Applied Mathematical Sciences, Vol. 5, 2011, no. 24, 1153 - 1161

Exceptional Family of Elements and Solvability of Mixed Variational Inequality Problems

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Abstract

In this paper, we introduce a new kind of exceptional family for a class of mixed variational inequality problem (in short MVIPs) in a Hilbert space. Based on which and the topological degree theory, we obtain an alternative theorem and some existence theorems.

Mathematics Subject Classification: 49J40; 47J20

Keywords: mixed variational inequality, topological degree, resolvent operator, alternative theorem, exceptional family of elements, solution existence

1. Introduction

Variational inequality problems (in short VIs) have many successful practical applications in the last two decades. They have been used to formulate and investigate equilibrium models arising in economics, transportation and operations research. The development of solution existence has played an important role in theory, algorithms and applications for VIs. So far, a large number of solution existence results have been obtained by many authors.

Recently, exceptional families of elements (in short exceptional families) have been widely used to investigate the solvability of complementarity problems and variational inequality problems. Smith first introduced in [1] the notion of exceptional sequence of elements for continuous functions in order to investigate the solution existence of nonlinear complementarity problems. From then on, several formulations of exceptional families of elements were introduced by many authors, see for example [2-5] and references therein. By means of these formulations, several kinds of alternative theorems are obtained, which play an important role in studying solution existence of VIs. Han introduced in [5] a kind of exceptional family for VI over a general unbounded closed convex subset K of $\mathbb{R}^n$. By using the topological degree theory of single-valued mappings, He showed a characterization theorem that produces very general existence and compactness theorems for VI.

In this paper, we consider a class of mixed variational inequality problems (in short MVIPs) involving a nonlinear proper convex lower semicontinuous functions. Because of the presence of this term, the projection method and its variant forms cannot be extended and modified to study the solution existence for MVIPs. So, we use a new method instead of the projection method, that is, the resolvent operator. With the properties of resolvent operators and topological degree theory, we propose a new kind of of exceptional family. Based on which, we obtain an alternative theorem and some existence theorems of MVIPs.

2. Preliminaries

Let H be a real Hilbert space with inner product $\langle \cdot, \cdot \rangle$ and norm $\| \cdot \|$ and K be a nonempty unbounded closed convex set of H. Let $\partial\varphi$ be the subdifferential mapping, which is a maximal monotone operator, of a proper convex lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$. Given an open bounded set $D \subset H$, the closure and boundary of D are denoted by $\bar{D}$ and ∂D, respectively. We denote by $C(\bar{D})$ the family of all continuous operators from $\bar{D}$ into H and by $deg(f, D, y)$ the topological degree associated with f, D and y for $f \in C(\bar{D})$ and $y \in H \setminus f(\partial D)$.

For given nonlinear operators $f,\ g : H \to H$ with $g(H) \cap Dom\partial\varphi \neq \emptyset$. Consider the following problem of finding $x \in H$ such that

$$\langle f(x), g(y) - g(x) \rangle + \varphi(g(y)) - \varphi(g(x)) \geq 0, \quad \forall g(y) \in H. \tag{2.1}$$

The inequality (2.1) is called general mixed variational inequality. We denote by K^* its solution set.

The following are some special cases of the problem (2.1):

When $g \equiv I$, the identity operator, problem (2.1) is equivalent to finding $x \in H$ such that

$$\langle f(x), y - x \rangle + \varphi(y) - \varphi(x) \geq 0, \quad \forall y \in H.$$

which is called a mixed variational inequality. For its applications, numerical methods and formulations, see [6-7] and references therein.

When φ is the indicator function of a closed convex set K in H, that is

$$\varphi(x) \equiv I_K(x) = \begin{cases} 0, & \text{if } x \in K, \\ +\infty, & \text{otherwise.} \end{cases}$$

problem (2.1) is equivalent to finding $x \in H,\ g(x) \in K$ such that

$$\langle f(x), g(y) - g(x)\rangle \geq 0, \quad \forall g(y) \in K. \tag{2.2}$$

For $g \equiv I$, the identity operator, problem (2.2) collapses to: find $x \in K$ such that

$$\langle f(x), y - x\rangle \geq 0, \quad \forall y \in K, \tag{2.3}$$

which is called classical variational inequality. The solution existence of the problem were studied in [2-5].

The following basic concepts and results are taken from [5-6].

Definition 2.1. Let $A : H \to 2^H$ be monotone set-valued mapping. The resolvent operator associated with A is defined by

$$J_A(u) = (I + \rho A)^{-1}(u), \quad \forall u \in H.$$

where I is the identity operator and $\rho > 0$ is an any constant. It is well know that if A is maximal monotone then its resolvent operator $J_A(u)$ is a single-valued nonexpansive mapping and is defined everywhere on H.

Lemma 2.1. Let D be an open bounded subset of H and let $f \in C(\bar{D})$ be one-to-one mapping. If $p \in D$, then $deg(f, D, p) = \pm 1$.

Lemma 2.2. For a given $z \in H, u \in H$ satisfying the inequality

$$\langle u - z, v - u\rangle + \rho\varphi(v) - \rho\varphi(u) \geq 0, \quad \forall v \in H.$$

if and only if

$$u = J_\varphi(z),$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator of the subdifferential mapping $\partial\varphi$, which is a maximal monotone set-valued mapping and I is the identity operator.

Lemma 2.3. $x^* \in K^*$ if and only if

$$x^* = g^{-1}[J_\varphi[g(x^*) - \rho f(x^*)]], \quad \forall \rho > 0.$$

where $J_\varphi = (I + \rho\partial\varphi)^{-1}$ is the resolvent operator.

Lemma 2.4. Let D be an open bounded set of H and $f : \bar{D} \subset H \to H$ be a continuous mapping. If $y \in H \setminus f(\partial D)$ and $deg(f, D, y) \neq 0$, then $f(x) = y$ has a solution in D.

Lemma 2.5. Let $D \subset H$ be an open bounded set and $H : \bar{D} \times [0, 1] \to H$ be a continuous mapping. If $y \notin \{H(x, t) : x \in \partial D, t \in [0, 1]\}$, then $deg(H(\cdot, t), D, y)$ remains a constant as t varies over [0,1].

3. Exceptional family and alternative theorem

In this section, we introduce a new kind of exceptional family and propose an alternative theorem.

Definition 3.1. Let $\hat{x} \in H$ be an any given point and $\partial\varphi$ be the subdifferential mapping, which is a maximal monotone operator, of a proper convex lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$, A family of elements $\{x^r\}_{r>0} \subset H$ is said to be an exceptional family for problem (2.1) with respect to $\hat{x}$, if

(i) $\|x^r\| \to \infty$, as $r \to \infty$;

(ii) for any $r > \|g^{-1}[J_\varphi[g(\hat{x})]]\|$, there exists a real number $\alpha_r > 0$ such that

$$x^r = g^{-1}[J_\varphi[g(\hat{x}) - \alpha_r f(x^r)]], \tag{3.1}$$

where J_φ is the resolvent operator.

Theorem 3.1. (alternative theorem) Let $\partial\varphi$ be a subdifferential mapping, which is a maximal monotone operator, of a proper convex lower semicontinuous function $\varphi : H \to R \cup \{+\infty\}$ and let $f,\ g : H \to H$ be two continuous functions. Then one of the following consequences holds:

(i) $K^* \neq \emptyset$.

(ii) for every point $\hat{x} \in H$, there exists an exceptional family of problem (2.1) with respect to $\hat{x}$.

Proof. Assume that $K^* = \emptyset$. we will show that, the assumption (ii) holds.

Define a homotopy $M : H \times [0, 1] \to H$ by

$$M(x, t) = g(x) - J_\varphi[t(g(x) - f(x)) + (1 - t)g(\hat{x})]. \tag{3.2}$$

For any $r > 0$, let $D_r = \{x \in H : \|x\| < r\}$, where $r > 0$. We first prove that for every $r > \|g^{-1}[J_\varphi(g(\hat{x}))]\|$, there exist $x^r \in \partial D_r$ and $t_r \in [0, 1]$ such that $0 = M(x^r, t_r)$.

In fact, suppose to the contrary that there exists $r_0 > \|g^{-1}[J_\varphi(g(\hat{x}))]\|$ such that

$$0 \notin M(x, t) : \forall x \in \partial D_{r_0}, \forall t \in [0, 1] \tag{3.3}$$

Since $M : \bar{D}_{r_0} \times [0, 1] \subset H \to H$ is continuous, by the continuity of $f,\ g$ and the property of the resolvent operator $J_\varphi(\cdot)$, it follows from Lemma 2.5 and (3.3) that $deg(M(\cdot, t), D_{r_0}, t)$ is a constant on [0,1]. By (3.2), we know that $M(x, 0) = g(x) - J_\varphi[g(\hat{x})]$ is a single-valued mapping. According to Lemma 2.1, it follows that

$$|deg(M(\cdot, 0), D_{r_0}, 0)| = 1.$$

Hence

$$|deg(M(\cdot, 1), D_{r_0}, 0)| = |deg(M(\cdot, 0), D_{r_0}, 0)| = 1.$$

Since $M(x, 1) = g(x) - J_\varphi[g(x) - f(x)]$ is given by (3.2), by Lemma 2.4, we know that there exists at least one point x^* such that $M(x^*, 1) = 0$. This contradicts $K^* = \emptyset$.

Secondly, we will prove there exists an exceptional family. Take arbitrary $r > \|g^{-1}[J_\varphi(g(\hat{x}))]\|$. According to the above discuss, we know that there exist $x^r \in \partial D_r$ and $t_r \in [0,1]$ such that

$$g(x^r) = J_\varphi[t_r(g(x^r) - f(x^r)) + (1-t_r)g(\hat{x})]. \tag{3.4}$$

If $t_r = 1$, then (3.4) reduces to $g(x^r) = J_\varphi[g(x^r) - f(x^r)]$, which implies that $x^r \in K^*$. This contradicts $K^* = \emptyset$. If $t_r = 0$, then $g(x^r) = J_\varphi[g(\hat{x})]$ and $\|x^r\| = r = \|g^{-1}[J_\varphi(g(\hat{x}))]\|$, which contradicts $r > \|g^{-1}[J_\varphi(g(\hat{x}))]\|$. Hence, $t_r \in (0,1)$.

From (3.4), we can deduce that

$$(1-t_r)(g(\hat{x}) - g(x^r)) - t_r f(x^r) \in \partial\varphi(g(x^r)).$$

Hence, by the property of subdifferential mapping $\partial\varphi$ we have

$$\langle (1-t_r)(g(\hat{x}) - g(x^r)) - t_r f(x^r), g(y) - g(x^r)\rangle \le \varphi(g(y)) - \varphi(g(x^r)), \forall g(y) \in H,$$

which is equivalent to for any $g(y) \in H$ we have

$$\langle g(x^r) - [g(\hat{x}) - \frac{t_r}{1-t_r} f(x^r)], g(y) - g(x^r)\rangle + \frac{1}{1-t_r}\varphi(g(y)) - \frac{1}{1-t_r}\varphi(g(x^r)) \ge 0.$$

By lemma 2.2, we have

$$g(x^r) = J_\varphi[g(\hat{x}) - \frac{t_r}{1-t_r} f(x^r)]. \tag{3.5}$$

Put $\alpha_r := \frac{t_r}{1-t_r} > 0$, we have

$$x^r = g^{-1}[J_\varphi(g(\hat{x}) - \alpha_r f(x^r))]. \tag{3.6}$$

On the other hand, we have $\|x^r\| \to \infty$ as $r \to \infty$ and $x^r \in H$. It follows from $\alpha_r > 0$ and (3.6) that $\{x^r\}_{r>0} \subset H$ is an exceptional family for problem (2.1) with respect to $\hat{x}$.

4. Existence Theorems

In this section, we study solution existence by means of exceptional family for general mixed variational inequality.

Theorem 4.1. Let

(i) the hypotheses in theorem 3.1 hold;

(ii)for any family of elements $\{x^k\} \subset H$ with $\|x^k\| \to \infty$ as $k \to \infty$, if there exists $x^{k_0} \in \{x^k\}$, $\alpha_{k_0} > 0$ and $g(y) \in H$, such that

$$\|g(y) - g(\hat{x})\| < \|g(x^{k_0}) - g(\hat{x})\| \text{ and } \langle f(x^{k_0}), g(x^{k_0}) - g(y)\rangle + \frac{1}{\alpha_{k_0}}(\varphi(g(x^{k_0})) - \varphi(g(y)) \ge 0.$$

Then $K^* \neq \emptyset$.

Proof. Assume to the contrary that $K^* = \emptyset$. By Theorem 3.1, there exists an exceptional family with respect to $\hat{x}$, denoted by $\{x^r\}$. By Definition 3.1, we know that $\{x^r\} \subseteq H$ with $\|x^r\| \to \infty$, as $r \to \infty$, and there exists $\{\alpha_r\} > 0$ such that (3.1) hold. Let

$$u^r := g(\hat{x}) - \alpha_r f(x^r);$$

then, $x^r = g^{-1}[J_\varphi(u^r)]$ and

$$f(x^r) = \frac{1}{\alpha_r}(g(\hat{x}) - u^r). \tag{4.1}$$

From $x^r = g^{-1}[J_\varphi(u^r)]$, we can deduce that

$$u^r - g(x^r) \in \partial\varphi(g(x^r)),$$

hence

$$\langle u^r - g(x^r), g(y) - g(x^r)\rangle \leq \varphi(g(y)) - \varphi(g(x^r)), \quad \forall g(y) \in H. \tag{4.2}$$

By our assumption, we know that there exist $x^{k_0} \in \{x^r\}$, $\alpha_{k_0} \in \{\alpha_r\}$ and $g(y) \in H$ with

$$\|g(y)-g(\hat{x})\| < \|g(x^{k_0})-g(\hat{x})\| \text{ and } \langle f(x^{k_0}), g(x^{k_0})-g(y)\rangle+\frac{1}{\alpha_{k_0}}(\varphi(g(x^{k_0}))-\varphi(g(y)) \geq 0.$$

combing (4.1) and (4.2), we know that

$$f(x^{k_0}) = \frac{1}{\alpha_{k_0}}(g(\hat{x})-u^{k_0}) \text{ and } \langle u^{k_0}-g(x^{k_0}), g(y)-g(x^{k_0})\rangle \leq \varphi(g(y))-\varphi(g(x^{k_0})).$$

Thus,

$$\begin{aligned}
0 &\leq \langle f(x^{k_0}), g(x^{k_0}) - g(y)\rangle + \tfrac{1}{\alpha_{k_0}}(\varphi(g(x^{k_0})) - \varphi(g(y)))\\
&= \tfrac{1}{\alpha_{k_0}}\langle g(\hat{x}) - u^{k_0}, g(x^{k_0}) - g(y)\rangle + \tfrac{1}{\alpha_{k_0}}(\varphi(g(x^{k_0})) - \varphi(g(y)))\\
&= -\tfrac{1}{\alpha_{k_0}}\langle u^{k_0} + g(x^{k_0}) - g(x^{k_0}) - g(\hat{x}), g(x^{k_0}) - g(y)\rangle + \tfrac{1}{\alpha_{k_0}}(\varphi(g(x^{k_0})) - \varphi(g(y)))\\
&= \tfrac{1}{\alpha_{k_0}}\langle u^{k_0} - g(x^{k_0}), g(y) - g(x^{k_0})\rangle + \tfrac{1}{\alpha_{k_0}}(\varphi(g(x^{k_0}))\\
&\quad -\varphi(g(y))) - \tfrac{1}{\alpha_{k_0}}\langle g(x^{k_0}) - g(\hat{x}), g(x^{k_0}) - g(y)\rangle\\
&\leq -\tfrac{1}{\alpha_{k_0}}\langle g(x^{k_0}) - g(\hat{x}), g(x^{k_0}) - g(y)\rangle\\
&= -\tfrac{1}{\alpha_{k_0}}\langle g(x^{k_0}) - g(\hat{x}), g(x^{k_0}) - g(\hat{x}) + g(\hat{x}) - g(y)\rangle\\
&= -\tfrac{1}{\alpha_{k_0}}[\|g(x^{k_0}) - g(\hat{x})\|^2 - \langle g(x^{k_0}) - g(\hat{x}), g(y) - g(\hat{x})\rangle]\\
&= \tfrac{1}{\alpha_{k_0}}[-\|g(x^{k_0}) - g(\hat{x})\|^2 + \langle g(x^{k_0}) - g(\hat{x}), g(y) - g(\hat{x})\rangle]\\
&\leq \tfrac{1}{\alpha_{k_0}}[-\|g(x^{k_0}) - g(\hat{x})\|^2 + \|g(x^{k_0}) - g(\hat{x})\|\|g(y) - g(\hat{x})\|]\\
&= \tfrac{1}{\alpha_{k_0}}\|g(x^{k_0}) - g(\hat{x})\|[\|g(y) - g(\hat{x})\| - \|g(x^{k_0}) - g(\hat{x})\|]\\
&< 0.
\end{aligned}$$

This is a contradiction derived from our assumption. Then $K^* \neq \emptyset$.

Note that the assumption (ii) in Theorem 4.1 can be equivalent to follows:

(ii)* There exists a constant $\rho > 0$ such that for any $x \in H$ with $\|g(x) - g(\hat{x})\| > \rho$, there exist $g(y) \in H$ and $\alpha > 0$ satisfying

$$\|g(y) - g(\hat{x})\| < \|g(x) - g(\hat{x})\|,$$

and

$$\langle f(x), g(x) - g(y)\rangle + \frac{1}{\alpha}(\varphi(g(x)) - \varphi(g(y)) \geq 0.$$

Consequently, we have the following theorem.

Theorem 4.2. Let

(i) the hypotheses in Theorem 3.1 hold;

(ii) for any $x \in H$ with $\|g(x) - g(\hat{x})\| > \rho$ there exist $g(y) \in H$ and $\alpha > 0$ satisfying

$$\|g(y)-g(\hat{x})\| < \|g(x)-g(\hat{x})\| \text{ and } \langle f(x), g(x)-g(y)\rangle+\frac{1}{\alpha}(\varphi(g(x))-\varphi(g(y)) \geq 0.$$

Then $K^* \neq \emptyset$.

This implies the next result.

Theorem 4.3. Let

(i) the hypotheses in Theorem 3.1 hold;

(ii) there exists a nonempty bounded subset D of H such that, for every $g(x) \in H \setminus D$, there exist a $g(y) \in D$ and $\alpha > 0$ satisfying

$$\langle f(x), g(x) - g(y)\rangle + \frac{1}{\alpha}[\varphi(g(x)) - \varphi(g(y))] \geq 0,$$

Then $K^* \neq \emptyset$.

Theorem 4.4. Let

(i) the hypotheses in Theorem 3.1 hold;

(ii) if there exist a vector $\hat{x} \in H$ and $\alpha > 0$ such that the set

$$N(x^*) = \{g(x) \in H : \langle f(x), g(x) - g(x^*)\rangle + \frac{1}{\alpha}[\varphi(g(x)) - \varphi(g(x^*))] < 0\}$$

is bounded and nonempty, then $K^* \neq \emptyset$.

Proof. Take arbitrarily $\widehat{x} \in H$. If F satisfies the assumptions (i) and (ii) in the theorem, then there exists $x^* \in H$ such that the set $N(x^*)$ is bounded. Consequently, there exists $\rho_0 > 0$ such that $N(x^*) \subset B(0, \rho_0)$, where $B(0, \rho_0) = \{g(x) \in H : \|g(x)-g(\hat{x})\| \leq \rho_0\}$. Put $\rho_* = max\{\rho_0, \|g(x^*)-g(\hat{x})\|\}$. For any $g(x) \in \{z \in H : \|z-g(\hat{x})\| > \rho_*\}$, we have $g(x) \notin N(x^*)$, which implies that $\langle f(x), g(x) - g(x^*)\rangle + \frac{1}{\alpha}[\varphi(g(x)) - \varphi(g(x^*))] \geq 0\}$. Hence, the assumption in Theorem 4.2 is satisfied and then $K^* \neq \emptyset$.

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Received: September, 2010