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On Property (β) of Generalized Cesàro Sequence Space

C. Mongkolkeha[a], S. Panoi[b] and W. Sanhan[a1]

[a] Department of Mathematics, Faculty of Liberal Arts and Science, Kasetsart University
Kamphaeng-Saen Campus, Nakhonpathom 73140, Thailand

[b] Department of Mathematics, Faculty of Science, Khonkean University
Khonkean 40002, Thailand

Abstract

In this paper we study some geometric properties of generalized Cesàro sequence spaces $ces_{(p)}$. The main result of this study is to show that the space $ces_{(p)}$ has property (β).

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1 Introduction

The investigations of metric geometry of normed spaces such properties of nomed speces which are invariant with respect to linear isometries, date back to 1936, when J. A. Clarkson introduced the notion of uniformly convex spaces in the paper uniformly convex spaces in Trans. Amer. Math. Soc40(1936), and it was shown that L_p with $1 < p < \infty$ are examples of such space. Metric geometry of normed spaces has applications in areas of mathematics,among others in approximation theory, fixed point theory, probability teory, ergodic theory, optimization theory, control theory, operator theory. J. A. Clarkson [1] introduced uniform convexity property (UC) or (UR) of Banach spaces which ensures, for example, the existence and unicity of nearest

[1]Corresponding author; W. Sanhan
Email addresses: winate_s@yahoo.com, cm.mongkol@hotmail.com

points in best approximation problems. There are many convexity properties of Banach spaces which are also important geometric properties related to uniform convexity property and they are more general than (UC). Two of them are known as locally uniformly rotund (LUR) and rotundity (R). It is well known that

$$(UR) \Longrightarrow (LUR) \Longrightarrow (R).$$

Roughly speaking, we can classify metric geometric propertics of normed spaces dividing then into the following groups: Rotundity properties, smoothness properties, monotomicity properties (in the class of mormed lattices only), Complex rotundity properties and other geometric properties related to the fixed point theory(nearly uniform rotundity, uniform Kadec-Klee property, nearly uniform smoothness, noncreaseness, uniform noncreaseness, Opial property,uniform Opial property). Among geometric properties, property (β), property (H), drop property (D) and nearly uniform convexity (NUC) are very important. Rolewicz [10] showed that property (β) follows from (UC) and that the property (β) implies (NUC) and (NUC) implies property (D). He also proved in [9] that a Banach space X has property (D), then X is reflexive. Montesinos extended this result by showing that X has property (D) if and only if X is reflexive and property (H). It is also known that (UKK) implies property (H). Summarizing the above discussion we have

$$\begin{array}{l} \qquad\qquad\quad (D) \Longrightarrow (Reflexive) \\ \qquad\qquad\quad\ \Uparrow \end{array}$$

$(UC) \Longrightarrow (\beta) \Longrightarrow (NUC) \Longrightarrow (UKK) \Longrightarrow$ property (H). The converse of this implications are not true in general, such as in [5], Kutzarova provided an example of (NUC) space which does not have property (β).

A lot of mathematicians are interested in Cesàro sequence space. Y. cui and H. Hudzik [3] indicated that Cesàro sequence space is $(kNUC)$ and has uniform Opail property where $p > 1$. Y. Cui and C. Meng [4] pointed that Cesàro sequence space has the Banach-Saks of type p if $p > 1$, and it was shown that has property (β). S. Suantai [14] revealed that generalize Cesàro sequence space has property (H). In 2003, S. Suantai [15] introduced a new modular space which is a generalization of Cesàro sequence space, and then, some geometric properties on such new modular space were considered eqiuipped with the Luxemburg norm. At this point, it is interesting to study geometric properties of the Cesàro sequence space. In this paper, we study property (β) of generalization Cesaro sequence space $ces_{(p)}$ equipped with the Luxemburg norms, where $p = (p_k)$, $1 < p < \infty$ is a positive real sequence.

2 Preliminary Notes

Definition 2.1. *For a real vector space X, a function $\rho : X \to [0,\infty]$ is called a* modular *if it satisfies the following conditions :*

(i) $\rho(x) = 0$ *if and only if* $x = 0$;

(ii) $\rho(\alpha x) = \rho(x)$ *for all scalar* α *with* $|\alpha| = 1$;

(iii) $\rho(\alpha x+\beta y) \leq \rho(x)+\rho(y)$, *for all* $x, y \in X$ *and all* $\alpha, \beta \geq 0$ *with* $\alpha+\beta = 1$.

The modular ρ is called convex if

(iv) $\rho(\alpha x + \beta y) \leq \alpha\rho(x) + \beta\rho(y)$, *for all* $x, y \in X$ *and all* $\alpha, \beta \geq 0$ *with* $\alpha + \beta = 1$.

If ρ is modular in X, we define

$$X_\rho = \{x \in X : \rho(\lambda x) \to 0 \text{ as } \lambda \to 0\},$$

and

$$X^*_\rho = \{x \in X : \rho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

It clear that $X_\rho \subseteq X^*_\rho$. If ρ is a convex modular, for $x \in X_\rho$ we define

$$\|x\| = \inf\{\lambda > 0 : \rho(\frac{x}{\lambda}) \leq 1\}. \tag{1}$$

Orlicz [6] proved that if ρ is convex modular on X, then $X_\rho = X^*_\rho$ and $\|.\|$ is a norm on X_ρ for which it is a Banach space. The norm $\|.\|$ defined as in (1) is called the Luxemburg norm.

A modular ρ is said to satisfy the $\delta_2 -$ *condittion*$(\rho \in \delta_2)$ if for any $\varepsilon > 0$ there exist a constants $K \geq 2$ and $a > 0$ such that

$$\rho(2u) \leq K\rho(u) + \varepsilon$$

for all $u \in X_\rho$ with $\rho(u) \leq a$.

If ρ satisfies the $\delta_2 -$ *condittion* for any $a > 0$ with $K \geq 2$ dependent on a, we say that ρ the *stong* $\delta_2 -$ *condittion*$(\rho \in \delta_2^s)$.

The following results are very important for our consideration.

Proposition 2.2. *If $\rho \in \delta_2^s$, then for any $L > 0$ and $\varepsilon > 0$, there exists $\delta = \delta(L, \varepsilon) > 0$ such that*

$$|\rho(u + v) - \rho(u)| < \varepsilon$$

whenever $u, v \in X_\rho$ *with* $\rho(u) \leq L$, *and* $\rho(v) \leq \delta$.

Proof. See[2, Lemma 2.1]. □

Proposition 2.3. *If* $\rho \in \delta_2^s$, *then for any* $x \in X_\rho$ *we have* $\|x\| = 1$ *if and only if* $\rho(x) = 1$.

Proof. See[2, Corollary 2.2]. □

Proposition 2.4. *If* $\rho \in \delta_2$, *then for any sequence* (x_n) *in* X_ρ, $\|x_n\| \to 0$ *if and only if* $\rho(x_n) \to 0$.

Proof. See[2, Lemma 2.3]. □

Proposition 2.5. *If* $\varrho \in \delta_2^s$, *then for any* $\varepsilon \in (0,1)$ *there exists* $\delta \in (0,1)$ *such that* $\varrho(x) \leq 1-\varepsilon$ *implies* $\|x\| \leq 1-\delta$.

Proof. See[12, Theorem 1.3]. □

Let l^0 be the space of all real sequences and $p = (p_k)$ a bounded sequence of positive real numbers with $p_k \geq 1$ for all $k \in \mathbb{N}$, the Cesàro sequence space (ces_p, for short) is defined by

$$ces_p = \{x \in l^0 : \sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=0}^{n} |x(i)| \right)^p < \infty\},$$

equipped with the norm

$$\|x\| = \left(\sum_{n=1}^{\infty} \left(\frac{1}{n} \sum_{i=1}^{n} |x(i)| \right)^p \right)^{\frac{1}{p}}$$

and

$$\|x\|_0 = \left(\sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_{r} |x(i)| \right)^p \right)^{\frac{1}{p}},$$

where $\sum\limits_{r}$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$.

It is known that these two norms are equivalent and ces_p is Banach with respect to each of the two norms. The Generalized Cesàro Sequence Space $ces_{(p)}$ is defined by

$$ces_{(p)} = \{x \in l^0 : \varrho(\lambda x) < \infty \text{ for some } \lambda > 0\},$$

where $\varrho(x) = \sum\limits_{r=0}^{\infty} \left(\frac{1}{2^r} \sum\limits_{r} |x(i)| \right)^{p_r}$ and $\sum\limits_{r}$ denotes a sum over the ranges $2^r \leq i < 2^{r+1}$. We consider $ces_{(p)}$ equipped with the Luxemburg norm

$$\|x\| = \inf\{\varepsilon > 0 : \varrho(\tfrac{x}{\varepsilon}) \leq 1\}.$$

3 Main Results

In this section, we study property (β) of Generalized Cesàro sequence $ces_{(p)}$. First we give some relationships between a modular ϱ and its luxemburg.

Proposition 3.1. *The ϱ is convex modular on $ces_{(p)}$.*

Proof. Let $x, y \in ces_{(p)}$. It is obvious that $\varrho(x) = 0$ if and only if $x = 0$ and $\varrho(\alpha x) = \varrho(x)$ for scalar α with $|\alpha| = 1$.

Let $\alpha \geq 0$, $\beta \geq 0$ with $\alpha + \beta = 1$. By the convexity of the function $t \mapsto |t|^{p_r}$, for all $r \in \mathbb{N}$, we have

$$\begin{aligned}
\varrho(\alpha x + \beta y) &= \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_{r} |\alpha x(i) + \beta y(i)| \right)^{p_r} \\
&\leq \sum_{r=0}^{\infty} \left(\alpha \frac{1}{2^r} \sum_{r} |x(i)| + \beta \frac{1}{2^r} \sum_{r} |y(i)| \right)^{p_r} \\
&\leq \alpha \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_{r} |x(i)| \right)^{p_r} + \beta \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_{r} |y(i)| \right)^{p_r} \\
&= \alpha \varrho(x) + \beta \varrho(y).
\end{aligned}$$

□

Proposition 3.2. *For $x \in ces_{(p)}$, the modular ϱ on $ces_{(p)}$ satisfies the following properties:*

(i) if $0 < a < 1$, then $a^M \varrho(\frac{x}{a}) \leq \varrho(x)$ and $\varrho(ax) \leq a\varrho(x)$;

(ii) if $a > 1$, then $\varrho(x) \leq a^M \varrho(\frac{x}{a})$;

(iii) if $a \geq 1$, then $\varrho(x) \leq a\varrho(x) \leq \varrho(ax)$.

Proof. (i) Let $0 < a < 1$. Then we have

$$\begin{aligned}
\varrho(x) &= \sum_{r=0}^{\infty} \left(\frac{1}{2^r} \sum_{r} |x(i)| \right)^{p_r} \\
&= \sum_{r=0}^{\infty} \left(\frac{a}{2^r} \sum_{r} |\frac{x_n}{a}| \right)^{p_r} \\
&= \sum_{r=0}^{\infty} a^{p_r} \left(\frac{1}{2^r} \sum_{r} |\frac{x_n}{a}| \right)^{p_r}
\end{aligned}$$

$$
\begin{aligned}
&\geq \sum_{r=0}^{\infty} a^{M}\left(\frac{1}{2^r}\sum_{r}|\frac{x_n}{a}|\right)^{p_r}\\
&= a^{M}\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|\frac{x_n}{a}|\right)^{p_r}\\
&= a^{M}\varrho(\frac{x}{a}).
\end{aligned}
$$

By convexity of modular ϱ, we have $\varrho(ax) \leq a\varrho(x)$, so (i) is obtained.
(ii) Let $a > 1$. Then

$$
\begin{aligned}
\varrho(x) &= \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|x(i)|\right)^{p_r}\\
&= \sum_{r=0}^{\infty} a^{p_r}\left(\frac{1}{2^r}\sum_{r}|\frac{x_n}{a}|\right)^{p_r}\\
&\leq a^{M}\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|\frac{x_n}{a}|\right)^{p_r}\\
&= a^{M}\varrho(\frac{x}{a}).
\end{aligned}
$$

Hence (ii) is satisfies. (iii) follows from the convexity of ϱ. $\square$

Proposition 3.3. *For any $x \in ces_{(p)}$, we have*

(i) if $\|x\| \leq 1$, then $\varrho(x) \leq \|x\|$;

(ii) if $\|x\| > 1$, then $\varrho(x) \geq \|x\|$;

(iii) $\|x\| = 1$ if and only if $\varrho(x) = 1$;

(iv) $\|x\| < 1$ if and only if $\varrho(x) < 1$;

(v) $\|x\| > 1$ if and only if $\varrho(x) > 1$.

Proof. (i) Let $\varepsilon > 0$ be such that $0 < \varepsilon < 1 - \|x\|$, so $\|x\| + \varepsilon < 1$. By the definition of $\|.\|$, then there exits $\lambda > 0$ such that $\|x\| + \varepsilon > \lambda$ and $\varrho(\frac{x}{\lambda}) \leq 1$. By Proposition (i) and (iii), we have

$$
\begin{aligned}
\varrho(x) &\leq \varrho\left(\frac{(\|x\|+\varepsilon)}{\lambda}x\right)\\
&= \varrho\left((\|x\|+\varepsilon)\frac{x}{\lambda}\right)\\
&\leq (\|x\|+\varepsilon)\varrho\left(\frac{x}{\lambda}\right)\\
&\leq \|x\|+\varepsilon,
\end{aligned}
$$

which implies that $\varrho(x) \le \|x\|$. Hence (i) is satisfies.
(ii) Let $\varepsilon > 0$ such that $0 < \varepsilon < \frac{\|x\|-1}{\|x\|}$, then $0 < (1-\varepsilon)\|x\| \le \|x\|$. By definition of $\|.\|$ and Proposition 3.2(i), we have $1 < \varrho(\frac{x}{(1-\varepsilon)\|x\|}) < \frac{1}{(1-\varepsilon)\|x\|}\varrho(x)$, so $(1-\varepsilon)\|x\| < \varrho(x)$ for all $\varepsilon \in (0, \frac{\|x\|-1}{\|x\|})$ which implies that $\|x\| \le \varrho(x)$.
(iii) Assume that $\|x\| = 1$. Let $\varepsilon > 0$ then there exits $\lambda > 0$ such that $1+\varepsilon > \lambda > \|x\|$ and $\varrho(\frac{x}{\lambda}) \le 1$. By Proposition 3.2$(ii)$, we have $\varrho(x) \le \lambda^M \varrho(\frac{x}{\lambda}) \le \lambda^M < (1+\varepsilon)^M$, so $(\varrho(x))^{\frac{1}{M}} < 1+\varepsilon$ for all $\varepsilon > 0$ which implies that $\varrho(x) \le 1$. If $\varrho(x) < 1$, let $a \in (0,1)$ such that $\varrho(x) < a^M < 1$. From Proposition 3.2(i), we have $\varrho(\frac{x}{a}) \le \frac{1}{a^M}\varrho(x) < 1$. Hence $\|x\| < a < 1$, which contradiction. Thus, we have $\varrho(x) = 1$.

Conversely, assume that $\varrho(x) = 1$. By definition of $\|.\|$, we conclude that $\|x\| \le 1$. If $\|x\| < 1$, then we have by (i) that $\varrho(x) \le \|x\| < 1$ which contradiction, so we obtain that $\|x\| = 1$.
(iv) follows from (i) and (iii).
(v) follows from (iii) and (iv). □

Proposition 3.4. *For any $x \in ces_{(p)}$, we have*

(i) if $0 < a < 1$ and $\|x\| > a$, then $\varrho(x) > a^M$;

(ii) if $a \ge 1$ and $\|x\| < a$, then $\varrho(x) < a^M$.

Proof. (i) Let $0 < a < 1$ and $\|x\| > a$. Then $\|\frac{x}{a}\| > 1$, by proposition 3.3(v), we have $\varrho(\frac{x}{a}) > 1$. Hence by proposition 3.2(i), we have $\varrho(x) \ge a^M \varrho(\frac{x}{a}) > a^M$, so we obtain (i).
(ii) Suppose $a \ge 1$ and $\|x\| < a$. Then $\|\frac{x}{a}\| < 1$, by Proposition 3.3(iv), we have $\varrho(\frac{x}{a}) < 1$. If $a = 1$, it is obvious that $\varrho(x) < 1 = a^M$. If $a > 1$, then by Proposition 3.2(ii), we obtain that $\varrho(x) \le a^M \varrho(\frac{x}{a}) \le a^M$. □

Proposition 3.5. *Let (x_n) be sequence in $ces_{(p)}$.*

(i) If $\|x_n\| \to 1$ as $n \to \infty$, then $\varrho(x_n) \to 1$ as $n \to \infty$.

(ii) If $\varrho(x_n) \to 0$ as $n \to \infty$, then $\|x_n\| \to 0$ as $n \to \infty$.

Proof. (i) Assume that $\|x_n\| \to 1$ as $n \to \infty$. Let $\varepsilon \in (0,1)$. Then there exists $N \in \mathbb{N}$ such that $1-\varepsilon < \|x_n\| < 1+\varepsilon$ for all $n \ge N$.By Proposition 3.4, we have $(1-\varepsilon)^M < \varrho(x_n) < (1+\varepsilon)^M$ for all $n \ge N$, which implies that $\varrho(x_n) \to 1$ as $n \to \infty$.
(ii) Suppose that $\|x_n\| \nrightarrow 0$ as $n \to \infty$. Then there exists $\varepsilon \in (0,1)$ and a subsequence (x_{n_k}) of (x_n) such that $\|x_{n_k}\| > \varepsilon$ for all $k \in \mathbb{N}$. By Proposition 3.4(i) we obtain $\varrho(x_{n_k}) > (\varepsilon)^M$ for all $k \in \mathbb{N}$. This implies that $\varrho(x_n) \nrightarrow 0$
as $n \to \infty$. □

Proposition 3.6. *For any $L > 0$ and $\varepsilon > 0$ there exists $\delta > 0$ such that $|\varrho(u+v) - \varrho(u)| < \varepsilon$ whenever $u, v \in ces_{(p)}$ with $\varrho(u) \leq L$ and $\varrho(v) \leq \delta$.*

Proof. Since $p = (p_r)$ is bounded, it is easy to see that $\varrho \in \delta_2^s$. Hence, the Proposition is obtined directly from Proposition 2.2. $\square$

Proposition 3.7. *For every sequence $(x_n) \in ces_{(p)}$ we have $\|x_n\| \to 0$ if and only if $\varrho(x_n) \to 0$.*

Proof. It following directly from Proposition 2.4 because $\varrho \in \delta_2^s$. $\square$

Proposition 3.8. *For every sequence $x \in ces_{(p)}$. the following condition holds. For any $\varepsilon \in (0,1)$, there exists $\delta \in (0,1)$ such that $\varrho(x) \leq 1 - \varepsilon$ implies $\|x_n\| \leq 1 - \delta$.*

Proof. Since $p = (p_r)$ is bounded, it is easy to see that $\varrho \in \delta_2^s$. Hence, the Proposition is obtined directly from Proposition 2.5 $\square$

Proposition 3.9. *If $\lim_{n\to\infty} \inf p_n > 1$ then for any $x \in ces_{(p)}$ there exists $k_0 \in \mathbb{N}$ and $\theta \in (0,1)$ such that $\varrho(\frac{x^k}{2}) \leq \frac{1-\theta}{2}\varrho(x^k)$ for all $k \in \mathbb{N}$ with $k \geq k_0$, where $x^k = (\overbrace{0, 0, ..., 0,}^{k-1} \sum_{2^r \leq i \leq k} |x(i)|, x(k+1), x(k+2), ...)$ and $2^r \leq k < 2^{r+1}$.*

Proof. Let $k \in \mathbb{N}$ be fixed. So there exist $r_k \in \mathbb{N}$ such that $2^{r_k} \leq k < 2^{r_k+1}$. Let α be a real number such that $1 < \alpha \leq \lim_{k\to\infty} \inf p_k$. Then there exists $k_0 \in \mathbb{N}$ such that $\alpha < p_k$ for all $k \geq k_0$. Let $\theta \in (0,1)$ be a real such that $(\frac{1}{2})^\alpha \leq \frac{1-\theta}{2}$. Then for each $x \in ces_{(p)}$ and $k \geq k_0$, we have

$$\begin{aligned}
\varrho(\frac{x^k}{2}) &= \sum_{r=r_k}^{\infty} \left(\frac{1}{2^r} \sum_r |\frac{x(i)}{2}| \right)^{p_r} \\
&= \sum_{r=r_k}^{\infty} \left(\frac{1}{2}\right)^{p_r} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^{p_r} \\
&\leq \left(\frac{1}{2}\right)^{\alpha} \sum_{r=r_k}^{\infty} \left(\frac{1}{2^r} \sum_r |x(i)| \right)^{p_r} \\
&< \frac{1-\theta}{2} \varrho(x^k).
\end{aligned}$$

$\square$

Theorem 3.10. *The space $ces_{(p)}$ has property (β).*

Proof. Let $\varepsilon > 0$ and $(x_n) \subset B(ces_{(p)})$ with $sep(x_n) \geq \varepsilon$ and $x \in B(ces_{(p)})$. For any $y \in B(ces_{(p)})$ and for each $N \in \mathbb{N}$ we can find $r_0 \in \mathbb{N}$ such that $2^{r_0} \leq N < 2^{r_0+1}$, we define $y^N = (\overbrace{0, 0, ..., 0,}^{N-1} \sum_{2^{r_0} \leq i \leq N} |y(i)|, y(N+1), y(N+2), ...)$. Since for each $i \in \mathbb{N}$, $(x_n(i))_{n=1}^{\infty}$ is bounded. By using the diagonal method, we have that for each $N \in \mathbb{N}$ we can find subsequence (x_{n_j}) of (x_n) such that $(x_{n_j}(i))$ converges for each $i \in \mathbb{N}$ with $1 \leq i \leq N$. Therefore, for each $N \in \mathbb{N}$ there exists $s_N \in \mathbb{N}$ such that $sep((x_{n_j}^N)_{j > s_N}) \geq \varepsilon$. Hence there is a sequence of positive integers $(s_N)_{N=1}^{\infty}$ with $s_1 < s_2 < s_3 < ...$ such that $\|x_{s_N}^N\| \geq \frac{\varepsilon}{2}$ for all $N \in \mathbb{N}$. By Proposition 3.7 there exists $\eta > 0$ such that $\varrho(x_{s_N}^N) \geq \eta$ for all $N \in \mathbb{N}$. Hence

$$\sum_{r=r_0}^{\infty} \left(\frac{1}{2^r} \sum_r |x_{s_N}^N(i)| \right)^{p_r} \geq \eta \tag{2}$$

for all $N \in \mathbb{N}$. By Proposition 3.9, there exist $k_0 \in \mathbb{N}$ and $\theta \in (0, 1)$ such that

$$\varrho(\frac{u^N}{2}) \leq \frac{1-\theta}{2} \varrho(u^N) \tag{3}$$

for all $u \in ces_{(p)}$ and $N \geq k_0$. By Proposition 3.8, there exist $\delta > 0$ such that for any $y \in ces_{(p)}$.

$$\varrho(y) \leq 1 - \frac{\theta\eta}{4} \Longrightarrow \|y\| \leq 1 - \delta. \tag{4}$$

From Proposition 3.6, there exists δ_0 such that

$$|\varrho(u+v) - \varrho(u)| < \frac{\theta\eta}{4} \tag{5}$$

whenever $\varrho(u) \leq 1$ and $\varrho(v) \leq \delta_0$.

Since $x \in B(ces_{(p)})$, we have that $\varrho(x) \leq 1$. Then there exits $k \geq k_0$ such that $\varrho(x^k) \leq \delta_0$. We put $u = x_{s_k}^k$ and $v = x^k$,

$$\varrho(\tfrac{u}{2}) = \sum_{r=r_0}^{\infty} \left(\tfrac{1}{2^r} \sum_r |\tfrac{x_{s_k}(i)}{2}| \right)^{p_r} < 1 \text{ and } \varrho(\tfrac{v}{2}) = \sum_{r=r_0}^{\infty} \left(\tfrac{1}{2^r} \sum_r |\tfrac{x(i)}{2}| \right)^{p_r} < \delta_0.$$

From (5), we have

$$
\begin{aligned}
\sum_{r=r_0}^{\infty}\left(\frac{1}{2^r}\sum_{r}|\frac{x(i)+x_{s_k}(i)}{2}|\right)^{p_r} &= \varrho(\frac{u+v}{2}) \\
&\leq \varrho(\frac{u}{2})+\frac{\theta\eta}{4} \\
&\leq \frac{1-\theta}{2}\left(\varrho(u)\right)+\frac{\theta\eta}{4}. \qquad (6)
\end{aligned}
$$

By (2), (3), (6) and convexity of function $f(t)=|t|^{p_r}$, for all $r\in\mathbb{N}$, we have

$$
\begin{aligned}
\varrho(\frac{x+x_{s_k}}{2}) &= \sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_r|\frac{x(i)+x_{s_k}(i)}{2}|\right)^{p_r} \\
&= \sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|\frac{x(i)+x_{s_k}(i)}{2}|\right)^{p_r} + \sum_{r=r_0}^{\infty}\left(\frac{1}{2^r}\sum_r|\frac{x(i)+x_{s_k}(i)}{2}|\right)^{p_r} \\
&\le \frac{1}{2}\left(\sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|x(i)|\right)^{p_r} + \sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|x_{s_k}(i)|\right)^{p_r}\right) \\
&\quad + \sum_{r=r_0}^{\infty}\left(\frac{1}{2^r}\sum_r|\frac{x_{s_k}(i)}{2}|\right)^{p_r} + \frac{\theta\eta}{4} \\
&\le \frac{1}{2}\left(\sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|x(i)|\right)^{p_r} + \sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|x_{s_k}(i)|\right)^{p_r}\right) \\
&\quad + \frac{1-\theta}{2}\sum_{r=r_0}^{\infty}\left(\frac{1}{2^r}\sum_r|x_{s_k}(i)|\right)^{p_r} + \frac{\theta\eta}{4} \\
&\le \frac{1}{2}\sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|x(i)|\right)^{p_r} + \frac{1}{2}\sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|x_{s_k}(i)|\right)^{p_r} \\
&\quad + \frac{1-\theta}{2}\sum_{r=r_0}^{\infty}\left(\frac{1}{2^r}\sum_r|x_{s_k}(i)|\right)^{p_r} + \frac{\theta\eta}{4} \\
&\le \frac{1}{2}\sum_{r=0}^{r_0-1}\left(\frac{1}{2^r}\sum_r|x(i)|\right)^{p_r} + \frac{1}{2}\sum_{r=0}^{\infty}\left(\frac{1}{2^r}\sum_r|x_{s_k}(i)|\right)^{p_r} \\
&\quad - \frac{\theta}{2}\sum_{r=r_0}^{\infty}\left(\frac{1}{2^r}\sum_r|x_{s_k}(i)|\right)^{p_r} + \frac{\theta\eta}{4} \\
&\le \frac{1}{2}+\frac{1}{2}-\frac{\theta\eta}{2}+\frac{\theta\eta}{4} \\
&= 1-\frac{\theta\eta}{4}.
\end{aligned}
$$

So it follow from (4) that

$$\|\frac{x+x_{s_k}}{2}\| \le 1-\delta.$$

Therefore, the space $ces_{(p)}$ has property (β). $\square$

Since $(UC) \Rightarrow (\beta) \Rightarrow (NUC) \Rightarrow (D) \Rightarrow (Rfx)$, the following results are obtained directly from Theorem 3.10.

Corollary 3.11. *The space $ces_{(p)}$ is nearly uniform convexity.*

Corollary 3.12. *The space $ces_{(p)}$ has drop property.*

Corollary 3.13. *The space $ces_{(p)}$ is reflexive.*

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