_1{}'\begin{pmatrix} 0 & \sinh\varphi_2\cos\varphi_3 & \cosh\varphi_2 \\ \sinh\varphi_2\cos\varphi_3 & 0 & -\sinh\varphi_2\sin\varphi_3 \\ -\cosh\varphi_2 & -\sinh\varphi_2\sin\varphi_3 & 0 \end{pmatrix} \qquad (5.13)\\
&\quad + \varphi_2{}'\begin{pmatrix} 0 & \sin\varphi_3 & 0 \\ \sin\varphi_3 & 0 & \cos\varphi_3 \\ 0 & \cos\varphi_3 & 0 \end{pmatrix} + \varphi_3{}'\begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\\
&= \begin{pmatrix} 0 & M_1 & 0 \\ M_1 & 0 & M_2 \\ 0 & M_2 & 0 \end{pmatrix},
\end{aligned}
$$

which leads to

$$
M_1 = \varphi_1{}'\sinh\varphi_2\cos\varphi_3 + \varphi_2{}'\sin\varphi_3,\ \ M_2 = -\varphi_1{}'\sinh\varphi_2\sin\varphi_3 + \varphi_2{}'\cos\varphi_3, \qquad (5.14)
$$

therefore

$$
\varphi_1{}' = \frac{M_1\cos\varphi_3 - M_3\sin\varphi_3}{\sinh\varphi_2}, \qquad (5.15)
$$

substituting (4.4), (5.10) and (5.11) into (5.15) yields $\varphi_1{}' = K$ and integrating with respect to t yields $\varphi_1 = Kt + \beta$ where β is a constant of integration and for $\beta = 0$ yields $\varphi_1 = Kt$. An other hand from (5.5) yields :

$$
\begin{aligned}
T &= \begin{pmatrix} \cosh\varphi_1\cos\varphi_3 - \cosh\varphi_2\sin\varphi_1\sin\varphi_3 \\ \sinh\varphi_2\sin\varphi_3 \\ \sinh\varphi_1\cos\varphi_3 + \cos\varphi_1\cosh\varphi_2\sin\varphi_3 \end{pmatrix}, \qquad (5.16)\\
N &= \begin{pmatrix} -\sin\varphi_1\sinh\varphi_2 \\ \cosh\varphi_2 \\ \cos\varphi_1\sinh\varphi_2 \end{pmatrix},\\
B &= \begin{pmatrix} -\cos\varphi_1\sin\varphi_3 - \sin\varphi_1\cosh\varphi_2\cos\varphi_3 \\ \sinh\varphi_2\cos\varphi_3 \\ -\sin\varphi_1\sin\varphi_3 + \cos\varphi_1\cosh\varphi_2\cos\varphi_3 \end{pmatrix},
\end{aligned}
$$

substituting (5.10), (5.11) and $\varphi_1 = Kt$ into (5.6) and choosing the natural frame to be positively oriented (5.5) yields $R = (T \mid N \mid B)$.

□

Lemma 5.2. *The optimal path $\alpha(t) \in E^3$ defined by the differential equation $\frac{d\alpha(t)}{dt} = R\overrightarrow{e_2}$, with $M_1 = r\cos ct$, $M_2 = c$ and $M_3 = r\sin ct$ are described by $\alpha(t) = \frac{1}{K^2}\left[r\cos Kt, ct, r\sin Kt\right]^T$.*

Proof. From $\frac{d\alpha(t)}{dt} = \frac{1}{K}\begin{pmatrix} -r\sin Kt \\ c \\ r\cos Kt \end{pmatrix}$, by integrating yields:

$$\alpha(t) = \frac{1}{K}\int \begin{pmatrix} -r\sin Kt \\ c \\ r\cos Kt \end{pmatrix} dt = \frac{1}{K^2}\left(r\cos Kt, ct, r\sin Kt\right)^T. \qquad \square$$

6. Conclusion

An application of the Maximum Principle to this optimal control problem results in a system of first order differential equations that yields coordinate free necessary conditions for optimality. In this paper, Frenet frame of curve is applied to the Lorentz-Minkowski space to solve optimal control problem. A particular set of curves is analyzed that satisfies these necessary conditions and provides analytic solutions for the corresponding optimal motions. In this study, the coordinate free maximum principle and the theory of integrable Hamiltonian systems are used to minimize the cost function which is equivalent to integrate the Lorentz inner product of Darboux vector with respect to Frenet frame of the curve.

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