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Reflection of Subsonic Phase Boundaries [1] [2]

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Abstract

The multidimensional stability of subsonic phase transitions in a non-isothermal van der Waals fluid is considered. The arguments are based on the existence result of planar waves in our previous work. We prove that the planar wave is uniformly stable in the sense of Majda under both one dimensional and multidimensional perturbations.

Mathematics Subject Classification: 35L45, 35L50

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1 Introduction

In this paper, the attention is paid to the reflection phenomenon of subsonic phase transitions in a van der Waals fluid. The phase transition is an important phenomenon in physics, mechanics and fluid dynamics. In a fluid with a non-monotonic state equation, say van der Waals fluid, multiple phases usually coexist and phase boundaries propagates. There has been a rich literature devoted to the existence and stability of phase transitions in one space variable, cf. [1, 2] and references therein. For multidimensional phase transitions,one can refer to the work [3, 4, 5, 6].

The purpose of this paper is to establish the existence of the solutions for the problem of the reflection of subsonic phase transitions. We shall study the case when a plane subsonic phase boundary hit a curved rigid wall. To determine the reflected wave, we propose the assumption that there is only one discontinuity in the reflected wave and find the reflected wave is a shock

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wave. The initial boundary value problem is similar to the one for reflection of shock wave in [7].

The remainder of this paper is arranged as follows. In section 2 we recall the admissibility criterion for subsonic phase transition and formulate the problem of the reflection of subsonic phase transition. In section 3, we will determine the reflected wave by studying a simple planar case and give the initial data for the multidimensional case. In section 4, we give the problem of the reflected wave and establish the existence of such solutions.

2 Problems

2.1 Admissibility criterion for phase transitions

For an isothermal van der Waals fluid, the following well-known Euler equations

$$\partial_t \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix} + \partial_x \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \\ \rho uv \end{pmatrix} + \partial_y \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p(\rho) \end{pmatrix} = 0 \tag{1}$$

represents the conservation of mass and momentum, where ρ and (u, v) are the density and velocity of the fluid respectively. The pressure law $P(\tau) = p(1/\tau)$, with $\tau = 1/\rho$ being the specific volume, is given by

$$P(\tau) = \frac{RT}{\tau - b} - \frac{a}{\tau^2} \quad (\tau > b) \tag{2}$$

where T denotes the positive constant temperature, R is the perfect gas constant, and a, b are positive constants. When the temperature T satisfies $\frac{a}{4bR} < T < \frac{8a}{27bR}$, there are $\tau_* < \tau^*$ such that

$$\begin{cases} P'(\tau) < 0, & \text{if } b < \tau < \tau_* \text{ of } \tau > \tau^*, \\ P'(\tau) > 0, & \text{if } \tau_* < \tau < \tau_*. \end{cases} \tag{3}$$

Thus the state of $\tau \in (b, \tau_*)$ represents the liquid phase while that of $\tau \in (\tau^*, +\infty)$ is the vapor phase. Generally, these two phases are likely to coexist and one may observe the propagation of phase boundaries.

Denote by $U = (\rho, u, v)^T$,

$$F_0(U) = \begin{pmatrix} \rho \\ \rho u \\ \rho v \end{pmatrix}, F_1(U) = \begin{pmatrix} \rho u \\ \rho u^2 + p(\rho) \\ \rho uv \end{pmatrix}, F_2(U) = \begin{pmatrix} \rho v \\ \rho uv \\ \rho v^2 + p(\rho) \end{pmatrix},$$

and

$$A_1(U) = (F_0'(U))F_1'(U) = \begin{pmatrix} u & \rho & 0 \\ \frac{c^2}{\rho} & u & 0 \\ 0 & 0 & u \end{pmatrix}, A_2(U) = (F_0'(U))F_2'(U) = \begin{pmatrix} v & 0 & \rho \\ 0 & v & 0 \\ \frac{c^2}{\rho} & 0 & v \end{pmatrix}$$

where $c = (p'(\rho))^{\frac{1}{2}}$ is the sound speed.

If a piecewise smooth function

$$U(t,x,y) = \begin{cases} U_+(t,x,y) & x > \varphi(t,y) \\ U_-(t,x,y) & x < \varphi(t,y) \end{cases}$$

where $U_\pm \in C^1 \{\pm(x - \varphi(t,y)) > 0\}$ belonging to different phase and $\varphi \in C^2$ being the phase boundary, is said to be a subsonic phase transition, it has to satisfy system (1) in the region where the function is smooth. On the phase boundary $\{x = \varphi(t,y))\}$ satisfy the Rankine-Hugoniot condition

$$\varphi_t[F_0(U)] - [F_1(U)] + \varphi_y[F_2(U)] = 0 \tag{4}$$

with $[\cdot]$ representing the difference of a function on the discontinuity, and the subsonic condition

$$M_\pm = \frac{1}{c_\pm}\left|\frac{u_\pm - \varphi_y v_\pm - \varphi_t}{(1+\varphi_y^2)^{\frac{1}{2}}}\right| < 1, \tag{5}$$

where $c_\pm = (p'(\rho))^{\frac{1}{2}}$ are the sound speeds. Due to (5), the Rankine-Hugoniot condition (4) is not enough to determine the boundary value to grantee the well-posedness of the problem [1]. Following [1, 4], we have the following boundary condition derived from the viscosity-capillarity admissibility criterion:

$$\left[e'(\rho) + \frac{(u - \varphi_y v - \varphi_t)^2}{2(1+\varphi_y^2)}\right] = -\nu a(j,\nu), \quad \text{on } x = \varphi(t,y) \tag{6}$$

where $e(\rho) = \rho E(\rho)$ is the free energy per unit volume with $E(\rho)$ satisfying $d_\rho E(\rho) = p(\rho)/\rho^2$, $j = \rho_\pm(u_\pm - \varphi_y v_\pm - \varphi_t)/(1+\varphi_y^2)^{\frac{1}{2}}$ is the mass transfer flux across the phase boundary, and

$$a(j,\nu) = j\int_{-\infty}^{+\infty} \tau'^2(\xi; j,\nu)d\xi \tag{7}$$

where $\tau(\xi; j,\nu)$ is the viscosity-capillarity profile satisfying

$$\begin{cases} \tau'' = \nu j\tau' + \pi - P(\tau) - j^2\tau \\ \lim_{\xi\to-\infty}\tau = \frac{1}{\rho_-}|_{x=\varphi} \quad \lim_{\xi\to+\infty}\tau = \frac{1}{\rho_+}|_{x=\varphi} \end{cases}$$

with τ', τ'' being the first and second order derivatives of τ with respect to ξ, $\pi = p(\rho_\pm) + j^2/\rho_\pm$ valued at $x = \varphi$.

2.2 Formulation of the problem

Let Σ be a physical boundary in $\mathbb{R}^2$ parameterized by $x = \eta(y)$, where $\eta(y) \in C^\infty$ and $\eta(0) = \eta'(0) = 0$, $\eta(y) \le 0$. The outside of Σ, $\{x > \eta(y)\}$, is filled with the fluid. On Σ, the velocity of the fluid in the normal direction is zero. Therefore the boundary condition on Σ reads

$$u - \eta_y v = 0. \tag{8}$$

Let $\mathcal{P}$ be a plane subsonic phase boundary moving towards Σ. For $t < 0$, on both sides of $\mathcal{P}$ the flow fields are constant. Denote by $U_a = (\rho_a, u_a, v_a)$ on the left and $U_b = (\rho_b, u_b, v_b)$ on the right. Without loss of generality, we assume $\tau_a \in (b, \tau_*)$ and $\tau_b \in (\tau^*, +\infty)$. We assume that

$$P''(\tau) > 0 \text{ for } \tau \in [\tau_b, +\infty). \tag{9}$$

$\mathcal{P}$ moves towards Σ with a constant velocity $V < 0$. We parameterize $\mathcal{P}$ by $\{x = Vt\}$.

At $t = 0$, $\mathcal{P}$ meets Σ at the point $(0, 0)$. Obviously, before the phase boundary $\mathcal{P}$ intersects with the surface Σ, the velocity of $\mathcal{P}$ and the flow field on both side of $\mathcal{P}$ remain constant. Therefore the intersection of $\mathcal{P}$ with Σ is

$$\Gamma = \left\{ x = \eta(y), t = \frac{1}{V}\eta(y) \right\}.$$

For $t > \frac{1}{V}\eta(y)$, we assume only one new discontinuity $\mathcal{S}$ issuing from Γ. Denote by $x = \psi(t, y)$ the equation of $\mathcal{S}$ and

$$U(t,x,y) = \begin{cases} U_-(t,x,y) & \eta(y) < x < \psi(t,y) \\ U_+(t,x,y) & x > \psi(t,y) \end{cases}$$

the flow fields on the left side and the right side of $\mathcal{S}$. $U_\pm(t,x,y)$ and $\psi(t,y)$ should satisfy the following conditions:

(1)$U_\pm(t,x,y)$ satisfy the system in their corresponding domains.

(2)$U_\pm(t,x,y)$ satisfy the Rankine-Hugoniot condition on $\{x = \psi(t,y)\}$.

(3)$u_- - \eta_y v_- = 0$ on $x = \eta(y)$.

The next purpose is to determine what kind of discontinuity $\mathcal{S}$ is and prove the local existence of $U_\pm$ and ψ.

3 Reflected waves

3.1 A simple case

In order to determine the reflected wave, first we study a simple case, the case when the physical boundary is a plane. Assume $\eta(y) \equiv 0$ and $v_b = 0$. Then the reflected wave is also a plane wave and the flow field behind the discontinuity S is constant. Denote by $V' > 0$ the speed of $\mathcal{S}$. From (4) we have that the incident wave satisfies the following jump conditions:

$$\frac{u_a - V}{\tau_a} = \frac{u_b - V}{\tau_b}, \quad P_a + \frac{(u_a - V)^2}{\tau_a} = P_b + \frac{(u_b - V)^2}{\tau_b} \tag{10}$$

We can determine the flow field behind $\mathcal{S}$ from the Rankine-Hugoniot conditions and boundary condition (8) as follows:

$$u_- = v_- = 0, \quad \frac{u_- - V'}{\tau_-} = \frac{u_b - V'}{\tau_b}, \quad P_- + \frac{(u_- - V')^2}{\tau_-} = P_b + \frac{(u_b - V')^2}{\tau_b} \tag{11}$$

From (10) and (11), we have

$$\frac{\tau_b}{\tau_-} = 1 - \frac{\tau_a u_b^2}{(\tau_a - \tau_b)VV'} < 1$$

Therefore $\tau_- \in (\tau_b, +\infty)$. Eliminating V' from (11), we have that τ_- solves the following equation

$$P(\tau_-) = P_- = -\frac{u_b^2}{\tau_- - \tau_b} + P_b. \tag{12}$$

Denote by $F(\tau) = P(\tau) - P_b + \frac{u_b^2}{\tau - \tau_b}$. By a simple calculation, we have

$$F'(\tau) = P'(\tau) - \frac{u_b^2}{(\tau - \tau_b)^2} < 0 \text{ for } \tau \in (b, \tau_*) \cup (\tau^*, +\infty).$$

On the other hand, we have

$$\lim_{\tau \to \tau_b^+} F(\tau) = +\infty, \quad \lim_{\tau \to +\infty} F(\tau) = -P_b < 0.$$

Therefore there exists a unique $\tau_- \in (\tau_b, +\infty)$ satisfying (12). Thus we have the speed of the reflected wave $\mathcal{S}$:

$$V' = \tau_- \sqrt{\frac{P_b - P_-}{\tau_- - \tau_b}}$$

and relative velocity of the fluid on the front side of S:

$$u_b - V' = -\tau_b \sqrt{\frac{P_b - P_-}{\tau_- - \tau_b}}.$$

By calculation, we get the sound speed on each sides of S

$$c_- = \tau_- \sqrt{-P'(\tau_-)} \quad , c_b = \tau_b \sqrt{-P'(\tau_b)}.$$

Due to the convexity (10), we have the following Lax entropy condition:

$$0 < V' < c_-, \quad V' > u_b + c_b$$

which indicates $\mathcal{S}$ is a front shock wave.

3.2 Multidimensional case

As we mentioned in the above subsection, the reflected wave is a front shock wave, for any point $(t, x, y) \in \Gamma$, we shall determine the initial speed, $\sigma(y) > 0$, of the reflected shock $\mathcal{S}$ and the initial flow field behind $\mathcal{S}$.

First, for $(t, x, y) \in \Gamma$, the flow field behind the shock front (ρ_-, u_-, v_-) and σ satisfy the boundary condition (8) and the Rankine-Hugoniot conditions as follows:

$$\begin{cases} u_- - \eta_y v_- = 0 \\ \sigma(\rho_- - \rho_b) - (\rho_- u_- - \rho_b u_b) + \eta_y \rho_- v_- = 0 \\ \sigma(\rho_- u_- - \rho_b u_b) - (p_- - p_b) - (\rho_- u_-^2 - \rho_b u_b^2) + \eta_y \rho_- u_- v_- = 0 \\ \sigma \rho_- v_- - \rho_- u_- v_- + \eta_y (p_- - p_b) + \eta_y \rho_- v_-^2 = 0 \end{cases}. \tag{13}$$

Substituting the first equation of (13) into the other three and eliminating σ, we have that τ_- satisfies the following equation

$$P(\tau_-) = -\frac{u_b^2}{(\tau_- - \tau_b)(1 + \eta_y^2)} + P_b. \tag{14}$$

Similar to the planar case, we have

$$\frac{\tau_b}{\tau_-} = 1 - \frac{\tau_a u_b^2}{(\tau_a - \tau_b) V \sigma} < 1$$

which implies that $\tau_- = 1/\rho_- \in (\tau_b, +\infty)$. Therefore there exists a unique $\tau_-(y) \in (\tau_b, +\infty)$ satisfying (14). Therefore the initial shock speed is determined as $\sigma(y) = \tau_-(y) u_b / (\tau_-(y) - \tau_b)$. The initial velocity of the fluid behind the shock front is

$$u_-(y) = \frac{\eta_y^2 u_b^2}{1 + \eta_y^2}, \quad v_-(y) = \frac{\eta_y u_b^2}{1 + \eta_y^2}.$$

4 Existence of Reflected Shocks

First we give the problem for the reflected shock as follows

$$\begin{cases} \partial_t U_- + A_1(U_-) \partial_x U_- + A_2(U_-) \partial_y U_- = 0 & \text{in } \eta(y) < x < \psi(y, t) \\ \psi_t [F_0(U)] - [F_1(U)] + \psi_y [F_2(U)] = 0 & \text{on } x = \psi(y, t) \\ u_- - \eta_y v_- = 0 & \text{on } x = \eta(y) \end{cases}. \tag{15}$$

One can refer to [7] for the detail of the proof of the local existence of the solution to the problem (15). Here we only give the main result.

Theorem 4.1 *If for all fixed* $(x_0, y_0) \in \{x = \eta(y)\}$ *all the planar shock wave*

$$U = \begin{cases} U_-(0, x_0, y_0) & x < \sigma(y_0)t \\ U_b & x > \sigma(y_0)t \end{cases}$$

are uniformly stable in the sense of Majda [8], then there exists a solution to problem (15) locally in time.

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