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Application of Homotopy and Homotopy Perturbation Methods to Differential Equations of Heat Transfer and Shear Deformation of Beams

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Abstract

In this paper, different physical differential equations related to heat transfer and shear deformation of beams are solved by new but powerful analytical methods: Liao's Homotopy method (H.M), Homotopy method with Pade approximation and the He's Homotopy-Perturbation Method (HPM).
Nonlinear convective–radiative cooling equation, nonlinear heat equation with cubic nonlinearity and the shear deformation of sandwich beams are used as examples to illustrate the solution procedures. Comparison of the applied methods with exact solutions reveals that both methods are greatly effective.

Keywords: Homotopy, Homotopy-Perturbation, Heat Transfer, Shear Deformation of Beam

1 Introduction

Most scientific problems and phenomena occur nonlinearly. Except in a limited number of these problems, we have difficulty in finding their exact analytical solutions. Therefore, approximate analytical solutions were introduced. It is well

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known that perturbation methods [1] provide the most versatile tools available in nonlinear analysis of engineering problems. The major drawback in the traditional perturbation technique is the over dependence on the existence of small parameter. Recently, some promising approximate analytical solutions are proposed, among which homotopy method [2–9] and homotopy-perturbation method (HPM) [10-15] are the most effective and convenient ones for both weakly and strongly nonlinear equations. A complete review on various asymptotic methods can be found in [16-17].

This paper will apply the HM and the HPM to some nonlinear heat transfer and shear deformation equations. Both of these methods have been shown to effectively, easily and accurately solve a large class of linear and nonlinear problems with components converging rapidly to accurate solutions.

2 Homotopy analysis method

In this section we employ the homotopy analysis method to the discussed problems. To show the basic idea, let us consider the following differential equation:

$$N[u(t)] = 0 \tag{1}$$

, where N is a nonlinear operator, t denotes independent variable, u(t) is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [18] constructs the so-called zero-order deformation equation

$$(1-p)L(\phi(r,t,p) - f_0(r,t)) = hH(t)pN[\phi(r,t,p)] \tag{2}$$

, where $p \in [0-1]$ is the embedding, h≠0 is a non-zero auxiliary parameter, H(t)≠0 is an auxiliary function, L is an auxiliary linear operator, u0(t) is an initial guess of u(t), φ(t;p) is a unknown function, respectively. It is important that one has great freedom to choose auxiliary things in HM. Obviously, when p=0 and p=1, it holds

$$\phi(r,t,0) = f_0(r,t) \qquad \phi(r,t,1) = f(r,t) \tag{3}$$

, respectively. Thus as p increases from 0 to 1, the solution φ(t; p) varies from the initial guess u0(t) to the solution u(t). Expanding φ (t; p) in Taylor series with respect to p, one has:

$$\phi(r,t,p) = f_0(r,t) + \sum_{k=1}^{\infty} f_k(r,t)p^k \tag{4}$$

where:

$$f_k(r,t) = \frac{1}{k!}\frac{\partial^k \phi(r,t,p)}{\partial p^k}\bigg|_{p=0} \tag{5}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter h, and the auxiliary function are so properly chosen, the series (4) converges at p=1, one has

$$f(r,t) = f_0(r,t) + \sum_{k=1}^{\infty} f_k(r,t) \tag{6}$$

, which must be one of solutions of original nonlinear equation, as proved by Liao [15]. As h= −1 and H(t)=1, Eq. (2) becomes

$$(1-p)L(\phi(r,t,p) - f_0(r,t)) = hH(r,t)pN[\phi(r,t,p)] \tag{7}$$

which is used mostly in the homotopy perturbation method (HPM), whereas the solution obtained directly, without using Taylor series. The comparison between HM and HPM can be found in [18].

According to definition (4), the governing equation can be deduced from the zero-order deformation Eq. (2). Define the vector

$$\vec{u}_n = \{u_0(t), u_1(t), \ldots, u_n(t)\} \tag{8}$$

Differentiating Eq. (2) m times with respect to the embedding parameter p and then setting p=0 and finally dividing them by m!, we have the so-called mth-order deformation equation:

$$L(\phi_n(r,t,p) - \chi_n \phi_{n-1}(r,t,p)) = hH(r,t)R_N[\phi_{n-1}(r,t,p)] \tag{9}$$

Where:

$$\chi_m = \begin{cases} 0 & m \le 1 \\ 1 & otherwise \end{cases} \tag{10}$$

$$R_n(\phi_{n-1}(r,t,p)) = \frac{1}{(n-1)!} \frac{\partial^{n-1} N[\phi(r,t,p)]}{\partial p^{n-1}} \Bigg|_{p=0} \tag{11}$$

It should be emphasized that um(t) for m≥1 is governed by the linear Eqs. (2) or (7). with the linear boundary conditions that come from the original problem, which can be easily solved by symbolic computation software such as Maple.

3.0 THE APPLICATION OF HM IN HEAT TRANSFER

In order to assess the accuracy of HM, here we will consider the two following examples.

3.1. Cooling of a system by convection and radiation

Consider the problem of combined convective–radiative cooling of a lumped system. Let the system have volume *V*, surface area *A*, density ρ, specific heat *c*, emissivity *E* and the initial temperature T_i. At *t*=0, the system is exposed to an environment with convective heat transfer with the coefficient of *h* and the temperature T_a. The system also loses heat through radiation and the effective sink temperature is T_s. The cooling equation and the initial conditions are as follows:

$$\rho V c \frac{dT}{d\tau} + hA(T - T_a) + E\sigma A\left(T^4 - T_s^4\right) = 0, \qquad \tau = 0, \quad T = T_i \tag{12}$$

To solve the equation, we do the following changes of parameters:

$$u = \frac{T}{T_i}, \quad u_a = \frac{T_a}{T_i}, \quad u_s = \frac{T_s}{T_i}, \quad t = \frac{\tau}{\rho V c_a / hA}, \quad \varepsilon = \frac{E\sigma T_i^3}{h} \tag{13}$$

After parameter change, the heat transfer equation will result the following:

$$\frac{du}{dt} + (\theta - \theta_a) + \varepsilon(\theta^4 - \theta_s^4) = 0, \qquad t = 0, \quad \theta = 1 \tag{14}$$

For simplicity, we assume the case $u_a = u_s = 0$. So we have

$$\frac{d\theta}{dt} + \theta + \varepsilon\theta^4 = 0, \qquad t = 0, \quad \theta = 1 \tag{15}$$

The linear part of the equation (15) is chosen as the linear operator *L* in Eq. (9).

$$\mathrm{L}(t, q) := \left(\frac{\partial}{\partial t} \phi(t, q)\right) + \phi(t, q) \tag{16}$$

The non-linear operator *N* is the same as Eq. (15):

$$\mathrm{N}(t, q) := \left(\frac{\partial}{\partial t} \phi(t, q)\right) + \phi(t, q) + \varepsilon\, \phi(t, q)^4 \tag{17}$$

Let take *H*=1 and assume a zero-order solution which satisfies the boundary conditions.

$$u0 := \mathrm{e}^{(-t)} \tag{18}$$

If we apply Eq. (9) to the above initial approximation, integrating twice with respect to *t* and applying the homogenous boundary conditions, we obtain first-order solution:

$$u1 := \left(-\frac{1}{3} h\, \varepsilon\, \mathrm{e}^{(-3\,t)} + \frac{h\, \varepsilon}{3}\right) \mathrm{e}^{(-t)} \tag{19}$$

Similarly the solution to higher-order boundary value problem is:

$$u2 := \left(-\frac{1}{3} h\, \varepsilon\, \mathrm{e}^{(-3\,t)} + \frac{h\, \varepsilon}{3}\right) \mathrm{e}^{(-t)} + \mathrm{e}^{(-t)} \left(\frac{1}{3} h^2\, \varepsilon + \frac{1111111111}{5000000000} h^2\, \varepsilon^2\right) + \frac{1}{15000000000} h^2\, \varepsilon\, \mathrm{e}^{(-3\,t)} (-5000000000 + 3333333333\, \varepsilon\, \mathrm{e}^{(-3\,t)} - 6666666666\, \varepsilon)\, \mathrm{e}^{(-t)} \tag{20}$$

$$\vdots$$

Similarly, the other terms of the HP series solution can be obtained. For choosing the appropriate value of *h,* the variation of first and second derivatives of *u* at *t*=0 are plotted against the parameter *h*, Fig. 1. As it is seen from this figure, the proper interval for h to get converged series is $h \in [-2 \quad 0]$.

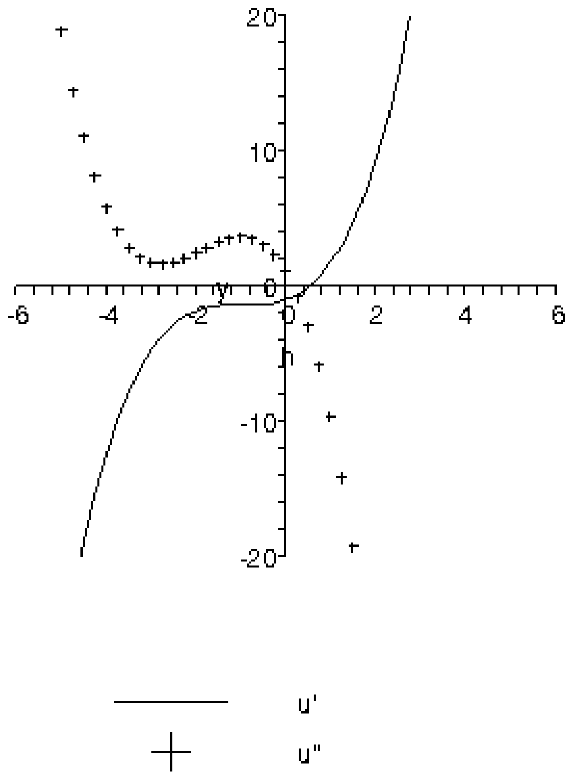

Fig. 1 The h-curves for $\varepsilon = 0.4$, 3rd order approximation

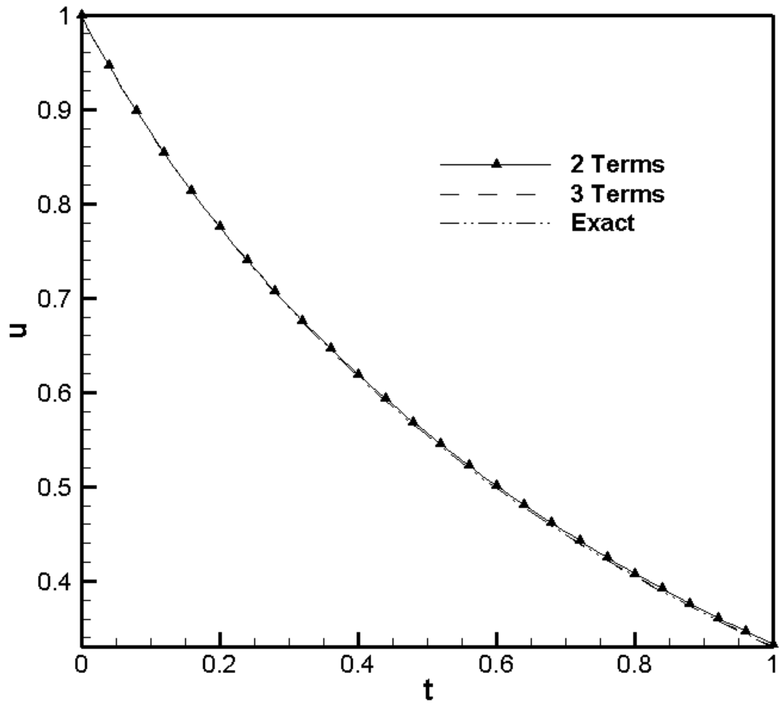

Fig. 2: Comparison of Homotopy method (2 and 3 terms) and exact solution

Adding the first three terms and setting h=-1 (to get similar solution as HPM method used in [19]), we get an analytical expression for velocity filed:

$$\boldsymbol{u}(\boldsymbol{t}) = \boldsymbol{e}^{-t} + \frac{1}{3}\varepsilon\left(\boldsymbol{e}^{-4t} - \boldsymbol{e}^{-t}\right) - \frac{2}{9}\varepsilon^2\left(-\boldsymbol{e}^{-7t} + 2\boldsymbol{e}^{-4t} - \boldsymbol{e}^{-t}\right) \quad (21)$$

The exact solution for this problem is:

$$\frac{1}{3}\ln\frac{1+\varepsilon\theta^3}{(1+\varepsilon)\theta^3} = \boldsymbol{t} \quad (22)$$

Figure 2 shows the comparison between H.M. solution with 2 and 3 terms and the exact one. As it is clear, using only two terms of H.M. series gives accurate result.

3.2. Nonlinear heat equation with cubic nonlinearity

Next, we consider the nonlinear heat transfer equation with cubic nonlinearity in the form of:

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} - 2u^3 \quad (23)$$

with the following initial condition:

$$u(x,0) = \frac{1+2x}{x^2+x+1} \quad (24)$$

The linear operator for H.M. is:

$$\mathrm{L}(t, x, q) := \frac{\partial}{\partial t}\phi(t, x, q) \quad (25)$$

The non-linear operator N is:

$$\mathrm{N}(t, x, q) := \left(\frac{\partial}{\partial t}\phi(t, x, q)\right) - \left(\frac{\partial^2}{\partial x^2}\phi(t, x, q)\right) + 2\,\phi(t, x, q)^3 \quad (26)$$

Let take H=1 and assume a zero-order solution of the form:

$$u0 := \frac{1+2x}{x^2+x+1} \quad (27)$$

Then, we obtain first-order solution:

$$u1 := \frac{6h(1+2x)t}{(x^2+x+1)^2} \quad (28)$$

Similarly the solution to higher-order problem is:

$$u2 := \frac{6h(1+2x)t}{(x^2+x+1)^2} + \frac{6h^2(1+2x)(x^2t+xt+t+6t^2)}{(x^2+x+1)^3} \quad (29)$$

$$\vdots$$

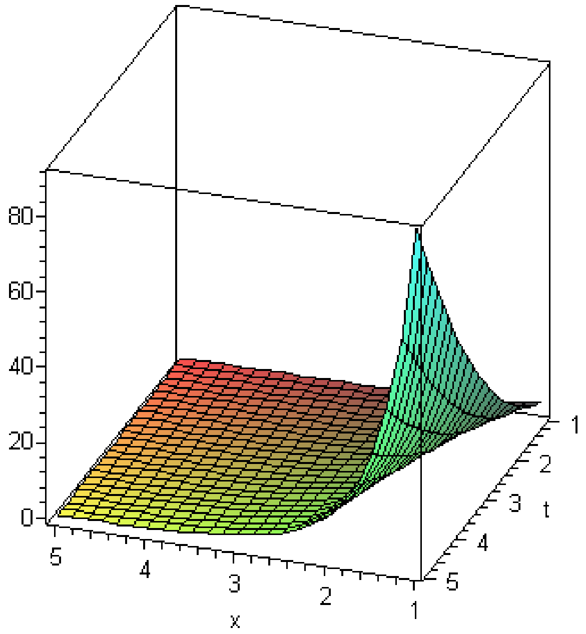

Fig. 3: Solution of Eq. (23) with HM.

Similarly, the other terms of the HM series solution can be obtained. Using three terms of the homotopy seris gives:

$$u := -\frac{216\,(1+2\,x)\,t^3}{(x^2+x+1)^4} + \left(\frac{72\,(1+2\,x)}{(x^2+x+1)^3} + \frac{6\,(1+2\,x)\,(-6\,x^2-6\,x-6)}{(x^2+x+1)^4}\right)t^2 - \frac{6\,(1+2\,x)\,t}{(x^2+x+1)^2} + \frac{1+2\,x}{x^2+x+1} \tag{30}$$

Figure 3 shows the solution of H.M. for equation (23). The observed variation of u is mainly due to cubic nonlinearity term in the heat transfer equation.

4. Shear Deformation of Beams

4.1 Homotopy Perturbation method

In the HPM, the solution is considered in the form below:

$$u = \sum_{i=0} p^i u_i = u_0 + pu_1 + p^2 u_2 + \ldots \tag{31}$$

If $p \to 1$, then (33) becomes the approximate solution of the form

$$f = \lim_{p \to 1} u = \sum_{i=0}^{\infty} u_i \tag{32}$$

It is well known that the series (34) is convergent for most of the cases and also the rate of convergence is dependent on $L(u)$. Here, we apply both HPM and HM methods to a representative O.D.E and compare our results.

Consider the following third-order linear differential equation with its boundary conditions at three different points:

$$y''' - k^2 y' + a = 0 \qquad (33)$$
$$y'(0) = y'(1) = 0, \qquad y(0.5) = 0$$

, where the physical constants are k = 5 and a = 1. The function $y(x)$ shows the shear deformation of sandwich beams. The analytical solution of this problem is written as:

$$y(x) = \frac{a}{k^3}(\sinh\frac{k}{2} - \sinh kx) + \frac{a}{k^2}(x - 0.5) + \frac{a}{k^3}\tanh\frac{k}{2}(\cosh(kx) - \cosh\frac{k}{2}) \qquad (34)$$

Using the transformation $dy/dx = u(x)$, $du/dx = q(x)$, $dq/dx = k^2u - a$, we can rewrite the boundary value problem (33) as a system of differential equations:

$$\frac{du}{dx} = q(x) \qquad (35)$$
$$\frac{dq}{dx} = k^2 u - a$$

with $u(0) = 0$, $q(0) = A$, which can be written as a system of integral equations:

$$u(x) = 0 + \int_0^x q(t)dt$$
$$q(x) = A + \int_0^x (k^2 u - a)dt \qquad (36)$$

Using (31) for (36), we have:

$$u_0 + pu_1 + p^2 u_2 + ... = 0 + p\int_0^x (v_0 + pv_1 + p^2 v_2 + ..)dx$$
$$v_0 + pv_1 + p^2 v_2 + ... = A + p\int_0^x (k^2(u_0 + pu_1 + p^2 u_2 + ..) - a)dx \qquad (37)$$

Comparing the coefficient of like powers of p, we have

$$p^{(0)} : \begin{cases} u_0 = 0 \\ v_0 = A \end{cases}$$
$$p^{(1)} : \begin{cases} u_1 = Ax \\ v_1 = -ax \end{cases}$$
$$p^{(2)} : \begin{cases} u_2 = -\frac{1}{2}x^2 \\ c_1 = \frac{25}{2}Ax^2 \end{cases} \qquad (38)$$

$$p^{(3)}:\begin{cases} u_3 = \dfrac{25}{6}Ax^3 \\ c_1 = -\dfrac{25}{6}x^3 \end{cases}$$

$\vdots$

Combining all the terms, (31) gives

$$u := A\,x - \frac{1}{2}x^2 + \frac{25}{6}A\,x^3 - \frac{25}{24}x^4 + \frac{125}{24}A\,x^5 - \frac{125}{144}x^6 + \frac{3125}{1008}A\,x^7 - \frac{3125}{8064}x^8 + \frac{78125}{72576}A\,x^9 - \frac{15625}{145152}x^{10} + \frac{390625}{1596672}A\,x^{11} - \frac{390625}{19160064}x^{12} + \frac{9765625}{249080832}A\,x^{13} - \frac{9765625}{3487131648}x^{14} \quad (39)$$

Using the boundary conditions at $\boldsymbol{u}(1) = 0,\ \boldsymbol{y}(0.5) = 0$, we have

$$A=0.197 \quad (40)$$

The series solution is:

$$u := 0.1974\,x - 0.5000\,x^2 + 0.8224\,x^3 - 1.0417\,x^4 + 1.0280\,x^5 - 0.8681\,x^6 + 0.6119\,x^7 - 0.3875\,x^8 + 0.2125\,x^9 - 0.1076\,x^{10} + 0.0483\,x^{11} - 0.0204\,x^{12} + 0.0077\,x^{13} - 0.0028\,x^{14} \quad (41)$$

Then, by integrating u, according to *dy/dx* = $u(x)$, it is obtained:

$$y(x) := 0.0207\,x^{10} - 0.0431\,x^9 + 0.0745\,x^8 - 0.1240\,x^7 + 0.1669\,x^6 - 0.2083\,x^5 + 0.2002\,x^4 - 0.1667\,x^3 + 0.0961\,x^2 - 0.0111 \quad (42)$$

Higher accuracy can be obtained by evaluating more terms of the solution $u(x)$. In Fig. 4, the error between exact and HPM series solution is plotted. The error is of order 10^{-3}, which confirms the accuracy of HPM solution.

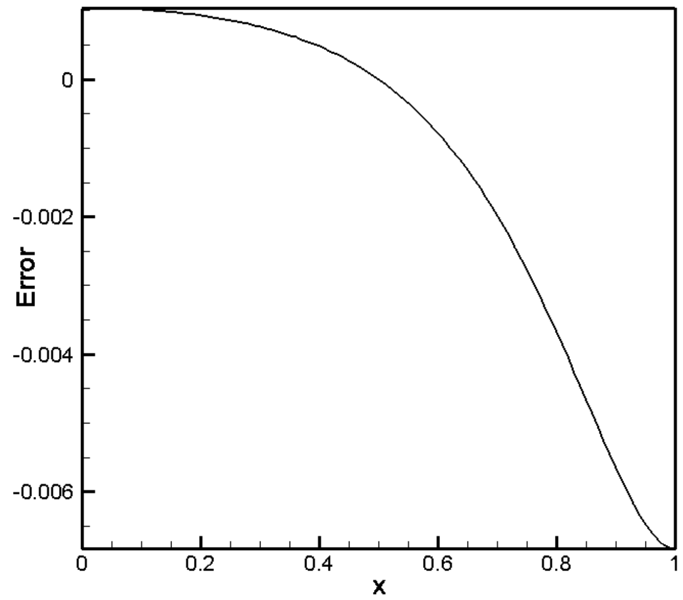

Fig. 4: Error between exact and HPM method.

4.2 Homotopy method

In this section, we apply HM to solve Eq. (33) to get another approximate solution for shear deformation equation. Under the rule of solution expression, we assume the shape of the solution as a polynomial of *x*: $y = \sum a_n x^n$. We take the zero-order solution $y_0 = a + a_1 x + a_2 x^2$. As it is customary in Homotopy method, all the non-homogenous boundary conditions must be satisfied by y_0, but applying the above three boundary conditions results in the entire coefficient to become zero; therefore the form taken for the solution expression must be modified and not include the *x* coefficient; therefore we take the solution expression as: $y = a_0 + \sum_{n=1} a_n x^{2n} + \sum_{n=1} b_n x^{2n+1}$ and $y_0 = a + a_1 x^2 + a_2 x^3$. Applying the boundary conditions give:

$$y_0 = -0.01 + 0.6x^2 - 0.04x^3 \tag{43}$$

We assume the auxiliary function *H* to be 1 and *L* operator as $\frac{d^2}{dx^2}$. If *L* operator contains the first order derivative, the first exponent of *x* will reappear in the solution, which is in contrast with the rule of solution expression. According to higher order deformation equation, other terms of the Homotopy series solution are obtained as:

$$y_1 = h(\frac{19}{50}x^2 - 0.5x^3 + 0.25x^4) \quad (44)$$

$$y_2 = h(h(-\frac{3}{2}x^2 + x^3) - 25h(\frac{19}{150}x^3 - \frac{1}{8}x^4 + \frac{1}{20}x^5)) + h(\frac{19}{50}x^2 - 0.5x^3 + 0.25x^4)$$

$$\vdots$$

Adding up 6 terms of the above series, the homotopy series solution is obtained. In order to find the suitable value of *h*, we evaluate the derivatives of *y* at x=0 and plot them vs. *h*, as shown in Fig.5. Both of second and third derivatives of y suggest the best value of h between [-0.2 0.2].

4.3.Homotopy pade approximation

Using the polynomial base functions, we have obtain the [1, 1] and [2, 2] Homotopy Pade approximate solutions. The details of Homotopy Pade method can be found in [18]. In table 1, a comparison is made between exact, homotopy solution (h=0.01) and the above Pade approximations. Interestingly, the solution of homotopy method and Pade [2, 2] are the more accurate results and [1, 1] Pade approximation gives relatively poor results in comparison with the other solutions.

Table 1: Comparison between Exact, Homotopy and Pade [1, 1] and Pade [2, 2] solutions

X	Exact Solution	Homotopy (h=0.01)	Pade [1,1]	Pade [2,2]
0	-0.01210708561	-0.01000000000	-0.01000000000	-0.01000000000
0.1	-0.01126850725	-0.009420303776	-0.008784528162	-0.009457481016
0.2	-0.00922220621	-0.007851492010	-0.006069954170	-0.008047775751
0.3	-0.00646686795	-0.005546215970	-0.002663935610	-0.006038525263
0.4	-0.00332019477	-0.002753459667	0.001018082195	-0.003620938828
0.5	0.	0.0002814374429	0.004730585111	-0.0009232585198
0.6	0.00332019479	0.003316757912	0.008309100261	0.001972066839
0.7	0.00646686795	0.006114383634	0.01163327602	0.005012991626
0.8	0.0092222062	0.008439773525	0.01461608543	0.008166308162
0.9	0.0112685072	0.01006194107	0.01721079921	0.01141010380
1.0	0.0121070856	0.01075343198	0.01943255819	0.01472958201

5 Conclusion

In this study, different approximate solution methods such as Homotopy method, Homotopy method combined with Pade approximation and Homotopy Perturbation method are used to solve some physical differential equations such as heat transfer problem and shear deformation of beams. Comparison of the applied methods with exact solutions reveals that both methods are greatly effective. It was observed that even using a few terms of H.M. and H.P.M. series gives accurate results. These methods also do not require large computer memory and discretization.

References

[1] A. Aziz, T.Y. Na, Perturbation Method in Heat Transfer, Hemisphere Publishing Corporation, Washington, (1984).

[2] J.H. He, Approximate analytical solution for seepage flow with fractional derivatives in porous media, Comput. Methods Appl. Mech. Eng. 167 (1998) 57–68.

[3] J.H. He, Approximate solution of nonlinear differential equations with convolution product nonlinearities, Comput. Methods Appl. Mech. Eng. 167 (1998) 69–73.

[4] J.H. He, Variational iteration method—a kind of non-linear analytical technique: some examples, Internat. J. Non-linear Mech. 34 (1999) 699–708.

[5] J.H. He, Variational iteration method for autonomous ordinary differential systems, Appl. Math. Comput. 114 (2000) 115–123.

[6] J.H. He, X.H. Wu, Construction of solitary solution and compacton-like solution by variational iteration method, Chaos, Solitons Fractals 29 (2006) 108–113.

[7] S. Abbasbandy, S. Liao. A new modification of false position method based on homotopy analysis method, Appl.Math.& Mech. 2008, 29(2) 223-228

[8] A. Rezania; A. Ghorbali; G. Domairry; H. Bararnia. Consideration of transient heat conduction in a semi-infinite medium using homotopy analysis method, Appl.Math.& Mech. 2008, 29(12) 1625-1632

[9] Jing ZHU, Lian-Cun ZHENG, Xin-Xin ZHANG. Analytical solution to stagnation-point flow and heat transfer over a stretching sheet based on homotopy analysis, Appl.Math. Mech. -Engl. Ed. 2009, 30(4) 463-474

[10] J.H. He, Homotopy perturbation technique, Comput. Methods Appl. Mech. Eng. 178 (1999) 257–262.

[11] J.H. He, A coupling method of a homotopy technique and a perturbation technique for non-linear problems, Internat. J. Non-linear Mech. 35 (2000) 37–43.

[12] J.H. He, Homotopy perturbation method: a new nonlinear analytical technique, Appl. Math. Comput. 135 (2003) 73–79.

[13] J.H. He, The homotopy perturbation method for nonlinear oscillators with discontinuities, Appl. Math. Comput. 151 (2004) 287–292.

[14] J.H. He, Application of homotopy perturbation method to nonlinear wave equations, Chaos, Solitons Fractals 26 (2005) 695–700.

[15] J.H. He, Homotopy perturbation method for bifurcation of nonlinear problems, Internat. J. Nonlinear Sci. Numer. Simul. 6 (2005) 207–208.

[16] E.M. Abulwafa, M.A. Abdou,AA. Mahmoud, The solution of nonlinear coagulation problem with mass loss, Chaos, Solitons Fractals 29 (2006) 313–330.

[17] A.A. Soliman,A numerical simulation and explicit solutions of KdV-Burgers' and Lax's seventh-order KdV equations, Chaos, Solitons Fractals 29 (2006) 294–302.

[18] S.J. Liao, The proposed homotopy analysis technique for the solution of nonlinear problems. Ph.D. Thesis, Shanghai Jiao Tong University, (1992).

[19] D.D. Ganji, A. Sadighi, Application of homotopy-perturbation and variational iteration methods to nonlinear heat transfer and porous media equations, Journal of Computational and Applied Mathematics 207 (2007) 24.

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