Applied Mathematical Sciences, Vol. 5, 2011, no. 22, 1089 - 1102

Precise Asymptotics of Generalized Stochastic Order Statistics for Extreme Value Distributions

Reza Hashemi and Molood Abdollahi

Department of Statistics, Razi University
67149, Kermanshah, Iran
rhashemi@razi.ac.ir

Abstract

In this paper we mainly investigate the precise asymptotics of the generalized stochastic order statistics in the extreme value distributions based on the suitable counting process. In this work, we present the relations among the boundary function, weighted function, convergence rate and limit value of series, this will be enable us to introduce a unified approach to several kinds of ordered random variables.

Keywords: Generalized stochastic order statistics, precise asymptotics, extreme value distributions

1 Introduction and Main Results

Since Hsu and Robbins (1947) introduced the concept of complete convergence, there have been extensions in two directions. Let $\{X, X_k, k \geq 1\}$ be a sequence of i.i.d. random variables, $S_n = \sum_{k=1}^{n} X_k, n \geq 1$, and $\varphi(x)$ and $f(x)$ be the positive functions defined on $[0, \infty)$. One extension is to discuss the moment conditions, from which it follows that

$$\sum_{n=1}^{\infty} \varphi(n) P(|S_n| \geq \epsilon f(n)) < \infty, \quad \epsilon > 0, \tag{1}$$

where $\sum_{n=1}^{\infty} \varphi(n) = \infty$. In this direction, one can refer to Hsu and Robbins (1947), Erdös (1949,1950) and Baum and Katz (1965), etc. They respectively studied the cases in which $\varphi(n) \equiv 1$, $f(n) = n$ and $\varphi(n) = n^{r/p-2}$, $f(n) = n^{1/p}$, where $0 < p < 2$, $r \geq p$.
Another extension departs from the convergence rate and limit value of $\sum_{n=1}^{\infty} \varphi(n) P(|S_n| > \epsilon f(n))$ as $\epsilon \downarrow a$, $a \geq 0$. A first result in this direction was Heyde (1975), who proved

that

$$\lim_{\epsilon \downarrow 0} \epsilon^2 \sum_{n=1}^{\infty} P(|S_n| \geq \epsilon n) = EX^2, \tag{2}$$

where $EX = 0$ and $EX^2 < \infty$. For analogous results in the more general case, see Chen (1978), Gut and Spătaru (2000), Gut (2002), Spătaru (1999).
Research in precise asymptotics has often focused on the part $\sum S_n$ and its corresponding objects, but the samples maximum $M_n = \max_{i \leq n} X_i$, $n \geq 1$ has relatively neglected. An important objective within the extreme value theory is the description of M_n and its applications. On the other hand, M_n is the special case of order statistics. Moreover, order statistics and record values play a crucial role in statistics and its applications; both modes describe random variables arranged in order of magnitude. In this work, we will study the precise asymptotics for generalized order statistics introduced by Kamps (1995), which permits a unified approach to several models of ordered random variables, e.g. (ordinary) order statistics, record values, sequential order statistics, progressive censoring, etc. We first introduce some concepts, notation and basic properties.

Definition 1.1 *The generalized inverse of the distribution function, df, F*

$$\overleftarrow{F}(t) = \inf\{x \in R : F(x) \geq t\}, \quad 0 < t < 1,$$

is called the quantile function of the df F. The quantity $x_t = \overleftarrow{F}(t)$ defines the t-quantile of F.

Definition 1.2 *Say that X (or F) belongs to the maximum domain of attraction of the extreme value distribution $G(x)$, if there exist normalizing constants $c_{1n} > 0$, centering constants $d_n \in R$, $n \geq 1$, and df $G(x)$, satisfying $c_{1n}^{-1}(M_n - d_n) \xrightarrow{d} Z \sim G$, denoted by $X \in MDA(G)$(or $F \in MDA(G)$).*

By the famous Fisher-Tippett theorem (see Embrechts et al, 1997, Theorem 3.2.7), extreme value distributions have only three types: Fréchet distributions, Gumbel distributions and Weibull distributions. It is well known that the standard Fréchet distribution, standard Gumbel distribution, and standard Weibull distributions have $G_{1,\alpha}$, $G_{2,\alpha}$, $G_{3,\alpha}$ forms respectively, where

$$\begin{aligned}
G_{1,\alpha}(x) &= \exp(-x^{-\alpha}) I(x > 0), \quad \text{for some } \alpha > 0 \\
G_{2,\alpha}(x) &= \begin{cases} \exp(-(-x)^{\alpha}) & \text{if } x < 0 \\ 1 & \text{if } x > 0 \end{cases} \\
G_{3,\alpha}(x) &= \exp(-\exp(-x)), \quad x \in R.
\end{aligned}$$

Definition 1.3 *Let F be a df with right endpoint $x_F \leq \infty$. Suppose there exists some $z < x_F$ such that F has representation*

$$\bar{F}(x) = c \exp\{-\int_z^x \frac{1}{a(t)} dt\}, \quad z < x < x_F,$$

where c is some positive constant, $a(.)$ is a positive and absolutely continuous function (with respect to the Lebesque measure) with density a' and $\lim_{x\uparrow x_F} a'(x) = 0$. Then F is called a Von Mises function, the function $a'(.)$ is the auxiliary function of F.

Definition 1.4 *Let $n \in N$, $K > 0$, $m_1, \ldots, m_{n-1} \in R$, $M_r = \sum_{j=r}^{n-1} m_j$, $1 \le r \le n-1$, be parameters such that $\gamma_r = k + n - r + M_r \ge 1$ for all $r \in \{1, \ldots, n-1\}$, and let $\widetilde{m} = (m_1, \ldots, m_{n-1})$, if $n \ge 2$, $\widetilde{m} \in R$ arbitrary, if $n = 1$. If the random variables $U(r, n, \widetilde{m}, k)$, $r = 1, \ldots, n$, possess a joint density function of the form*

$$f^{U(1,n,\widetilde{m},k),\ldots,U(n,n,\widetilde{m},k)}(u_1, \ldots, u_n) = k \left(\prod_{j=1}^{n-1} \gamma_j\right) \left(\prod_{i=1}^{n-1} (1-u_i)^{m_i}\right) (1-u_n)^{k-1},$$

on the cone $0 \le u_1 \le \ldots \le u_n \le 1$, then they are called uniform generalized order statistics.

Let F be an arbitrary distribution function. The random variables $X(r, n, \widetilde{m}, k) = F^{-1}(U(r, n, \widetilde{m}, k))$, $r = 1, \ldots, n$, are called generalized order statistics based on F in the particular case $m_1 = m_2 = \ldots = m_{n-1} = m$ variables are denoted by $U(r, n, m, k)$ and $X(r, n, m, k)$, $r = 1, \ldots, n$.
The parameters k, m_j, $j = 1, \ldots, n-1$, determine the model of ordered random variables. For example, in the case $k = 1$, $m_j = -1$ one gets record values. A further example can be found in Kamps (1995).
The paper of Nasri-Roudsari (1996) started to develop the extreme value theory of generalized order statistics. It was shown that well known results of extreme value theory for ordinary order statistics in the (weak) domain of attraction and normalizing constants carry over to generalized order statistics. In particular, the possible limit distribution of extreme generalized order statistics (called generalized extreme value distribution) were established (Nasri-Roudsari, 1996, Theorem 3.3) and it was shown that an underlying distribution function of an i.i.d. sequence $X_1, X_2, \ldots$ belongs to the (weak) domain of attraction of an extreme value distribution (Nasri-Roudsari, 1996, Corollary 3.7). As in Marohn (2002) and Nasri-Roudsari and Cramer (1999), we assume in the following that the underlying parameters $m_1, \ldots, m_{n-1}$ of the generalized order statistics are equal, i.e. $m_i = m > -1$ such a condition seems to be restrictive; nevertheless, various interesting models are still included. For a discussion we refer to the Yan (2008), in his paper the precise asymptotics of generalized stochastic order statistics for the Frechet distribution which is one of the three possible distributions in the extreme value distributions is presented. In this paper we are mainly investigate the generalization of the last mentioned paper to all of the extreme value distributions. Our main results are as follows:
Let $g_1(x)$ and $h(x)$ be positive and differentiable functions defined on $[n_0, \infty)$, which are both strictly increasing to ∞, $\varphi(x) = {g'}_1(h(x))h'(x)$ is monotone, in the case

$\varphi(x)$ be monotone nondecreasing, we assume $\lim_{n\to\infty}(\varphi(n+1)/\varphi(n)) = 1$. In addition assume that the underlying distribution function $F \in MDA(G_{2,\alpha})$, $\alpha > 0$, and counting process $N(t)$, $t > 0$, satisfies

$$\begin{gathered} t^{-1}N(t) \xrightarrow{P} \lambda > 0, \quad t > 0 \\ \sup_{t>0} \frac{E(N(t))^{\tau}}{t^{\tau}} < \infty, \quad \text{for fixed } r \geq 1 \end{gathered} \tag{3}$$

where $\tau = \frac{k}{m+1} + r - 1$. And assume that $g(x)$, $x \geq n_0$, satisfy the following conditions: footnotesize

$\forall \epsilon > 0$,

$$G_0(\epsilon) := \begin{cases} \frac{1}{\Gamma(\tau)} \int_{-\infty}^{\lambda(-\epsilon h(n_0))^{\alpha(m+1)}} g_1\left(-\epsilon^{-1}(y\lambda^{-1})^{1/\alpha(m+1)}\right) y^{\tau-1} e^{-y} dy & \text{if } \alpha(m+1) \text{ is odd} \\ \frac{1}{\Gamma(\tau)} \int_{\lambda(-\epsilon h(n_0))^{\alpha(m+1)}}^{\infty} g_1\left(-\epsilon^{-1}(y\lambda^{-1})^{1/\alpha(m+1)}\right) y^{\tau-1} e^{-y} dy & \text{if } \alpha(m+1) \text{ is even} \end{cases} \tag{4}$$

such that

$$\begin{gathered} G_0(\epsilon) < \infty, \\ \text{and } \lim_{\epsilon \downarrow 0} G_0(\epsilon) = \infty. \end{gathered} \tag{5}$$

And for all $G(\epsilon) \sim (G_0(\epsilon))^{-1}$, $\epsilon \downarrow 0$,

$$\begin{cases} \lim_{M\to\infty} \lim_{\epsilon\downarrow 0} G(\epsilon) \int_{-\infty}^{\lambda(-\epsilon g_1^{-1}(MG_0(\epsilon))^{\alpha(m+1)}} g_1\left(-\epsilon^{-1}(y\lambda^{-1})^{1/\alpha(m+1)}\right) y^{\tau-1} e^{-y} dy = 0 & \text{if } \alpha(m+1) \text{ is odd} \\ \lim_{M\to\infty} \lim_{\epsilon\downarrow 0} \int_{\lambda(-\epsilon g_1^{-1}(MG_0(\epsilon))^{\alpha(m+1)}}^{\infty} g_1\left(-\epsilon^{-1}(y\lambda^{-1})^{1/\alpha(m+1)}\right) y^{\tau-1} e^{-y} dy = 0 & \text{if } \alpha(m+1) \text{ is even,} \end{cases} \tag{6}$$

for some $\theta > 0$,

$$\lim_{M\to\infty} \lim_{\epsilon\downarrow 0} G(\epsilon) \epsilon^{-(\alpha-\theta)(k+(r-1)(m+1))} \int_{g_1^{-1}(G_0(\epsilon)M)}^{\infty} y^{-(\alpha-\theta)(k+(r-1)(m+1))} dg_1(y) = 0 \tag{7}$$

where $g_1^{-1}(x)$, $h^{-1}(x)$ are the inverse functions of $g_1(x)$ and $h_1(x)$, $M > 1$.

Theorem 1.1 *For the appropriate choice of functions φ and h as above, if the counting process $N(t)$ satisfies (3), G_0 satisfies (4) and (5) and G satisfies the equations (6) and (7), then we can write*

$$\lim_{\epsilon\downarrow 0} G(\epsilon) \sum_{n=n_0}^{\infty} \varphi(x) P\Big(X(N(n)-r+1, N(n), m, k) - d_n \geq \epsilon h(n) c_n\Big) = 1, \tag{8}$$

where d_n and c_n can be chosen as in the following theorem(2.1).

Now let G', satisfies the following conditions: $\forall\epsilon > 0$,

$$G'_0(\epsilon) := \frac{1}{\Gamma(\tau)}\int_0^{\lambda\exp\{-(m+1)\epsilon h(n_0)\}} g_1\Big(\frac{\epsilon^{-1}}{m+1}ln(\frac{\lambda}{y})\Big)y^{\tau-1}e^{-y}dy < \infty, \tag{9}$$

$$\lim_{\epsilon\downarrow 0} G'_0(\epsilon) = \infty, \tag{10}$$

$$G'(\epsilon) \sim (G'_0(\epsilon))^{-1}, \quad \epsilon\downarrow 0, \tag{11}$$

$$\lim_{M\to\infty}\lim_{\epsilon\downarrow\emptyset} G'(\epsilon)\int_0^{\lambda\exp\{-(m+1)\epsilon g_1^{-1}(G'_0(\epsilon)M)\}} g_1\Big(\frac{\epsilon^{-1}}{m+1}ln(\frac{\lambda}{y})\Big)y^{\tau-1}e^{-y}dy = 0, \tag{12}$$

$$\begin{cases} \lim_{M\to\infty}\lim_{\epsilon\downarrow\emptyset} G'(\epsilon)\int_{h^{-1}og_1^{-1}(G'(\epsilon)M)}^{\infty} \varphi(x)\Big(c(\epsilon h(x)c'_x + d'_x)\Big)^{k+(m+1)(r-1)} \\ \times\exp\left\{-(k+(m+1)(r-1))\int_z^{\epsilon h(x)c'_x+d'_x}\frac{g(t)}{a(t)}\right\}dx = 0, \\ or \\ \lim_{M\to\infty}\lim_{\epsilon\downarrow\emptyset} G'(\epsilon)\int_{h^{-1}og_1^{-1}(G'(\epsilon)M)}^{\infty} \varphi(x) \\ \times\exp\left\{(k+(m+1)(r-1))\Big(t - \int_z^{(\epsilon h(x)c'_x+d'_x)+t\tilde{a}(\epsilon h(x)c'_x+d'_x)}\frac{1}{a(u)}\Big)du = 0\right\}, \end{cases} \tag{13}$$

where $z < x < x_F$, and $g_1^{-1}(x)$, $h^{-1}(x)$ are the inverse functions of $g_1(x)$ and $h(x)$, and $M > 1$, $c(.)$ and $g(.)$ are measurable functions satisfying $c(x) \longrightarrow c > 0$, $g(x) \longrightarrow 1$ as $x \uparrow x_F$, and $a(x)$ is a positive, absolutely continuous function (with respect to the Lebesque measure).

Theorem 1.2 *Let φ defined as above, and assume that the underlying distribution function $F \in MDA(G_{3,\alpha})$, $\alpha > 0$, and the counting process $N(t)$, $t > 0$, satisfies the conditions defined in (3). In addition choosing G' as in the equation (13). Then we can write*

$$\lim_{\epsilon\downarrow 0} G'(\epsilon)\sum_{n=n_0}^{\infty}\varphi(x)P\Big(X(N(n)-r+1, N(n), m, k) - d'_n \geq \epsilon h(n)c'_n\Big) = 1, \tag{14}$$

where d'_n and c'_n as in the following proposition.

Remark 1.1 *We argue that the theorem of Yan, Wang and Cheng (2006) can be viewed as a particular case of theorems (1.1) and (1.2).*

2 Some Lemmas and Theorems

In this section we review some necessary theorems and lemmas.

Theorem 2.1 (Theorem 3.3.12 in Embrechts) *The df F belongs to the maximum domain of attraction of $G_{2,\alpha}$, $\alpha > 0$, if and only if $x_F < \infty$ and $\bar{F}(x_F - x^{-1}) = x^{-\alpha}L(x)$ for some slowly varying function L. If $F \in MDA(G_{2,\alpha})$, then $c_n^{-1}(M_n - x_F) \xrightarrow{d} G_{2,\alpha}$, where the normalizing constants c_n can be chosen as $c_n = x_F - \overleftarrow{F}(1 - n^{-1})$ and $d_n = x_F$ and $F \in MDA(G_{2,\alpha}) \Longleftrightarrow x_F < \infty$, $\bar{F}(x_F - x^{-1}) \in R_{-\alpha}$.*

Proposition 2.1 **(Proposition 3.3.25 in Embrechts)** *Suppose the df F is a von Misess function. Then $F \in MDA(G_{3,\alpha})$. A possible choice of norming constants is*

$$d_n = \overleftarrow{F}(1 - n^{-1}) \text{ and } c_n = a(d_n), \tag{15}$$

where a is the auxiliary of F.

Theorem 2.2 (Theorem 3.3.26 in Embrechts) *The df F with right endpoint $x_f \leq \infty$ belongs to the maximum domain of attraction of $G_{3,\alpha}$, $\alpha > 0$, if and only if there exists some $z < x_F$ such that F has representation*

$$\bar{F}(x) = c(x)\exp\left\{-\int_z^x \frac{g(t)}{a(t)}dt\right\}, \quad z < x < x_F, \tag{16}$$

where c and g are measurable functions satisfying $c(x) \longrightarrow c > 0$, $g(x) \longrightarrow 1$ as $x \uparrow x_F$, and $a(x)$ is a positive, absolutely continuous function (with respect to Lebesque measure) with density $a'(x)$ having $\lim_{x\uparrow x_F} a'(x) = 0$. We can choose in this case,

$$d'_n = \overleftarrow{F}(1 - n^{-1}) \text{ and } {c'}_n = a({d'}_n). \tag{17}$$

A possible choice for the function a is

$$a(x) = \int_x^{x_f} \frac{\bar{F}(t)}{\bar{F}(x)}dt, \quad x < x_F, \tag{18}$$

motivated by von Misess functions, we call the function a in (17) an auxiliary function for F.
For a r.v. X the function $a(x)$ defined in (18) is nothing but the mean excess function

$$a(x) = E(X - x|X > x), \quad x < x_F; \tag{19}$$

(see also section 3.4 of Embrechts).

Theorem 2.3 (Theorem 3.3.27 in Embrechts) *The df F belongs to the maximum domain of attraction of $G_{3,\alpha}$, $\alpha > 0$, if and only if there exists some positive function $\tilde{a}$ such that as $x \uparrow x_F$,*

$$\frac{\bar{F}(x + t\tilde{a}(x))}{\bar{F}(x)} = e^{-t}, \quad t \in R. \tag{20}$$

A possible choice is $\tilde{a} = a$ as given in (18).

LEMMA 2.1 *Suppose that the counting process $N(t)$, $t > 0$, satisfies $t^{-1}N(t) \stackrel{P}{\longrightarrow} \lambda > 0$, $t \longrightarrow \infty$, $\sup_{t>0} \frac{E(N(t))^\tau}{t^\tau} < \infty$, for fixed $r \geq 1$, and $F \in MDA(G)$. Then there exist $\alpha_n > 0$, $\beta_n \in R$, $n \geq 1$ such that when $n \longrightarrow \infty$,*

$$\Delta_{n,r} = \sup_x \Big| P\Big(\alpha_n^{-1}(X(N(n) - r + 1, N(n), m, k) - \beta_n) > x\Big) - \bar{H}_{r,m,k,\lambda}(x) \Big| \longrightarrow 0. \tag{21}$$

Here $H_{r,m,k,\lambda}(x) = \frac{1}{\Gamma(\tau)}\Gamma(\tau, \lambda(-\log G(x))^{m+1})$ and $\Gamma(\alpha, x)$ denotes the incomplete gamma function which is defined by $\Gamma(\alpha, x) = \int_x^\infty t^{\alpha-1}e^{-t}dt$, $\alpha > 0$, $x \geq 0$.

LEMMA 2.2 *Suppose that $F \in MDA(G_{2,\alpha})$, $\alpha > 0$. Then for $\forall \epsilon > 0$, $\forall \theta > 0$, $\forall r \geq 1$, there exists $0 < c < \infty$ such that*

$$P\Big(X(N(n) - r + 1, N(n), m, k) - d_n > \epsilon h(n)c_n\Big) \leq c(\epsilon h(n))^{-(\alpha-\theta)(k+(r-1))(m+1)}.$$

Proof. By Lemma 2.5 in Nasri-Roudsari (1996), for every $i \geq r$,

$$\begin{aligned}
P \quad & \Big(X(i - r + 1, i, m, k) - d_n \geq \epsilon h(n)c_n\Big) = \\
1 \quad - \quad & I_{1-(1-F(\epsilon h(n)c_n+d_n))^{M+1}}(i - r + 1, \frac{k}{m+1} + r - 1), \quad r = 1, \ldots, n \\
\leq \quad & c\Big(\bar{F}(\epsilon h(n)c_n + d_n)\Big)^{k+(m+1)(r-1)} (i - r + 1)^{\frac{k}{m+1}+r-1} \\
\leq \quad & c\Big(\bar{F}(\epsilon h(n)c_n + d_n)\Big)^{k+(m+1)(r-1)} i^{\frac{k}{m+1}+r-1}.
\end{aligned} \tag{22}$$

By (21) and the basic renewal theorem, we have

$$\begin{aligned} P & \Big(X(N(n)-r+1,N(n),m,k)-d_n \geq \epsilon h(n)c_n\Big) = \\ \sum_{i=r}^{\infty} & \Big(1-I_{1-(1-F(\epsilon h(n)c_n+d_n))^{M+1}}(i-r+1,\frac{k}{m+1}+r-1)\Big) \\ \leq & \quad c\Big(\bar{F}(\epsilon h(n)c_n+d_n)\Big)^{k+(m+1)(r-1)}\sum_{i=r}^{\infty} i^{\frac{k}{m+1}+r-i}P(N(n)=i) \\ \leq & \quad c\Big(\bar{F}(\epsilon h(n)c_n+d_n)\Big)^{k+(m+1)(r-1)}\sum_{i=r}^{\infty} E(N(n))^{\frac{k}{m+1}+r-1} \\ \leq & \quad c\Big(\bar{F}(\epsilon h(n)c_n+d_n)\Big)^{k+(m+1)(r-1)}\sum_{i=r}^{\infty} n^{\frac{k}{m+1}+r-1}. \end{aligned} \tag{23}$$

Since $F \in MDA(G_{2,\alpha})$, there exists a function $L \in R_0$, such that $\bar{F}(x_F - x^{-1}) = x^{-\alpha}L(x)$.
Suppose $x_F < \infty$ and define $F_\star(x) = F(x_F - x^{-1})$, $x > 0$, then $\bar{F}_\star \in R_{-\alpha}$, then $F_\star \in MDA(G_{2,\alpha})$ with normalizing constants $c^\star_n = \overleftarrow{F}(1-n^{-1})$ and $d^\star_n = 0$.
Then $\bar{F}_\star(x) = x^{-\alpha}L(x)$, $x > 0$ and $\bar{F}_\star(c^\star_n) \sim n^{-1}$, $n \longrightarrow \infty$, then $\bar{F}_\star(\epsilon h(x)c^\star_n) = \bar{F}(\epsilon h(x)c_n + d_n)$.
By (22), (23) and potter's theorem (Bingham(Theorem 1.5.6)), for $\forall\theta > 0$, we have

$$\begin{aligned} P & \Big(X(N(n)-r+1,N(n),m,k) > \epsilon h(x)c_n + d_n\Big) \\ \leq & \; c\left(\Big(\epsilon h(n)c^\star_n\Big)^{-\alpha}L\Big(\epsilon h(n)c^\star_n\Big)\right)^{k+(r-1)(m+1)}\Big(c_n^{\star^{\alpha}}(L(c^\star_n))^{-1}\Big)^{k+(r-1)(m+1)} \\ = & \; c\epsilon^{-\alpha(k+(r-1)(m+1))}(h(n))^{-\alpha(k+(r-1)(m+1))}\Big(\frac{L(\epsilon h(n)c^\star_n)}{L(c^\star_n)}\Big)^{k+(r-1)(m+1)} \\ \leq & \; c\epsilon^{-\alpha(k+(r-1)(m+1))}(h(n))^{-\alpha(k+(r-1)(m+1))}\Big(\epsilon h(n)\Big)^{(k+(r-1)(m+1))} \\ = & \; c\Big(\epsilon h(n)\Big)^{-(\alpha-\theta)(k+(r-1)(m+1))}. \end{aligned} \tag{24}$$

LEMMA 2.3 *Suppose that* $F \in MDA(G_{3,\alpha})$, $\alpha > 0$. *Then for* $\forall\epsilon > 0$, $\forall r \geq 1$, *there exists* $0 < c < \infty$, c'_n, d'_n *and* $c(.)$ *and* $g(.)$ *such that* $c'_n = a(d'_n)$ *and* $d'_n = \overleftarrow{F}(1-n^{-1})$ *and* $c(.)$, $g(.)$ *are measurable functions satisfying* $c(x) \longrightarrow c > 0$, $g(x) \longrightarrow 1$ *as* $x \uparrow x_F$, *and* $a(x)$ *is a positive, absolutely continuous function (with respect to the Lebesque measure) with density* $a'(x)$ *having limiting* $\lim_{x\uparrow x_F} a'(x) = 0$,

$x_F \le \infty$,

$$\begin{aligned} &P \quad \Big(X(N(n)-r+1, N(n), m, k) - d'_n \ge \epsilon h(n) c'_n\Big) \\ &\le \quad c\Big(c\big(\epsilon h(n)c'_n + d'_n\big)\Big)^{k+(m+1)(r-1)} \\ &\times \quad \exp\Big\{-(k+(m+1)(r-1))\int_z^{\epsilon h(n)c'_n + d'} \frac{g(t)}{a(t)}dt\Big\}, \quad z < t < x_F. \end{aligned}$$

Proof. By Lemma 2.5 in Nasri-Roudsari (1996), for $\forall i \ge r$,

$$\begin{aligned} &P \quad \Big(X(i-r+1, i, m, k) - d'_n \ge \epsilon h(n) c'_n\Big) \\ &= \quad 1 - I_{1-\Big(1-F(\epsilon h(n)c'_n + d'_n)\Big)^{m+1}}\Big(i-r+1, \frac{k}{m+1}+r-1\Big), \quad r = 1, \dots, n \\ &\le \quad c\Big(\bar{F}(\epsilon h(n)c'_n + d'_n)\Big)^{k+(m+1)(r-1)}(i-r+1)^{\frac{k}{m+1}+r-1} \\ &\le \quad c\Big(\bar{F}(\epsilon h(n)c'_n + d'_n)\Big)^{k+(m+1)(r-1)} i^{\frac{k}{m+1}+r-1}. \end{aligned} \tag{25}$$

By (21) and basic renewal theorem, we have

$$\begin{aligned} &P \quad \Big(X(N(n)-r+1, N(n), m, k) - d'_n \ge \epsilon h(n) c'_n\Big) \\ &= \quad \sum_{i=r}^{\infty}\Big(1 - I_{1-\Big(1-F(\epsilon h(n)c'_n + d'_n)\Big)^{m+1}}\Big(i-r+1, \frac{k}{m+1}+r-1\Big)\Big) \\ &\le \quad c\Big(\bar{F}(\epsilon h(n)c'_n + d'_n)\Big)^{k+(m+1)(r-1)} \sum_{i=r}^{\infty} i^{\frac{k}{m+1}+r-1} P\big(N(n) = i\big) \\ &\le \quad c\Big(\bar{F}(\epsilon h(n)c'_n + d'_n)\Big)^{k+(m+1)(r-1)} E(N(n))^{\frac{k}{m+1}+r-1} \\ &\le \quad c\Big(\bar{F}(\epsilon h(n)c'_n + d'_n)\Big)^{k+(m+1)(r-1)} n^{k+(m+1)(r-1)} \\ &= \quad c\Big(c(\epsilon h(n)c'_n + d'_n)\Big)^{k+(m+1)(r-1)} \\ &\times \quad \exp\Big\{-(k+(m+1)(r-1))\int_z^{\epsilon h(n)c'_n + d'} \frac{g(t)}{a(t)}dt\Big\}. \end{aligned} \tag{26}$$

LEMMA 2.4 *Suppose $F \in MDA(G_{3,\alpha})$, $\alpha > 0$. Then for $\forall \epsilon > 0$, $\forall r \ge 1$, there exists $0 < c < \infty$, and c'_n and d'_n and some positive function $\tilde{a}$ such that*

$$\begin{aligned} &P \quad \Big(X(N(n)-r+1, N(n), m, k) - d'_n \ge \epsilon h(n) c'_n\Big) \\ &\le \quad c\exp\Big(-(k+(m+1)(r-1))\int_z^{(\epsilon h(n)c'_n + d') + t\tilde{a}(\epsilon h(n)c'_n + d')} \frac{1}{a(u)}du\Big), \quad t \in R,\ u < x_F, \end{aligned}$$

where, a possible choice is $\tilde{a} = a$ as given in (18).
The proof is similar to proof of Lemma (2.3), we omit it.

3 Proof of Results

Proof of Theorem (1.1) is similar to proof of theorem (1.1) Jiago Yan, so we omit it.
Proof of Theorem (1.1). When $\varphi(x)$ is non increasing, we have

$$\sum_{n=n_0}^{\infty} \int_n^{n+1} \varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx = \int_{n_0}^{\infty} \varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx \quad (27)$$

$$\sum_{n=n_0+1}^{\infty} \varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n)) \leq \int_{n_0}^{\infty} \varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx \quad (28)$$

$$\int_{n_0}^{\infty} \varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx \leq \sum_{n=n_0}^{\infty} \varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n)). \quad (29)$$

By integration by parts and (27), (28), (29), we have

$$\begin{aligned}
\lim\nolimits_{\epsilon\downarrow 0} \quad & G'(\epsilon)\sum_{n=n_0}^{\infty} \varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n)) \\
= \quad & \lim_{\epsilon\downarrow 0} G'(\epsilon)\int_{n=n_0}^{\infty} \varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx \\
= \quad & \lim_{\epsilon\downarrow 0} G'(\epsilon)\int_{n=n_0}^{\infty} \bar{H}_{r,m,k,\lambda}(\epsilon h(x))dg_1(h(x)) \\
= \quad & \lim_{\epsilon\downarrow 0} G'(\epsilon)\int_{\epsilon h(n_0)}^{\infty} \bar{H}_{r,m,k,\lambda}(t)dg_1(\epsilon^{-1}t) \\
= \quad & \lim_{\epsilon\downarrow 0} G'(\epsilon)\Big[\bar{H}_{r,m,k,\lambda}(t)g_1(\epsilon^{-1}t)\big|_{\epsilon h(n_0)}^{\infty} + \int_{\epsilon h(n_0)}^{\infty} g_1(\epsilon^{-1}t)dH_{r,m,k,\lambda}(t)\Big] \\
= \quad & \lim_{\epsilon\downarrow 0} G'(\epsilon)\Big[\lim \bar{H}_{r,m,k,\lambda}(t)g_1(\epsilon^{-1}t) + \frac{\lambda^t}{\Gamma(\tau)}\int_0^{\lambda\exp\{-(m+1)\epsilon h(n_0)\}} g_1(\epsilon^{-1}t)e^{-\tau t}e^{-\lambda e^{-t}}dt\Big] \\
= \quad & \lim_{\epsilon\downarrow 0} G'(\epsilon)\Big[\lim_{t\longrightarrow\infty} \bar{H}_{r,m,k,\lambda}(t)g_1(\epsilon^{-1}t) + \frac{1}{\Gamma(\tau)}\int_0^{\lambda\exp\{-(m+1)\epsilon h(n_0)\}} g_1(\frac{\epsilon^{-1}}{m+1}ln\frac{\lambda}{y})y^{\tau-1}e^{-y}dy\Big] \\
= \quad & \lim_{\epsilon\downarrow 0} G'(\epsilon)\lim \bar{H}_{r,m,k,\lambda}(t)g_1(\epsilon^{-1}t) + 1.
\end{aligned}$$

$$\bar{H}_{r,m,k,\lambda}(t) = \frac{1}{\tau}\int_0^{\lambda e^{-t(m+1)}} x^{\tau-1}e^{-x}dx \sim \frac{\tau^{-1}}{\Gamma(\tau)}\Big(\lambda e^{-t(m+1)}\Big)^{\tau} e^{-\lambda e^{-t(m+1)}}, \quad t\longrightarrow\infty.$$

By (9), we have

$$\begin{aligned}
\frac{1}{(k+(m+1)(r-1))} & g_1 \quad (\epsilon^{-1}t)e^{-t(k+(m+1)(r-1))}e^{-\lambda e^{-t(m+1)}} \\
& = \quad g_1(\epsilon^{-1}t)e^{-\lambda e^{-t(m+1)}} \int_t^\infty e^{-u(k+(m+1)(r-1))}du \\
& \leq \quad \int_t^\infty g_1(\epsilon^{-1}t)e^{-\lambda e^{-t(m+1)}} e^{-u(k+(m+1)(r-1))}du \longrightarrow 0, t \longrightarrow \infty.
\end{aligned}$$

Hence we get

$$\lim_{\epsilon\downarrow 0} G'(\epsilon) \sum_{n=n_0}^{\infty} \varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n)) = 1. \tag{30}$$

When $\varphi(x)$ is nondecreasing, by assumption of Theorem (1.2), we know for $\forall\delta > 0$, there exists $n_1 \in N$, such that for $\forall n \geq n_1$,

$$\varphi(n) \leq \varphi(n+1) \leq (1+\delta)\varphi(n). \tag{31}$$

Then we have

$$\begin{aligned}
(1+\delta)^{-1} \quad & \sum_{n=n_1+1}^{\infty} \quad \varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n)) \\
& \leq \quad \int_{n_1}^\infty \varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx \\
& \leq \quad (1+\delta) \sum_{n=n_1}^{\infty} \varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n)).
\end{aligned} \tag{32}$$

Similarly, we get (30).
Let $b(\epsilon) = h^{-1}\left(g_1^{-1}(G_0(\epsilon)M)\right)$, $\forall ep > 0$, $\forall M > 1$.
By Lemma 2.3 in Wang and Yang (2003), for ϵ small enough, if $\varphi(x)$ is nondecreasing, then by (31) we have

$$4G_0(\epsilon)M \geq \sum_{n=n_0}^{[b(\epsilon)]} \varphi(n) \geq (1+\delta)^{-1} \sum_{n=n_0}^{[b(\epsilon)]+1} \varphi(n) - \varphi(n_1). \tag{33}$$

If $\varphi(x)$ is non increasing, then

$$4G_0(\epsilon)M \geq \sum_{n=n_0}^{[b(\epsilon)]+1} \varphi(n) - \varphi(n_0). \tag{34}$$

By Lemma (2.1) and Lemma 2.1 of Jiago Yan (2008) and Toeplicz's lemma, we have

$$\lim_{\epsilon\downarrow 0} G(\epsilon)\sum_{n=n_0}^{[b(\epsilon)]+1}\varphi(n)\Delta_{n,r}=0. \tag{35}$$

To prove (14), it suffices to show that

$$\lim_{\epsilon\downarrow 0} G(\epsilon)\sum_{n=[b(\epsilon)]+2}^{\infty}\varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n))=0. \tag{36}$$

$$\lim_{\epsilon\downarrow 0} G(\epsilon)\sum_{n=[b(\epsilon)]+2}^{\infty}\varphi(n)P\left(X(N(n)-r+1,N(n),m,k)-d'_n>\epsilon h(n)c'_n\right)=0. \tag{37}$$

If $\varphi(x)$ is nondecreasing, then for every ϵ small enough, we have

$$\int_{b(\epsilon)}^{\infty}\varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx\geq(1+\delta)^{-1}\sum_{n=[b(\epsilon)]+2}^{\infty}\varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n)). \tag{38}$$

If $\varphi(x)$ is nondecreasing, then

$$\int_{b(\epsilon)}^{\infty}\varphi(x)\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx\geq\sum_{n=[b(\epsilon)]+2}^{\infty}\varphi(n)\bar{H}_{r,m,k,\lambda}(\epsilon h(n). \tag{39}$$

By integration by parts, (38), (39), we get

$$\begin{aligned}
G(\epsilon)\int_{b(\epsilon)}^{\infty}\varphi(x)\quad & \bar{H}_{r,m,k,\lambda}(\epsilon h(x))dx\\
&= G(\epsilon)\int_{b(\epsilon)}^{\infty}\bar{H}_{r,m,k,\lambda}(\epsilon h(x))dg_1(h(x))\\
&= G(\epsilon)\int_{\epsilon h(b(\epsilon))}^{\infty}\bar{H}_{r,m,k,\lambda}(t)dg_1(\epsilon^{-1}t)\\
&\leq c\;G(\epsilon)\int_{\epsilon h(b(\epsilon))}^{\infty}g_1(\epsilon^{-1}t)dH_{r,m,k,\lambda}(t)\\
&= c\int_0^{\lambda\exp\{-(m+1)\epsilon g_1^{-1}(G'_0(\epsilon)M)\}}g_1\left(\frac{\epsilon^{-1}}{m+1}ln\frac{\lambda}{y}\right)y^{\tau-1}e^{-y}dy.
\end{aligned}\tag{40}$$

Hence by (12), we get (36).
By Lemma (2.3), (2.4), we prove (13).

$$
\begin{aligned}
G'(\epsilon) &\sum_{n>b(\epsilon)+1} \varphi(n) P\Big(X(N(n)-r+1, N(n), m, k) - d'_n > \epsilon h(n) c'_n\Big) \\
\leq\ & c\ G'(\epsilon) \sum_{n>b(\epsilon)+1} \varphi(n) \Big(c(\epsilon h(n) c'_n + d'_n)\Big)^{k+(m+1)(r-1)} \\
\times\ & \exp\Big\{-(k+(m+1)(r-1)) \int_z^{\epsilon h(n) c'_n + d'_n} \frac{g(t)}{a(t)} dt\Big\} \\
\leq\ & c\ G'(\epsilon) \int_{b(\epsilon)}^{\infty} \varphi(n) \Big(c(\epsilon h(n) c'_n + d'_n)\Big)^{k+(m+1)(r-1)} \\
\times\ & \exp\Big\{-(k+(m+1)(r-1)) \int_z^{\epsilon h(x) c'_x + d'_x} \frac{g(t)}{a(t)} dt\Big\} dx \\
\leq\ & c\ G'(\epsilon) \int_{h^{-1} o g^{-1}(G'_0(\epsilon) M)}^{\infty} \varphi(x) \Big(c(\epsilon h(x) c'_x + d'_x)\Big)^{k+(m+1)(r-1)} \\
\times\ & \exp\Big\{-(k+(m+1)(r-1)) \int_z^{\epsilon h(x) c'_x + d'_x} \frac{g(t)}{a(t)} dt\Big\} dx.
\end{aligned}
$$

By (13), we get (37).
The proof of other part of Theorem (1.2) is similar above, we omit it.

References

N.H. Bingham, C.M. Goldie, J.L. Teúgels, Regular Variation, Cambridge University Press, Cambridge, 1978.

L.E. Baum, M. Katz, Convergence Rates in the Law of Large Numbers, Trans. Amer. Math. Soc. 120 (1965) 108-123.

R. Chen, A Remark on the Tail Probability of a Distribution, J. Multivariate Anal. 8 (1978) 328-333.

P. Embrechts, C. Klüppelberg, T. Minkosch, Modeling Extremal Events for Insurance and Finance, Springer, Berlin, 1997.

P. Erdös, On a Theorem of Hsu and Robbins, Ann. Math. Statist. 20 (1949) 286-291.

P. Erdös, Remark on my Paper on a Theorem of Hsu and Robbins, Ann. MAth. Statist. 21 (1950) 138.

J. Galambos, The Asymptotic Theory of Extreme Order Statistics. Second ed., Krieger Publishing Co., 2001.

A. Gut, Precise Asymptotics for Record Times and the Associated Counting Process, Stochastic Process. Appl. 101 (2002) 233-239.

A. Gut, A. Spătaru, Precise Asymptotics in the Baum-Katz and Davis Law of Large Numbers, J. Math. Anal. Appl. 248 (2000) 233-246.

A. Gut, A. Spătaru, Precise Asymptotics in the Law of the Iterated Logarithm, Ann. Probab. 28 (2000) 1870-1883.

C.C. Heyde, A Supplement to the Strong Law of Large Numbers, J. Appl. Probab. 12 (1975) 173-175.

P.L. Hsu, H. Robbins, Complete Convergence and the Strong Law of Large Numbers, Proc. Natl. Acad. Sci. USA 33 (1947) 25-31.

U. Kamps, A concept of Generalized Order Statistics, J. Statist. Plann. Inference 48 (1995) 1-23.

F. Maron, Strong Domain of Attraction of Extreme Generalized Order Statistics, Extremes 5 (2002) 369-386.

D. Nasri-Roudsari, Extreme Value Theory of Generalized Order Statistics, J. Statist. Plann. Inference 55 (1996) 281-297.

D. Nasri-Roudsari, E. Cramer. On the Convergence Rates of Extreme Generalized Order Statistics, Extremes 2 (1999) 421-447.

A. Spătaru, Precise Asymptotics in Spitzer's Law of Large Numbers, J. Theoret. Probab. 12 (1999) 811-819.

Y.B. Wang, Y. Yang, A General Law of Precise Asymptotics for the Counting Process of Record Times, J. Math. Anal. Appl. 286 (2003) 753-764.

J.G. Yan, Y.B. Wang, F.Y. Cheng, Precise Asymptotics for Order Statistics of a Non-random Sample and a Random Sample, J. Systems Sci. Math. Sci. 26 (2) (2006) 237-244.

Received: July, 2010