Applied Mathematical Sciences, Vol. 5, 2011, no. 21, 1025 - 1036

Stability of Limit Cycle in a Prey Predator System with Pollutant

A. Sen

Department of Mathematics, Jadavpur University
Kolkata-700032, India
ayansen2003@yahoo.co.in

D. Mukhejee

Department of Mathematics, Vivekananda College
Thakurpukur, Kolkata, India

Bibhas Chandra Giri

Department of Mathematics, Jadavpur University
Kolkata-700032, India

Prasenjit Das

The Kidderpore Academy, 35 Ramkamal Street
Kolkata-700 023, India

Abstract

In this article dynamical behavior of a prey predator system in a polluted environment is studied. Stability of boundary and interior equilibrium point are derived. We find out the parameter that generates Hopf-bifurcation in the system. Supercritical and Subcritical nature of the periodic solution are also investigated. Analytical results derived are verified through computer simulations.

Keywords: Prey, Predator, Toxicant, Stability, Hopf- bifurcation

1. Introduction

Recently it is observed that pollution in the environment creates a major problem into the society. So the study of the effect of pollutant on the population is very much essential from the ecological point of view. Hallam and his co-workers [5] initiated the investigation of pollutant effect on population through mathematical modeling. Afterwards many investigations have been done on this issue [3, 4, 5, 6, 7, 9, 10, 11].

Pollutant like crude oil is not focused earlier. So the effect of this type of pollutant population requires thorough investigation. Oil spills from oil tankers on land surface and from distant oil spills, have recognized as a major environmental hazard. Bioremediation is one technique that may be useful to remove spilled oil under certain geographic and climatic conditions. Many compounds in crude oil are environmentally suitable but significant portion are toxigenic or mutagenic. Bioremediation is a technology that helps in converting the toxigenic compounds to nontoxic products without further disruption to the local environment. Biodegradation of crude oil by microbial seems to be natural process by which major portion of crude oil is used as organic carbon source, causing the breakdown of petroleum compounds to lower molecular compounds [2]. In other words, biodegradation of crude oil contaminants can be described as the conversion of chemical compounds by microorganism into energy, cell biomass and biological products. Due to the complex structure of organic compounds in crude oil, the uptake rate by a microbial depends on their distribution in crude oil. It is noted that only one microbial could not biodegrade crude oil thoroughly. In presence of either glycerol or rhamnolipid, both cell growth and biodegradation of crude oil can occur. Psudomonus aeruginosa [1] can convert crude oil into both cell mass and bio-surfactant. Here oil can be thought of nutrient for the micro-organism.

The main thrust of this paper is to study the biodegradation of oil (pollutant) by Pseudomonas aeruginosa (predator) which also consume prey (fish population) through mathematical modeling. In this proposed model, toxicant is harmful to prey populations while the predator populations can take it up with no deleterious effect. From ecological point of view, one can think that the prey is detoxifying the environment. The question of survival of detoxifying population is very much important from the biological point of view also.

This paper is organized as follows. In section 2, a prey predator model with the effect of toxicant is described. The stability of equilibrium point is described in section 3. In section 4, the Hopf bifurcation of the prey predator system is studied. In section 5, stability of the bifurcating periodic solution is analyzed. The theoretical results derived for the prey-predator system are verified by a numerical example in section 6. Finally, the paper is concluded in section 7.

2. The Model

Let x(t), y(t) and p(t) represent respectively the concentration of the prey in the total population, the concentration of the predator in the total population biomass and the concentration of toxicant in the environment at time t. The mass balance arguments drive to the following model:

$$\frac{dx}{dt} = x\left[r\left(1-\frac{x}{k}\right) - \frac{y}{1+x} - ap\right] \equiv E_1 \quad (1)$$

$$\frac{dy}{dt} = y\left[-d + \frac{\delta p}{c+p} + \frac{mx}{1+x}\right] \equiv E_2 \quad (2)$$

$$\frac{dp}{dt} = p^0 - pD - \frac{\delta p y}{c+p} \equiv E_3 \quad (3)$$

where $x_0(t) \geq 0$, $y_0(t) \geq 0$, $p_0(t) \geq 0$. Here parameter p^0 is toxicant input rate. The parameters δ and r represent the toxicant depletion rate in the environment and intrinsic growth rate of prey population respectively. k is the environmental carrying capacity of the prey population whereas the parameter a is the rate at which prey is diminished due to toxicant. The parameter d is the natural death rate of predator and c is the saturation constant. Further, the parameter m denotes conversion efficiency of predator. Hence predator detoxifies the pollutant.

Specific examples illustrate the above situation. Oil spills from oil tankers on land surface and from distant oil spills, have recognized as a major environmental hazard. This spilled oil not only destroy the habitats of aquatic animals and fish, but also create damage to the environment. It has been observed that *Pseudomomous aeruginosa* detoxify oil [Beal and Beatts 2000,Madigan et al. 2003]. This micro-organism produces a glycolipid emulsifier that reduces the surface tension of an oil interface and thus helps in removal of oil from water. Paramecium predates on the above bacteria.

Carbon monoxide is one of the important gaseous air pollutants. Recently, the leaves of several plants such as *Phaseolus vulgaris, Coleus blumei, Daucus carota* and others have been found to be capable of fixing Carbon monoxide [Subrahmanyam and Sambamurty 2000]. Some herbivore predates on the above plant populations. It may be noted that mammalian herbivore is affected by carbon monoxide such as cattle.

3. Stability of Equilibrium Points

In this section, we find the equilibrium points of the proposed model and analyze their behavior. The equilibrium points of system (1-3) are given by $E^{00}(0,0,\frac{p^0}{D})$,$E^{11}(k(1-\frac{ap^0}{rD}),0,\frac{p^0}{D})$,$E^{22}(0,\frac{1}{d}(p^0-\frac{cdD}{\delta-d}),\frac{cd}{\delta-d})$ and $E(x^*,y^*,p^*)$ where x^*,y^* and p^* can be obtained from the following equations:

$$x^*\left[r\left(1-\frac{x^*}{k}\right) - \frac{y^*}{1+x^*} - ap^*\right] = 0 \quad (4)$$

$$y^*\left[-d+\frac{\delta p^*}{c+p^*}+\frac{mx^*}{1+x^*}\right]=0 \quad (5)$$

$$p^0-p^*D-\frac{\delta p^*y^*}{c+p^*}=0 \quad (6)$$

Lemma 1. The solutions of system (1-3) which start in R^3_+ are bounded.
Proof: We define a function W = x+y+p.
The time derivative along a solution of (1),(2),(3) is
$\frac{dW}{dt}=\frac{dx}{dt}+\frac{dy}{dt}+\frac{dp}{dt}=x\left[r\left(1-\frac{x}{k}\right)-\frac{y}{1+x}\right]+y\left[-d+\frac{mx}{1+x}-ap\right]+p^0-pD-\frac{\delta px}{c+p}$
For each $\lambda>0$ the following inequality is fulfilled.
$\frac{dW}{dt}+\lambda W=x\left[r\left(1-\frac{x}{k}\right)-\frac{y}{1+x}\right]+y\left[-d+\frac{mx}{1+x}-ap\right]+p^0-pD+\frac{\delta px}{c+p}+\lambda(x+y+p)$
If we choose $\lambda<min[d,D]$, then right hand side is bounded for all $(x,y,p)\in R^3_+$. Thus we find a $M>0$ with $\frac{dW}{dt}+\lambda W\leq M$.
Applying a theorem on differential inequality [3], the differential inequality $\frac{dW}{dt}+\lambda W\leq M$, with initial condition at $t=0$ given by $W(0)$, gives $0\leq W(t)\leq W(0)e^{-\lambda t}+\frac{M}{\lambda}\left(1-e^{-\lambda t}\right)$ and for $t\longmapsto\infty$. Hence system (1),(2),(3) is bounded.

It is easy to see that the equilibrium point $E^{11}(k(1-\frac{ap^0}{rD}),0,\frac{p^0}{D})$ exists if $rD>ap^0$ holds and the equilibrium point $E^{22}(0,\frac{1}{d}(p^0-\frac{cdD}{\delta-d}),\frac{cd}{\delta-d})$ exist if $p^0(\delta-d)>cdD$ and $\delta>d$ hold simultaneously. To find out positive equilibrium point p^* we use the result of the cubic equation [12]. At the point $E(x^*,y^*,p^*)$ where $x^*=\frac{cd+p^*(d-\delta)}{(m-d)c+p^*(m-d+\delta)}$,$y^*=\frac{(p^0-p^*D)(c+p^*)}{\delta p^*}$ and p^* will be found out from the equation $a_0p^{*3}+a_1p^{*2}+a_2p^{*1}+a_3p^*=0$ where
$a_0=-k(m-d+\delta)[D(m-d+\delta)-m\delta]$
$a_1=-[2kDc(m-d)(m-d+\delta)+mk\delta[r(m-d+\delta)-ac(m-d)]-rm\delta(d-\delta)-kp^0(m-d+\delta)^2]$
$a_2=2kp^0c(m-d)(m-d+\delta)+rmc\delta[d-k(m-d)]-kc^2D(m-d)^2$
$a_3=kp^0c^2(m-d)^2$
Here $a_0<0$ when $(m-d+\delta)>0$ and $[D(m-d+\delta)-m\delta]<0$ i.e. when $d<m+\delta<d+\frac{m\delta}{D}$ and $d>\delta$ hold. We see that a_3 is always positive. Therefore the equation $a_0p^{*3}+a_1p^{*2}+a_2p^{*1}+a_3p^*=0$ has at least one positive root. y^* will be positive when $p^0>p^*D$. x^* will be positive when $m(c+p^*)>cd+p^*(d-\delta)$ holds. Hence, the equilibrium point E exists when $d<m+\delta<d+\frac{m\delta}{D}$,$d>\delta$,$p^0>p^*D$ and $m(c+p^*)>cd+p^*(d-\delta)$.
We now analyze the nature of the every equilibrium point. The Jacobian matrix at $E^{00}(0,0,\frac{p^0}{D})$ of system (1-3) is given by

$$J(E^{00})=\begin{pmatrix} r-\frac{ap^0}{D} & 0 & 0\\ 0 & -d+\frac{\delta p^0}{cD+p^0} & 0\\ 0 & -\frac{\delta p^0}{cD+p^0} & -D\end{pmatrix}$$

The eigenvalues of the above Jacobian matrix are $-D$, $r-\frac{ap^0}{D}$ and $-d+\frac{\delta p^0}{cD+p^0}$. Hence the system (1-3) is stable at the equilibrium point E^{00} when $rD<ap^0$ and $d(cD+p^0)>\delta p^0$and otherwise it is unstable.

Now, the Jacobian matrix at the equilibrium point $E^{11}(k(1-\frac{ap^0}{rD}),0,\frac{p^0}{D})$ of the system (1-3) is

$$J(E^{11}) = \begin{pmatrix} r\left(1-\frac{x^/}{k}\right)-\frac{rx^/}{k}-ap^/ & -\frac{x^/}{1+x^/} & -ax^/ \\ 0 & -d+\frac{\delta p^/}{c+p^/}+\frac{mx^/}{1+x^/} & 0 \\ 0 & -\frac{\delta p^0}{cD+p^0} & -D \end{pmatrix}$$

where $x^/ = k(r-\frac{ap^0}{rD})$ and $p^/ = \frac{p^0}{D}$.The eigenvalues of $J(E^{11})$ are $-D$, $-r+\frac{ap^0}{D}$ and $-d+\frac{\delta p^0}{cD+p^0}+\frac{mk(rD-ap^0)}{rD+(rD-ap^0)}$. This shows that the system (1-3) is stable at E^{11} if $rD > ap^0$ and $d > \frac{\delta p^0}{cD+p^0}+\frac{mk(rD-ap^0)}{rD+(rD-ap^0)}$ and unstable otherwise.

Similarly, the Jacobian matrix at the equilibrium point $E^{22}(0,\frac{1}{d}(p^0-\frac{cdD}{\delta-d}),\frac{cd}{\delta-d})$ of system (1-3) is

$$J(E^{22}) = \begin{pmatrix} r-y^{//}-ap^{//} & 0 & 0 \\ my^{//} & -d+\frac{\delta p^{//}}{c+p^{//}} & \frac{\delta y^{//}c}{(c+p^{//})^2} \\ 0 & -\frac{\delta p^{//}}{c+p^{//}} & -D-\frac{\delta y^{//}}{c+p^{//}}+\frac{\delta y^{//}p^{//}}{(c+p^{//})^2} \end{pmatrix}$$

where $y^{//} = \frac{1}{d}(p^0-\frac{cdD}{\delta-d})$ and $p^{//} = \frac{cd}{\delta-d}$. The eigenvalues $J(E^{22})$ are $r-\frac{p^0}{d}+\frac{c(D-ad)}{\delta-d}$ and $\frac{1}{2}[-\frac{p^0(\delta-d)^2+cd^2D}{cd\delta}+\sqrt{(\frac{p^0(\delta-d)^2+cd^2D}{cd\delta})^2-4\frac{p^0(\delta-d)^2-cdD(\delta-d)}{cd\delta}}]$

System (1-3) is stable at E^{22} if the relation $r < \frac{p^0}{d}-\frac{c(D-ad)}{\delta-d}$,$p^0(\delta-d) > cdD$ and $\delta > d$ hold.

From the above discussion, we state the stability criteria in the following theorem.

Theorem 1. The system (1-3) is

(i) stable at the equilibrium point E^{00} if $rD < ap^0$ and $\delta p^0 < d(cD+p^0)$ and unstable otherwise.

(ii) stable at the equilibrium point E^{11} if $rD > ap^0$ and $d > \frac{\delta p^0}{cD+p^0}+\frac{mk(rD-ap^0)}{rD+(rD-ap^0)}$ and unstable otherwise.

(iii)stable at the equilibrium point E^{22} if $r < \frac{p^0}{d}-\frac{c(D-ad)}{\delta-d}$,$p^0(\delta-d) > cdD$ and $\delta > d$ hold and unstable otherwise.

4. Analysis of Hopf bifurcation

In this section, we find the condition for which system (1-3) gives Hopf-bifurcation. Let us now consider the positive equilibrium point $E(x^*,y^*,p^*)$ of system (1-3) where x^*, y^* and p^* are given in (4-6).

The Jacobian matrix at the equilibrium point $E(x^*,y^*,p^*)$ is given by

$$J = \begin{pmatrix} x^*(-\frac{r}{k}+\frac{y^*}{(1+x^*)^2}) & -\frac{x^*}{1+x^*} & -ax^* \\ \frac{my^*}{(1+x^*)^2} & 0 & \frac{\delta y^* c}{(c+p^*)^2} \\ 0 & \frac{-\delta p^*}{c+p^*} & -D-\frac{\delta cy^*}{(c+p^*)^2} \end{pmatrix} \tag{7}$$

Eigenvalues of the above Jacobian matrix are the roots of the equation

$$\lambda^3 + A\lambda^2 + B + F = 0 \tag{8}$$

where

$$A = D + \frac{\delta c y^*}{(c+p^*)^2} + \frac{r x^*}{k} - \frac{x^* y^*}{(1+x^*)^2} \tag{9}$$

$$B = \frac{c y^* \delta^2 p^*}{(c+p^*)^3} - x^*[-\frac{r}{k} + \frac{y^*}{(1+x^*)^2}][D + \frac{\delta c y^*}{(c+p^*)^2}] + \frac{m x^* y^*}{(1+x^*)^3} \tag{10}$$

$$F = [D + \frac{\delta c y^*}{(c+p^*)^2}]\frac{m x^* y^*}{(1+x^*)^3} - \frac{m x^* y^* p^* \delta a}{(1+x^*)^2(c+p^*)} - x^*[-\frac{r}{k} + \frac{y^*}{(1+x^*)^2}]\frac{c y^* \delta^2 p^*}{(c+p^*)^3} \tag{11}$$

Let the roots of the equation (8) be $\sigma$and $\alpha \pm i\beta$ .Then we have
$2\alpha + \sigma = -A$
$\alpha^2 + \beta^2 + 2\alpha\sigma = B$
$(\alpha^2 + \beta^2)\sigma = -F$
System (1-3) is stable when $\alpha < 0$ and $\sigma < 0$; otherwise, it is unstable. On the other hand, system (1-3) gives Hopf bifurcation when $\alpha = 0$.
For $\alpha = 0$, we have

$$AB = F. \tag{12}$$

Moreover,

$$\frac{d\alpha}{dm} = \frac{\frac{dF}{dm} - (B\frac{dA}{dm} + A\frac{B}{dm})}{2(A^2 + B)} \neq 0 \tag{13}$$

Hence system (1-3) gives Hopf bifurcation when the above two conditions (12) and (13) hold simultaneously.
Theorem 2. At the equilibrium point $E(x^*, y^*, p^*)$ system (1-3) is
(i) stable when $\alpha < 0$ and $\sigma < 0$
(ii) unstable when any one or both of α and σ are positive and (iii) give Hopf bifurcation when $\alpha = 0$, i.e. $AB = F$ and $\frac{d\alpha}{dm} = \frac{\frac{dF}{dm} - (B\frac{dA}{dm} + A\frac{B}{dm})}{2(A^2+B)} \neq 0$ where $\sigma$and $\alpha \pm i\beta$ are the eigenvalues of the Jacobian matrix (7); A,B and F are given in (9) , (10) and (11) respectively.

5. Stability of the Bifurcating Periodic Solution

We now analyze the behavior of the periodic solution at the equilibrium point E and also find whether it is supercritical or subcritical in nature at that equilibrium point E. First of all we find the eigenvectors corresponding to different eigenvalues of the Jacobian matrix (7).
The eigenvector corresponding to the eigenvalue σ is $w[\frac{\sigma^2 - \sigma a_{33} - a_{23}a_{32}}{a_{21}a_{32}} \quad \frac{\sigma - a_{33}}{a_{32}} \quad 1]^T$ where $a_{11} = x^*(-\frac{r}{k} + \frac{y^*}{(1+x^*)^2}), a_{12} = -\frac{x^*}{1+x^*}, a_{13} = -ax^*, a_{21} = \frac{my^*}{(1+x^*)^2}, a_{22} = 0, a_{23} = \frac{\delta y^* c}{(c+p^*)^2}, a_{31} =$

$0, a_{32} = \frac{-\delta p^*}{c+p^*}, a_{33} = -D - \frac{\delta c y^*}{(c+p^*)^2}, w \in R$
We also know that the eigenvector corresponding to the eigenvalue $\alpha \pm i\beta$ is
$w[\frac{(\alpha \pm i\beta)^2 - (\alpha \pm i\beta)a_{33} - a_{23}a_{32}}{a_{21}a_{32}} \quad \frac{\alpha \pm i\beta - a_{33}}{a_{32}} \quad 1]^T, w \in R$

We know from **theorem 2 (iii)** that system (1-3) give Hopf bifurcation at E when $\alpha = 0$, i.e. $AB = F$ and $\frac{d\alpha}{dm} = \frac{\frac{dF}{dm} - (B\frac{dA}{dm} + A\frac{B}{dm})}{2(A^2+B)} \neq 0$ hold simultaneously. Here the method we use is based on the normal form theory [8]. Sine system (1-3) give Hopf bifurcation at E, to find the nature of periodic solution at the equilibrium point E we first rename the parameters of system (1-3) as $r^/$, $k^/$, $a^/$, $b^/$, $\delta^/$,$c^/$, $p^{0/}$ and $m^/$ respectively. Since system (1-3) has Hopf bifurcation at E then the eigenvectors corresponding to different eigenvalues of the Jacobian matrix (7) are given below.
The eigenvector corresponding to the eigenvalue σ is $w[\frac{\sigma^2 - \sigma a_{33} - a_{23}a_{32}}{a_{21}a_{32}} \quad \frac{\sigma - a_{33}}{a_{32}} \quad 1]^T$ where $a_{11} = x^{*/}(-\frac{r^/}{k^/} + \frac{y^{*/}}{(1+x^{*/})^2}), a_{12} = -\frac{x^{*/}}{1+x^{*/}}, a_{13} = -a^/ x^{*/}, a_{21} = \frac{m^/ y^{*/}}{(1+x^{*/})^2}, a_{22} = 0 a_{23} = \frac{\delta^/ y^{*/} c^/}{(c^/ + p^{*/})^2}, a_{31} = 0, a_{32} = \frac{-\delta^/ p^{*/}}{c^/ + p^{*/}}, a_{33} = -D^/ - \frac{\delta^/ c^/ y^{*/}}{(c^/ + p^{*/})^2}, w \in R$
and also know that the eigenvector corresponding to the eigenvalue $i\beta$ is
$w[\frac{(i\beta)^2 - (i\beta)a_{33} - a_{23}a_{32}}{a_{21}a_{32}} \quad \frac{i\beta - a_{33}}{a_{32}} \quad 1]^T, w \in R$
We now use the transformation
$x = x^* + b_{11}x_1 + b_{12}y_1 + b_{13}p_1$
$y = y^* + b_{21}x_1 + b_{22}y_1 + b_{23}p_1$
$p = p^* + b_{31}x_1 + b_{32}y_1 + b_{33}p_1$
Using the above transformation system (1-3) is then reduced to

$$\frac{dx_1}{dt} = \frac{E_3 B_2 - E_1 b_{22} + E_2 b_{12}}{B_2 - B_1} \equiv G^1 \quad (14)$$

$$\frac{dy_1}{dt} = \frac{(B_2 - B_1 + b_{11}b_{22} - b_{13}b_{22})E_1 + (b_{12}b_{13} - b_{11}b_{12})E_2 + (b_{13}B_1 - b_{11}B_2)E_3}{b_{12}(B_2 - B_1)} \equiv G^2 \quad (15)$$

$$\frac{dp_1}{dt} = \frac{E_1 b_{22} - E_2 b_{12} - E_3 B_1}{B_2 - B_1} \equiv G^3 \quad (16)$$

where
$b_{11} = -\frac{\beta^2 + a_{23}a_{32}}{a_{21}a_{32}}, b_{12} = \frac{\beta a_{33}}{a_{21}a_{32}}, b_{13} = \frac{\sigma^2 - \sigma a_{33} - a_{23}a_{32}}{a_{21}a_{32}}, b_{21} = -\frac{a_{33}}{a_{32}}, b_{22} = -\frac{\beta}{a_{32}}, b_{23} = \frac{\sigma - a_{33}}{a_{32}}, b_{31} = 1, b_{32} = 0, b_{33} = 1, B_1 = b_{11}b_{22} - b_{21}b_{12}, B_2 = b_{13}b_{22} - b_{23}b_{12}$
$E_1 = (x^* + b_{11}x_1 + b_{12}y_1 + b_{13}p_1) \times$
$\left[r^/ \left(1 - \frac{x^* + b_{11}x_1 + b_{12}y_1 + b_{13}p_1}{k^/}\right) - \frac{y^* + b_{21}x_1 + b_{22}y_1 + b_{23}p_1}{1 + x^* + b_{11}x_1 + b_{12}y_1 + b_{13}p_1} - a^/(p^* + b_{31}x_1 + b_{32}y_1 + b_{33}p_1)\right]$,
$E_2 = (y^* + b_{21}x_1 + b_{22}y_1 + b_{23}p_1)[-d^/ + \frac{\delta^/(p^* + b_{31}x_1 + b_{32}y_1 + b_{33}p_1)}{c^/ + p^* + b_{31}x_1 + b_{32}y_1 + b_{33}p_1} + \frac{m^/(x^* + b_{11}x_1 + b_{12}y_1 + b_{13}p_1)}{1 + x^* + b_{11}x_1 + b_{12}y_1 + b_{13}p_1}]$,
$E_3 = p^0 - (p^* + b_{31}x_1 + b_{32}y_1 + b_{33}p_1)D^/ - \frac{\delta^/(p^* + b_{31}x_1 + b_{32}y_1 + b_{33}p_1)(y^* + b_{21}x_1 + b_{22}y_1 + b_{23}p_1)}{c^/ + p^* + b_{31}x_1 + b_{32}y_1 + b_{33}p_1}$
Obliviously, (0, 0, 0) are the equilibrium point of new system (14-16). Then the Jacobian matrix of reduced system (14-16) is

$$J = \begin{pmatrix} \frac{\partial G^1}{\partial x_1} & \frac{\partial G^1}{\partial y_1} & \frac{\partial G^1}{\partial p_1} \\ \frac{\partial G^2}{\partial x_1} & \frac{\partial G^2}{\partial y_1} & \frac{\partial G^2}{\partial p_1} \\ \frac{\partial G^3}{\partial x_1} & \frac{\partial G^3}{\partial y_1} & \frac{\partial G^3}{\partial p_1} \end{pmatrix} \quad (17)$$

Here (0, 0, 0) is the equilibrium point of system (14-16) and
$\frac{\partial G^1}{\partial x_1} = \frac{\partial G^2}{\partial y_1} = \frac{\partial G^1}{\partial p_1} = \frac{\partial G^3}{\partial x_1} = \frac{\partial G^3}{\partial y_1} = \frac{\partial G^2}{\partial p_1} = 0$ and $-\frac{\partial G^1}{\partial y_1} = \frac{\partial G^2}{\partial x_1} = w_0$ and $D_1 = \frac{\partial G^3}{\partial p_1}$
We now calculate the followings quantities $g_{11}, g_{02}, g_{20}, G_{101}, G_{110}, G_{21}, h_{11}, h_{20}, w_{20}, w_{11} and g_{21}$ involving system parameters of system (14-16).The following quantities are evaluated at the point (0, 0, 0).

$g_{11} = \frac{1}{4}[\frac{\partial^2 G^1}{\partial x_1^2} + \frac{\partial^2 G^2}{\partial y_1^2} + i(\frac{\partial^2 G^2}{\partial x_1^2} + \frac{\partial^2 G^1}{\partial y_1^2})]$
$= \frac{1}{4}[\frac{B_2M_{13}-b_{22}M_{11}+b_{12}M_{12}}{B_2-B_1} + i\frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{11}+(b_{12}b_{13}-b_{11}b_{12})M_{12}+(b_{13}B_1-b_{11}B_2)M_{13}}{b_{12}(B_2-B_1)}]$
$g_{02} = \frac{1}{4}[\frac{\partial^2 G^1}{\partial x_1^2} - \frac{\partial^2 G^2}{\partial y_1^2} - 2\frac{\partial^2 G^2}{\partial x_1 \partial y_1} + i(\frac{\partial^2 G^2}{\partial x_1^2} - \frac{\partial^2 G^1}{\partial y_1^2}) + 2\frac{\partial^2 G^1}{\partial x_1 \partial y_1}]$
$= \frac{1}{4}[\frac{B_2M_{23}-b_{22}M_{21}+b_{12}M_{22}}{B_2-B_1} - 2\frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{24}+(b_{12}b_{13}-b_{11}b_{12})M_{25}+(b_{13}B_1-b_{11}B_2)M_{26}}{b_{12}(B_2-B_1)}]$
$+ \frac{i}{4}[2\frac{B_2M_{26}-b_{22}M_{24}+b_{12}M_{25}}{B_2-B_1} + 2\frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{21}+(b_{12}b_{13}-b_{11}b_{12})M_{22}+(b_{13}B_1-b_{11}B_2)M_{23}}{b_{12}(B_2-B_1)}]$
$g_{20} = \frac{1}{4}[\frac{\partial^2 G^1}{\partial x_1^2} - \frac{\partial^2 G^2}{\partial y_1^2} + 2\frac{\partial^2 G^2}{\partial x_1 \partial y_1} + i(\frac{\partial^2 G^2}{\partial x_1^2} - \frac{\partial^2 G^1}{\partial y_1^2}) - 2\frac{\partial^2 G^1}{\partial x_1 \partial y_1}]$
$= \frac{1}{4}[\frac{B_2M_{23}-b_{22}M_{21}+b_{12}M_{22}}{B_2-B_1} + 2\frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{24}+(b_{12}b_{13}-b_{11}b_{12})M_{25}+(b_{13}B_1-b_{11}B_2)M_{26}}{b_{12}(B_2-B_1)}]$
$+ \frac{i}{4}[-2\frac{B_2M_{26}-b_{22}M_{24}+b_{12}M_{25}}{B_2-B_1} - 2\frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{21}+(b_{12}b_{13}-b_{11}b_{12})M_{22}+(b_{13}B_1-b_{11}B_2)M_{23}}{b_{12}(B_2-B_1)}]$
$G_{21} = \frac{1}{8}[\frac{\partial^3 G^1}{\partial x_1^3} + \frac{\partial^3 G^1}{\partial x_1 \partial y_1^2} + \frac{\partial^3 G^2}{\partial^2 x_1 \partial y_1} + \frac{\partial^3 G^2}{\partial y_1^3}] + \frac{i}{8}[\frac{\partial^3 G^2}{\partial x_1^3} + \frac{\partial^3 G^2}{\partial x_1 \partial y_1^2} - \frac{\partial^3 G^1}{\partial^2 x_1 \partial y_1} - \frac{\partial^3 G^1}{\partial y_1^3}]$
$= \frac{1}{8}[\frac{B_2M_{41}-b_{22}M_{45}+b_{12}M_{43}}{B_2-B_1} + \frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{46}+(b_{12}b_{13}-b_{11}b_{12})M_{43}+(b_{13}B_1-b_{11}B_2)M_{42}}{b_{12}(B_2-B_1)}]$
$+ \frac{i}{8}[-\frac{B_2M_{42}-b_{22}M_{46}+b_{12}M_{43}}{B_2-B_1} + \frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{45}+(b_{12}b_{13}-b_{11}b_{12})M_{43}+(b_{13}B_1-b_{11}B_2)M_{41}}{b_{12}(B_2-B_1)}]$
$w_0 = -\frac{\partial G^1}{\partial y_1} = \frac{1}{B_2-B_1}[-\frac{\delta/p^* b_{22}B_2}{c/+p^*} + b_{22}x^*(\frac{r/b_{12}}{k/} + \frac{b_{22}}{1+x^*} - \frac{b_{22}y^*}{(1+x^*)^2}) + 2b_{12}(\frac{b_{22}m/b_{12}}{(1+x^*)} - \frac{m/b_{12}^2 y^*}{(1+x^*)^3)^3}]$
$h_{11} = \frac{1}{4}[\frac{\partial^2 G^3}{\partial x_1^2} + \frac{\partial^2 G^3}{\partial y_1^2}] = \frac{-B_1M_{13}-b_{12}M_{12}+b_{22}M_{11}}{4(B_2-B_1)}$
$h_{20} = \frac{1}{4}[\frac{\partial^2 G^3}{\partial x_1^2} - \frac{\partial^2 G^3}{\partial y_1^2} - 2i\frac{\partial^2 G^3}{\partial x_1 \partial y_1}] = \frac{1}{4(B_2-B_1)}[(b_{22}M_{21} - b_{12}M_{22} - B_1M_{23}) - 2i(b_{22}M_{24} - b_{12}M_{25} - B_1M_{26})]$
$D_1 = \frac{1}{(B_2-B_1)}[(b_{22}M_{31} - b_{12}M_{32} - B_1M_{33})]$
$D_1w_{11} = -h_{11}$
$(D_1 - 2iw_0)w_{20} = -h_{20}$
$G_{110} = \frac{1}{2}[\frac{\partial^2 G^1}{\partial x_1 \partial p_1} + \frac{\partial^2 G^1}{\partial y_1 \partial p_1} + i(\frac{\partial^2 G^2}{\partial x_1 \partial p_1} - \frac{\partial^2 G^2}{\partial y_1 \partial p_1})] =$
$\frac{1}{2}[\frac{B_2M_{55}-b_{22}M_{51}+b_{12}M_{53}}{B_2-B_1} + \frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{52}+(b_{12}b_{13}-b_{11}b_{12})M_{54}+(b_{13}B_1-b_{11}B_2)M_{56}}{b_{12}(B_2-B_1)}]$
$+ \frac{i}{2}[-\frac{B_2M_{56}-b_{22}M_{52}+b_{12}M_{54}}{B_2-B_1} + \frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{51}+(b_{12}b_{13}-b_{11}b_{12})M_{53}+(b_{13}B_1-b_{11}B_2)M_{55}}{b_{12}(B_2-B_1)}]$
$G_{101} = \frac{1}{2}[\frac{\partial^2 G^1}{\partial x_1 \partial p_1} - \frac{\partial^2 G^1}{\partial y_1 \partial p_1} + i(\frac{\partial^2 G^2}{\partial x_1 \partial p_1} + \frac{\partial^2 G^2}{\partial y_1 \partial p_1})] =$
$\frac{1}{2}[\frac{B_2M_{55}-b_{22}M_{51}+b_{12}M_{53}}{B_2-B_1} - \frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{52}+(b_{12}b_{13}-b_{11}b_{12})M_{54}+(b_{13}B_1-b_{11}B_2)M_{56}}{b_{12}(B_2-B_1)}]$
$+ \frac{i}{2}[\frac{B_2M_{56}-b_{22}M_{52}+b_{12}M_{54}}{B_2-B_1} + \frac{(B_2-B_1+b_{11}b_{22}-b_{13}b_{22})M_{51}+(b_{12}b_{13}-b_{11}b_{12})M_{53}+(b_{13}B_1-b_{11}B_2)M_{55}}{b_{12}(B_2-B_1)}]$
$g_{21} = G_{21} + 2G_{110}w_{11} + G_{101}w_{20}$

$M_{11} = 2b_{11}[-\frac{r^{/}b_{11}}{k^{/}} - \frac{b_{21}}{1+x^*} + \frac{b_{11}y^*}{(1+x^*)^2} - a^{/}] + 2x^*[\frac{b_{21}b_{11}}{(1+x^*)^2} - \frac{b_{11}^2y^*}{(1+x^*)^3}] + 2b_{12}[-\frac{r^{/}b_{12}}{k^{/}} - \frac{b_{22}}{1+x^*} + \frac{b_{12}y^*}{(1+x^*)^2}] + x^*[\frac{2b_{22}b_{12}}{(1+x^*)^2} - \frac{2b_{11}^2y^*}{(1+x^*)^3}]$

$M_{12} = 2b_{21}[\frac{c^{/}\delta^{/}}{(c^{/}+p^*)^2} + \frac{m^{/}b_{11}}{(1+x^*)^2}] + y^*[-\frac{2c^{/}\delta^{/}}{(c^{/}+p^*)} - 2m^{/}\frac{b_{11}^2}{(1+x^*)^2}] + \frac{2b_{22}m^{/}b_{12}}{(1+x^*)^2} - \frac{2m^{/}b_{12}^2y^*}{(1+x^*)^3}$

$M_{13} = 2\frac{\delta^{/}(y^*+b_{21}p^*)}{(c^{/}+p^*)^2} - 2\frac{\delta^{/}b_{11}}{c^{/}+p^*} - 2\frac{\delta^{/}p^*y^*}{(c^{/}+p^*)^3}$

$M_{21} = 2b_{11}[-\frac{r^{/}b_{11}}{k^{/}} - \frac{b_{21}}{1+x^*} + \frac{b_{11}y^*}{(1+x^*)^2} - a^{/}] + 2x^*[\frac{b_{21}b_{11}}{(1+x^*)^2} - \frac{b_{11}^2y^*}{(1+x^*)^3}] - 2b_{12}[-\frac{r^{/}b_{12}}{k^{/}} - \frac{b_{22}}{1+x^*} + \frac{b_{12}y^*}{(1+x^*)^2}] - x^*[\frac{2b_{22}b_{12}}{(1+x^*)^2} - \frac{2b_{11}^2y^*}{(1+x^*)^3}]$

$M_{22} = 2b_{21}[\frac{c^{/}\delta^{/}}{(c^{/}+p^*)^2} + \frac{m^{/}b_{11}}{(1+x^*)^2}] + y^*[-\frac{2c^{/}\delta^{/}}{(c^{/}+p^*)^3} - 2m^{/}\frac{b_{11}^2}{(1+x^*)^3}] - \frac{2b_{22}m^{/}b_{12}}{(1+x^*)^2} + \frac{2m^{/}b_{12}^2y^*}{(1+x^*)^3}$

$M_{23} = 2\frac{\delta^{/}(y^*+b_{21}p^*)}{(c^{/}+p^*)^2} - 2\frac{\delta^{/}b_{21}}{c^{/}+p^*} - 2\frac{\delta^{/}p^*y^*}{(c^{/}+p^*)^3}$

$M_{24} = b_{12}[-\frac{r^{/}b_{11}}{k^{/}} - \frac{b_{21}}{1+x^*} + \frac{b_{11}y^*}{(1+x^*)^2} - a^{/}] + b_{11}[-\frac{r^{/}b_{12}}{k^{/}} - \frac{b_{22}}{1+x^*} + \frac{b_{12}y^*}{(1+x^*)^2}] + x^*[\frac{b_{22}b_{11}+b_{12}b_{21}}{(1+x^*)^2} - 2\frac{b_{12}b_{11}y^*}{(1+x^*)^3}]$

$M_{25} = b_{22}[\frac{c^{/}\delta^{/}}{(c^{/}+p^*)^2} + \frac{m^{/}b_{11}}{(1+x^*)^2}] + m^{/}\frac{b_{12}b_{21}}{(1+x^*)^2} - 2\frac{y^*m^{/}b_{12}}{(1+x^*)^3}$

$M_{26} = -\frac{c^{/}\delta^{/}b_{22}}{(c^{/}+p^*)^2}$

$M_{31} = x^*[-\frac{r^{/}b_{13}}{k^{/}} - \frac{b_{23}}{1+x^*} + \frac{b_{13}y^*}{(1+x^*)^2} - a^{/}]$

$M_{32} = y^*[\frac{c^{/}\delta^{/}}{(c^{/}+p^*)^2} + \frac{m^{/}b_{13}}{(1+x^*)^2}]$

$M_{33} = -D^{/} - \frac{\delta^{/}(y^*+b_{23}p^*)}{c^{/}+p^*} + \frac{\delta^{/}p^*y^*}{(c^{/}+p^*)^2}$

$M_{41} = -6\frac{\delta^{/}(y^*+b_{21}p^*)}{(c^{/}+p^*)^3} + 6\frac{\delta^{/}b_{21}}{(c^{/}+p^*)^2} + 6\frac{\delta^{/}p^*y^*}{(c^{/}+p^*)^4}$

$M_{42} = 2\frac{\delta^{/}c^{/}b_{22}}{(c^{/}+p^*)^3}$

$M_{43} = 3b_{21}[-\frac{2\delta^{/}c^{/}}{(c^{/}+p^*)^3} - \frac{2m^{/}b_{11}^2}{(1+x^*)^3}] + 6y^*[\frac{\delta^{/}c^{/}}{(c^{/}+p^*)^4} + \frac{m^{/}b_{11}^3}{(1+x^*)^4}] - \frac{4m^{/}b_{22}b_{12}b_{11}}{(1+x^*)^3} - \frac{2m^{/}b_{21}b_{12}^2}{(1+x^*)^3} + \frac{6m^{/}b_{12}^2b_{11}y^*}{(1+x^*)^4}$

$M_{44} = b_{22}[-\frac{2\delta^{/}c^{/}}{(c^{/}+p^*)^3} - \frac{2m^{/}b_{11}^2}{(1+x^*)^3}] - \frac{4m^{/}b_{22}b_{12}b_{11}}{(1+x^*)^3} + 6\frac{m^{/}y^*b_{12}b_{11}^2}{(1+x^*)^4} - \frac{6m^{/}b_{22}b_{12}^2}{(1+x^*)^3} + 6\frac{m^{/}y^*b_{12}^3}{(1+x^*)^4}$

$M_{45} = 3b_{11}[\frac{2b_{21}b_{11}}{(1+x^*)^3} - \frac{2y^*b_{11}^2}{(1+x^*)^3}] + x^*[-6\frac{b_{21}b_{11}^2}{(1+x^*)^3} + 6\frac{b_{11}^3y^*}{(1+x^*)^4}] + 2b_{12}[\frac{b_{11}b_{22}+b_{12}b_{21}}{(1+x^*)^2} - 2\frac{b_{11}^2y^*b_{12}}{(1+x^*)^3}] + 2b_{11}[\frac{b_{12}b_{22}}{(1+x^*)^2} - \frac{b_{12}^2y^*}{(1+x^*)^3}] + x^*[-\frac{4b_{22}b_{12}b_{11}+2b_{12}^2b_{21}}{(1+x^*)^3} + \frac{6b_{12}^2b_{11}y^*}{(1+x^*)^4}]$

$M_{46} = b_{12}[\frac{2b_{21}b_{11}}{(1+x^*)^2} - \frac{2y^*b_{11}^2}{(1+x^*)^3}] + b_{11}[\frac{b_{22}b_{11}+b_{12}b_{21}}{(1+x^*)^2} - \frac{b_{12}b_{11}y^*}{(1+x^*)^3}] + b_{11}[\frac{b_{22}b_{11}+b_{12}b_{21}}{(1+x^*)^2} - 2\frac{b_{12}b_{11}y^*}{(1+x^*)^3}] + x^*[-2\frac{b_{22}b_{11}^2+b_{12}b_{21}b_{11}+b_{12}b_{11}b_{21}}{(1+x^*)^3} + 6\frac{b_{12}b_{11}^2y^*}{(1+x^*)^4}] + 2b_{12}[2\frac{b_{22}b_{12}}{(1+x^*)^2} - 2\frac{b_{12}^2}{(1+x^*)^3}] + x^*[-6\frac{b_{22}b_{12}^2}{(1+x^*)^3} + 6\frac{b_{12}^3}{(1+x^*)^4}]$

$M_{51} = b_{13}[-\frac{r^{/}b_{12}}{k^{/}} - \frac{b_{21}}{1+x^*} + \frac{y^*b_{11}}{(1+x^*)^2} - a^{/}] + b_{11}[-\frac{r^{/}b_{13}}{k^{/}} - \frac{b_{23}}{1+x^*} + \frac{y^*b_{13}}{(1+x^*)^2} - a^{/}] + x^*[\frac{b_{23}b_{11}+b_{13}b_{21}}{(1+x^*)^2} - 2\frac{y^*b_{13}b_{11}}{(1+x^*)^2}]$

$M_{52} = b_{13}[-\frac{r^{/}b_{12}}{k^{/}} - \frac{b_{22}}{1+x^*} + \frac{y^*b_{12}}{(1+x^*)^2}] + b_{12}[-\frac{r^{/}b_{13}}{k^{/}} - \frac{b_{23}}{1+x^*} + \frac{y^*b_{13}}{(1+x^*)^2} - a^{/}] + x^*[\frac{b_{23}b_{12}+b_{13}b_{22}}{(1+x^*)^2} - 2\frac{y^*b_{12}b_{13}}{(1+x^*)^3}]$

$M_{53} = b_{23}[\frac{c^{/}\delta^{/}}{(c^{/}+p^*)^2} + \frac{m^{/}b_{11}}{(1+x^*)^2}] + b_{21}[\frac{c^{/}\delta^{/}}{(c^{/}+p^*)^2} + \frac{m^{/}b_{13}}{(1+x^*)^2}] + y^*[-2\frac{c^{/}\delta^{/}}{(c^{/}+p^*)^3} - 2\frac{m^{/}b_{13}b_{11}}{(1+x^*)^2}]$

$M_{54} = m^{/}\frac{b_{23}b_{12}+b_{22}b_{13}}{(1+x^*)^2} + \frac{b_{22}\delta^{/}c^{/}}{(c^{/}+p^*)^2} - \frac{2m^{/}b_{13}b_{12}y^*}{(1+x^*)^3}$

$M_{55} = -\frac{\delta^{/}(b_{21}+b_{23})}{c^{/}+p^*} + \frac{\delta^{/}(2y^*+b_{23}p^*+p^*b_{21}-2p^*y^*)}{(c^{/}+p^*)^2}$

$M_{56} = -\frac{\delta^{/}c^{/}b_{22}}{(c^{/}+p^*)^2}$

From the above value of$g_{11}, g_{02}, g_{20}, G_{101}, G_{110}, G_{21}, h_{11}, h_{20}, w_{20}, w_{11} and g_{21}$ we can now calculate the value of $\mu_2, and \tau_2$ which are obtained from alone [8] where

$$c_1(0) = \frac{i}{2w_0}[g_{20}g_{11} - 2|g_{11}|^2 - \frac{1}{3}|g_{02}|^2] + \frac{g_{21}}{2} \tag{18}$$

$$\mu_2 = -\frac{Rec_1(0)}{\alpha^{/}(0)} \tag{19}$$

$$\tau_2 = -\frac{Imc_1(0) +_2 w^{/}(0)}{w_0} \tag{20}$$

$$\beta_2 = 2Rec_1(0) \tag{21}$$

If $\alpha^{/}(0) > 0$ the periodic solution which exists for $\mu_2 > 0$ is supercritical. On the other hand if $\alpha^{/}(0) > 0$, then the periodic solution which exists for $\mu_2 < 0$ is subcritical [8].

Theorem 4. Here μ_2 determines the direction of the Hopf-bifurcation. If $\mu_2 > 0(< 0)$ then the Hopf-bifurcation is supercritical (subcritical) and the bifurcating periodic solutions exist for $m < m^{/}(> m^{/})$; β_2 determines the stability of the bifurcating periodic solutions: the solutions are orbit ally stable (unstable) if $\beta_2 < 0(> 0)$; and τ_2 determines the period of the bifurcating periodic solutions: the period increases (decreases) if $\tau_2 > 0(< 0)$.

6. Application

From the above discussion we have seen that system (1-3) has a Hopf bifurcation at the equilibrium point E and we also find the stability condition at different equilibrium points of system (1-3). We take different example to describe the nature of the given system (1-3).

Example 1. First of all let us take the value of parameters $r = 5; k = 5; a = 0.25; d = 13.943; \delta = 2; m = 25.219; p^0 = 11.333; D = 1; c = 1$. For this choice of the value of parameters we see that the system (1-3) give Hopf bifurcation at the equilibrium point (1, 7, 2). Moreover we find the values $\mu_2 = 8.636, \tau_2 = -1.029, \beta_2 = -0.950\alpha^{/}(0) = 0.055$.

Example 2. When we chose parameter $r = 5; k = 5; a = 0.25; d = 13.943; \delta = 2; m = 26.219; p^0 = 11.333; D = 1; c = 1$ we see that the system give unstable limit cycle.

Example 3.But on the other hand if we choose the value of parameter $r = 5; k = 5; a = 0.25; d = 13.943; \delta = 2; m = 24.219; p^0 = 11.333; D = 1; c = 1$ we see that the system is stable. When m passes through the critical value $m^{/}(= 25.219)$ the equilibrium point of the system (1-3) losses its stability and a Hopf bifurcation occurs. Since $\mu_2 > 0$, the Hopf bifurcation is supercritical in nature. Since $\beta_2 < 0$, the individual periodic orbit is stable. Again $\tau_2 < 0$, the period decreases.

Example 4. By choosing the value of the parameter $r = 5; k = 5; a = 0.5; d = 13.943; \delta = 2; m = 24.5; p^0 = 11.333; D = 1; c = 1$ we see that the system is stable. Thus from the above discussion we see that the system (1-3) is stable and as well as it give Hopf bifurcation.

7. Conclusions

The nonlinear behavior of a prey predator system in a polluted environment has been studied in this paper. By choosing the rate of increase of predator with the help of prey as bifurcation parameter it has been shown that the Hopf bifurcation occurs in the system under consideration. The stability of the bifurcating periodic solution and the direction of Hopf bifurcation are also determined using the normal form theory. It is found by numerical example that the Hopf bifurcation in the proposed system is supercritical in nature.

References

[1] R. Beal W. B. Betts, W. B. Role of rhamnolipids biosurfactasts in the uptake and mineralization of hexadecane by Pseudomnous aeruginosa, J. Appl. Microbiol. 89,(2000) 158-168.

[2] A. Chhater, H. J. Purohit, R. Shanker,T. Chakrobortti P. Khanna. Bacterial consortia for crude oil spill remediation, Wat. Sci. Tech. 34,(1996) 187-193.

[3] B. Dubey J. Hursain Modelling the interaction of two biological species in a polluted environment, J. Math. Anal. Appl. 246,(2000) 58-79.

[4] H. I. Freedom J. B. Shukla [1991] Models for the effects of toxicant in a single - species and preadator - prey systems, J. Math. Biol. 30,(1991) 15-30.

[5] T. G. Hallam, C. E. Clerk G. S. Jorden Effects of toxicants on populations: A qualitative approach II .First order kinetics, J. Math. Biol. 18,(1983) 25-37.

[6] T. G. Hallam, C. E. Clerk R. R. Lassiter Effects of toxicants on populations: A qualitative approach-I. Equlibrium environment exposure, Ecal. Model. 18,(1983) 291-304.

[7] T. G. Hallam J. T. Deluna Effects of toxicants on populations: A qualitative approachesIII: Environmental and food chain pathways, J. Theoret. Biol. 109,(1984) 411-429.

[8] B. D. Hassard, N. D. Kazarinoff Y. H. Wan Theory and Application of Hopf Bifurcation Cambridge University Press, Cambridge, 1981.

[9] Z. Ma, G. Cui W. Wang Persistence and Existence of a population in a polluted environment, Math. Biosci. 101,(1990) 75.

[10] D. Mukherjee Effect of delay on two competing organism in a polluted environment Int. J. Biomathematics 1, (2008) 475-485.

[11] D. Mukherjee ,Persistence and Global stability of a population in a polluted environment with delay, J. Biol. System 10,(2002) 225-232.

[12] S. K. Mapa, Higher Algebra (Classical) Asoke prakasan, Calcutta-700007,2000.

Received: September, 2010