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Sine-Cosine Method for New Coupled ZK System[1]

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Abstract

The sine-cosine method is used to construct exact solutions of a new coupled ZK system. Traveling wave solutions are determined. It is shown that the sine-cosine method provides a powerful mathematical tool for solving a great many nonlinear partial differential equations in mathematical physics.

Keywords: The sine-cosine method; Solitary wave; Periodic wave; Coupled ZK equation

1 Introduction

The KdV equation is a model that governs the one-dimensional propagation of small-amplitude, weakly dispersive waves [4,5]. The nonlinear term uu_x in the KdV equation

$$u_t + auu_x + u_{xxx} = 0, \tag{1.1}$$

causes the steepening of wave form, whereas the dispersion effect term u_{xxx} in the same equation makes the wave form spread. The balance between this weak nonlinear steepening and dispersion gives rise to solitons. The KdV equation is therefore incapable of shock waves [6]. The KdV equation plays an important role in the development of the soliton theory, where nonlinearity and dispersion dominate, while dissipation effects are small enough to be neglected in the lowest order approximation [7,8]. Soliton is a localized wave that has an infinite support or a localized wave with exponential wings. Wadati [9-11] defined soliton as a nonlinear wave that has the following properties:

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(1) A localized wave propagates without change of its properties (shape, velocity, etc.).

(2) Localized waves are stable against mutual collisions and retain their identities. This means that soliton has the property of a particle.

The KdV equation is considered a spatially one-dimensional model. An extensive research work has been done in developing higher dimensional models, particularly those in the (2+1), two spatial and one time, dimensions [12]. The best known two-dimensional generalizations of the KdV equations are the Kadomtsov-Petviashivilli (KP) equation, and the Zakharov-Kuznetsov (ZK) equation. The Zakharov-Kuznetsov (ZK) equation given by

$$u_t + auu_x + b(u_{xx} + u_{yy})_x = 0, \tag{1.2}$$

is investigated in [4-6,13-16] by many distinct approaches.

Recently, a new hierarchy of nonlinear evolution equations was derived by Qin [1] by using a finite-dimensional integrable system. An interesting equation in this hierarchy is a new coupled KdV equation

$$\begin{cases} u_t = \beta u_{xxx} + \alpha(uv)_x + \gamma(vw)_x, \\ v_t = \beta v_{xxx} + \lambda(wu)_x, \\ w_t = \beta w_{xxx} + \lambda(uv)_x, \end{cases} \tag{1.3}$$

where $\alpha,\ \beta,\ \gamma,\ \lambda$ are arbitrary constants. Later, this new coupled equation was investigated by Wu [2], by using matrix transformation and Lax pair. Most recently, in the sense of the KP equation, Wazwaz [3] has extend the new coupled KdV equation to the new coupled ZK equation and studied the new coupled KdV equation and the new coupled ZK equation, by using the Hirota's bilinear method. The physical phenomena for this system was investigated thoroughly in [1,2,3].

Following the sense of the ZK Eq. (1.2) we can extend the coupled KdV system (1.3) to the new coupled ZK system in the form

$$\begin{cases} u_t - \alpha(uv)_x - \gamma(vw)_x - \beta(u_{xx} + u_{yy})_x = 0, \\ v_t - \lambda(wu)_x - \beta(v_{xx} + v_{yy})_x = 0, \\ w_t - \lambda(uv)_x - \beta(w_{xx} + w_{yy})_x = 0. \end{cases} \tag{1.4}$$

The derivation of this system is simply made by following the sense of the ZK equation.

The sine-cosine method will be mainly used to back up our analysis.

2 The method

In what follows, the method will be reviewed briefly. Full details can be found in [17,18] and the references therein.

For the method, we first use the wave variable $\xi = x + y - ct$ to carry a PDE in three independent variables

$$P(u, u_t, u_x, u_y, u_{xx}, u_{yx}, u_{yy}, u_{xxx}, ...) = 0, \tag{2.1}$$

into an ODE

$$Q(u, u', u'', u''', ...) = 0. \tag{2.2}$$

Eq. (2.2) is then integrated as long as all terms contain derivatives where integration constants are considered zeros.

The sine-cosine algorithm admits the use of the ansätz

$$u(x,y,t) = \begin{cases} \lambda \cos^{\beta}(\mu\xi), & |\ \mu\xi\ | < \frac{\pi}{2}, \\ 0, & otherwise, \end{cases} \tag{2.3}$$

or the ansätz

$$u(x,y,t) = \begin{cases} \lambda \sin^{\beta}(\mu\xi), & |\ \mu\xi\ | < \pi, \\ 0, & otherwise, \end{cases} \tag{2.4}$$

where λ, μ, and β are parameters that will be determined.

Substituting (2.3) or (2.4) into the integrated ODE gives a trigonometric equation of $\cos^{\beta}(\mu\xi)$ or $\sin^{\beta}(\mu\xi)$ terms. The parameters β, μ and λ, are then obtained by equating the exponents of each pair of cosine or sine, and by collecting all coefficients of the same power in $\cos^{\beta}(\mu\xi)$ or $\sin^{\beta}(\mu\xi)$, and set it equal to zero.

3 Using the sine-cosine method

Let

$$u(\mu\xi) = \lambda_1 \cos^{a_1}(\mu\xi), v(\mu\xi) = \lambda_2 \cos^{a_2}(\mu\xi), w(\mu\xi) = \lambda_3 \cos^{a_3}(\mu\xi). \tag{3.1}$$

Substituting (3.1) into (1.4) yields

$$\begin{cases} 2\beta[-\mu^2 a_1^2\lambda_1\cos^{a_1}(\mu\xi)+\mu^2\lambda_1 a_1(a_1-1)\cos^{a_1-2}(\mu\xi)] \\ =-c\lambda_1\cos^{a_1}(\mu\xi)-\alpha\lambda_1\lambda_2\cos^{a_1+a_2}(\mu\xi)-\gamma\lambda_2\lambda_3\cos^{a_2+a_3}(\mu\xi), \\ 2\beta[-\mu^2 a_2^2\lambda_2\cos^{a_2}(\mu\xi)+\mu^2\lambda_2 a_2(a_2-1)\cos^{a_2-2}(\mu\xi)] \\ =-c\lambda_2\cos^{a_2}(\mu\xi)-\lambda\lambda_1\lambda_3\cos^{a_1+a_3}(\mu\xi), \\ 2\beta[-\mu^2 a_3^2\lambda_3\cos^{a_3}(\mu\xi)+\mu^2\lambda_3 a_3(a_3-1)\cos^{a_3-2}(\mu\xi)] \\ =-c\lambda_3\cos^{a_3}(\mu\xi)-\lambda\lambda_1\lambda_2\cos^{a_1+a_3}(\mu\xi). \end{cases} \tag{3.2}$$

Eq. (3.2) is satisfied only if the following system of algebraic equations holds:

$$\lambda\gamma\neq 0,\ a_1\neq 1,\ a_2\neq 1,\ a_3\neq 1,\ a_1-2=a_1+a_2=a_2+a_3=a_3-2,\ a_2-2=a_1+a_3,$$

$$2\beta\mu^2a_1^2=2\beta\mu^2a_2^2=2\beta\mu^2a_3^2=c,\ 2\beta\mu^2\lambda_1a_1(a_1-1)=-\alpha\lambda_1\lambda_2-\gamma\lambda_2\lambda_3,$$
$$2\beta\mu^2\lambda_2a_2(a_2-1)=-\lambda\lambda_1\lambda_3,\ 2\beta\mu^2\lambda_3a_3(a_3-1)=-\lambda\lambda_1\lambda_2. \tag{3.3}$$

Solving the system (3.3) give

Case 1 $\quad a_1=a_2=a_3=-2,\ \mu^2=\frac{c}{8\beta},\ c=c,$

$$\lambda_1=\frac{3c}{2\lambda},\ \lambda_2=-\lambda_3=\frac{3c}{4\lambda\gamma}(\alpha\pm\sqrt{\alpha^2+4\lambda\gamma}). \tag{3.4}$$

Case 2 $\quad a_1=a_2=a_3=-2,\ \mu^2=\frac{c}{8\beta},\ c=c,$

$$\lambda_1=-\frac{3c}{2\lambda},\ \lambda_2=\lambda_3=\frac{3c}{4\lambda\gamma}(\alpha\pm\sqrt{\alpha^2+4\lambda\gamma}). \tag{3.5}$$

The results (3.4) and (3.5) can be easily obtained if we also use the sine method (2.4). Combining (3.4) and (3.5) with (2.3) and (2.4), the following triangular periodic solutions for (1.2):

Case 1

$$u_1(x,y,t)=\begin{cases}\frac{3c}{2\lambda}\sec^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct\mid<\sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise,\end{cases}$$

$$v_1(x,y,t)=\begin{cases}\frac{3c}{4\lambda}(\alpha\pm\sqrt{\alpha^2+4\lambda\gamma})\sec^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct\mid<\sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise,\end{cases}$$

$$w_1(x,y,t)=\begin{cases}-\frac{3c}{4\lambda}(\alpha\pm\sqrt{\alpha^2+4\lambda\gamma})\sec^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct\mid<\sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise,\end{cases}$$

$$c\beta > 0,\ \ \alpha^2 + 4\lambda\gamma \geq 0. \tag{3.6}$$

Case 2

$$u_2(x,y,t) = \begin{cases} -\frac{3c}{2\lambda}\sec^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$v_2(x,y,t) = \begin{cases} \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma})\sec^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$w_2(x,y,t) = \begin{cases} \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma})\sec^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$c\beta > 0,\ \ \alpha^2 + 4\lambda\gamma \geq 0. \tag{3.7}$$

Case 3

$$u_3(x,y,t) = \begin{cases} \frac{3c}{2\lambda}\csc^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$v_3(x,y,t) = \begin{cases} \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma})\csc^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$w_3(x,y,t) = \begin{cases} -\frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma})\csc^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$c\beta > 0,\ \ \alpha^2 + 4\lambda\gamma \geq 0. \tag{3.8}$$

Case 4

$$u_(x,y,t) = \begin{cases} -\frac{3c}{2\lambda}\csc^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$v_4(x,y,t) = \begin{cases} \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma})\csc^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$w_4(x,y,t) = \begin{cases} \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma})\csc^2\sqrt{\frac{c}{8\beta}}(x+y-ct), & \mid x+y-ct \mid < \sqrt{\frac{2\beta}{c}}\pi, \\ 0, & otherwise, \end{cases}$$

$$c\beta > 0,\ \ \alpha^2 + 4\lambda\gamma \geq 0. \tag{3.9}$$

However, for $c\beta < 0$, we obtain the following solitary solutions:

Case 5

$$\begin{cases} u_5(x,y,t) = \frac{3c}{2\lambda} sech^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ v_5(x,y,t) = \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) sech^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ w_5(x,y,t) = -\frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) sech^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \quad \alpha^2+4\lambda\gamma \geq 0. \end{cases} \tag{3.10}$$

Case 6

$$\begin{cases} u_6(x,y,t) = -\frac{3c}{2\lambda} sech^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ v_6(x,y,t) = \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) sech^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ w_6(x,y,t) = \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) sech^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \quad \alpha^2+4\lambda\gamma \geq 0. \end{cases} \tag{3.11}$$

Case 7

$$\begin{cases} u_7(x,y,t) = \frac{3c}{2\lambda} csch^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ v_7(x,y,t) = \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) csc^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ w_7(x,y,t) = -\frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) csc^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \quad \alpha^2+4\lambda\gamma \geq 0. \end{cases} \tag{3.12}$$

Case 8

$$\begin{cases} u_8(x,y,t) = -\frac{3c}{2\lambda} csch^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ v_8(x,y,t) = \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) csch^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \\ w_8(x,y,t) = \frac{3c}{4\lambda}(\alpha \pm \sqrt{\alpha^2+4\lambda\gamma}) sech^2\sqrt{\frac{-c}{8\beta}}(x+y-ct), \quad \alpha^2+4\lambda\gamma \geq 0. \end{cases} \tag{3.13}$$

4 Discussion

In this paper, we used the sine-cosine method to study a new coupled ZK equation. As a result, we obtained eight kinds of exact solutions including solitary waves and periodic waves. The method provided solitary wave solutions and triangular periodic wave solutions. Moreover, the obtained results in this work clearly demonstrate the reliability of the methods that were used.

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