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Analysis of Progressive Type-II Censoring in the Weibull Model for Competing Risks Data with Binomial Removals

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Abstract

In reliability and survival analysis, the cause of failure/death of items/subjects can be attributed to more than one source. It is well known as the competing risks problem in statistical literature. In this paper, we assume that the lifetime of each unit which fails due to a different cause of failure, follows a Weibull distribution with different parameters. It is further assumed that in the case of progressive censoring, the number of removal items/subjects at each failure time follows a binomial distribution, and the causes of failure are independent. Having these assumptions in mind and working with competing risks data, we obtain the maximum likelihood and approximate maximum likelihood estimators of the unknown parameters. The asymptotic distribution of the maximum likelihood estimators is used to determine the confidence intervals. Finally, Monte Carlo simulations applied to illustrate the approach.

Keywords: Progressive type-II censoring; Weibull model; Competing risks; Binomial removals; Iterative maximum likelihood

1 Introduction

In survival or reliability analysis, the subjects / items may fail due to different causes. These causes compete to fail the subjects / items. Analysis of data in this situation is called competing risks analysis. The data for this competing risk consist of the failure time and an indicator variable denoting the specific cause of failure of the individual or item. The cause of failure may be assumed to be independent or dependent. Although, the assumption of independence seems very restrictive, in the case of dependence, there are some identifiability issues of the underlying model. It is known (see for example Kalbfleisch and Prentice(1980) or Crowder(2001)) that without the information of the covariates, it is not possible to test the assumption of independent of failure time distributions just from the observed data. See Cox(1959), David and Moeschberger(1978), Sarhan et al.(2008) and Pareek et al.(2009) for exhaustive treatment of different competing risk models.

In medical or industrial applications, researchers have to treat the censored data because they usually do not have sufficient time to observe the lifetime of all subjects in the study. Furthermore, subjects / items may fail by cause other than the ones under study. There are numerous schemes of censoring. In this paper we deal with the statistical ananlysis of competing risks data under progressive type II censoring scheme when the lifetime distribution is assumed to follow independent but nonidentical Weibull's distribution with regard to the cause of failure. The censoring scheme defined as follows: Consider n individuals in a study and assume that there are K causes of failure which are known. At the time of each failure, zero or more surviving units may be removed from the study at random. The data from a progressive Type-II censored sample is as follows;

$$(X_{1:m:n}, \delta_1, R_1), \cdots, (X_{m:m:n}, \delta_m, R_m),$$

where $X_{1:m:n} < \cdots < X_{m:m:n}$ denote the m observed failure times, $\delta_1, \cdots, \delta_m$ denote the m causes of failure and $R_1, \cdots, R_m$ denote the number of units removed from the study at the failure times $X_{1:m:n} < \cdots < X_{m:m:n}$. An advantage of this censoring scheme is that it is more general than the complete data and the type-II right censoring cases, provided that $R_1 = R_2 = \cdots = R_m = 0$ and $R_1 = R_2 = \cdots = R_{m-1} = 0$,$R_m = n - m$ respectively. Secondly the appropriate choice of R_i, $i = 1, \cdots, m$ enables the researcher to terminate the study whenever required.

The statistical inference on the parameters of failure time distributions under progressive type-II censoring has been studied by sevral authors such as Cohen(1963), Mann(1971), Viveros and Balakrishnan(1994), Balakrishnan and Aggarwala(2000) and Balakrishnan(2007). Note that, in this scheme , $R_1, \cdots, R_m$ are all pre-fixed.

We suppose any test unit being removed from the life test is independent of the others but with the same removal probability p. Then, Tse et al.(2000) and Wu and Chang(2002) indicated that the number of test unite remove at each failure time has a binomial distribution.

The main aim of this paper is the analysis of competing risk model when the data are progressively type-II censored with binomial removals. It is assumed here that the causes of failures follow Weibull distributions. We obtain the maximum likelihood estimators (ML'E)of the unknown parameters. It is observed that the MLE's cannot be obtained in explicit form. It can be obtained by solving a one dimensional optimization problem. We propose a simple fixed point type algorithm to solve this optimization problem. Since the MLE's cannot be obtained in explicit form, we propose approximate maximum likelihood estimatores (AMLE's), which have explicit expressions. It may be mentioned that the AMLE's have been proposed in the literature by Balasooriya and balakrishnan(2000).
We obtain observed Fisher information matrix and using them to compute the confidence intervals of the unknown parameters.
The rest of the paper is organized as follows. In Section 2, we describe the model and present the notation and definition used throughout the paper. The MLE's and AMLE's are obtained in Sections 3 and 4, respectively. The Fisher information and asymptotic confidence bounds are presented in Section 5. Simulation results are presented in Section 6.

2 The Model's Assumptions

Assume that there exist n items whose lifetimes are independent and identically distributed. The experiment terminates as soon as the mth failure occurs, where m is fixed. The cause of each failure may attributed to two different causes. The lifetime of the ith item is denoted by $X_i, i = 1, \cdots, n$, and X_{ij} denotes the time of failure of the ith item by the cause jth where $j = 1, 2$, so $X_i = min\{X_{i1}, X_{i2}\}$, the random variable X_{ij} is assumed to follow a Weibull distribution with parameters (α, λ_j), $i = 1, \cdots, n$ and $j = 1, 2$ so the density, survival function and hazard rate of these random variables respectively are as follows
$f_j(x) = \alpha\lambda_j^{\alpha} x^{\alpha-1} \exp^{-(x\lambda_j)^{\alpha}}$
$\bar{F}_j(x) = \exp^{-(x\lambda_j)^{\alpha}}$
$h_j(x) = \alpha\lambda_j^{\alpha} x^{\alpha-1}$ where $\quad x > 0 \quad \alpha > 0 \quad \lambda > 0$

When the first item fails, we observe two values $X_{1:m:n}$ and $\delta_1 \in \{1, 2, *\}$ where $X_{1:m:n}$ denotes the first order statistics out of the m failed items, which in turn denotes the statistics from the whole sample. Then we randomly exclude R_1 items from the rest of the fair items. We assume that R_1 follows a binomial distribution with parameters p and

$n-m$.When the ith failure occurs $i=2,\cdots,m-1$, we observe two values, $X_{i:m:n}$ and $\delta_i \in \{1,2,*\}$, where $\delta_i=1$ denotes the cause of failure of ith item, it could be considered the first cause, and if $\delta_i=2$, it could denote the second cause, and if $\delta_i=*$, means the cause of unit i to fail is unknown, i.e. we don't know the real cause of failure for a test unit, but it must belong to cause 1 or cause 2. Then we randomly exclude R_i items from the rest of fair items, and we suppose that R_i follows independently binomial distributions with parameters $(n-m-\sum_{l=1}^{i-1} R_l, p)$. Finally the experiment terminates when the mth failure occurs, and we observe $X_{m:m:n}$ and $\delta_m \in \{1,2,*\}$. At this point in time, we exclude the rest of the items i.e $R_m = n-m-\sum_{l=1}^{m-1} R_l$.

3 The Likelihood function of the model

Using the above mentioned assumptions, the likelihood function for type-II progressively censored model under competing risk is as follows:
$L(\lambda_1,\lambda_2,\alpha,R) = L(\lambda_1,\lambda_2,\alpha|R=r)P(R=r)$.
So using Sarhan et al. (2008), we have:

$$\begin{aligned} L(\lambda_1,\lambda_2,\alpha|R=r) &= c\prod_{i=1}^{m}[f_1(x_{i:m:n})\bar{F}_2(x_{i:m:n})]^{I(\delta_i=1)}[f_2(x_{i:m:n})\bar{F}_1(x_{i:m:n})]^{I(\delta_i=2)} \\ &\quad [f_1(x_{i:m:n})\bar{F}_2(x_{i:m:n})+f_2(x_{i:m:n})\bar{F}_1(x_{i:m:n})]^{I(\delta_i=*)}[\bar{F}(x_{i:m:n})]^{r_i} \end{aligned}$$

The conditional likelihood function of the Weibull distribution is

$$L(\lambda_1,\lambda_2,\alpha|R=r) = c\alpha^m \lambda_1^{n_1\alpha}\lambda_2^{n_2\alpha}(\lambda_1^\alpha+\lambda_2^\alpha)^{n^*}\prod_{i=1}^{m} x_{i:m:n}^{\alpha-1}\exp\{-\sum_{i=1}^{m}(r_i+1)x_{i:m:n}^{\alpha}(\lambda_1^\alpha+\lambda_2^\alpha)\} \quad (3.1)$$

where $c = n(n-r_1-1)\cdots(n-r_1-r_2-\cdots-r_{m-1}-m+1)$
and $n_j = \sum_{i=1}^{m} I(\delta_i=j) \qquad j=1,2$ and $n^* = \sum_{i=1}^{m} I(\delta_i=*)$.
Suppose that the numbers of removed items are independent and have the identical probability mass function

$$P(R_1=r_1) = C_{r_1}^{n-m} p^{r_1}(1-p)^{n-m-r_1}$$

where $0 \le r_1 \le n-m$.

$$P(R_i=r_i|R_{i-1}=r_{i-1},\ldots,R_1=r_1) = C_{r_i}^{n-m-\sum_{l=1}^{i-1} r_l} p^{r_i}(1-p)^{n-m-\sum_{l=1}^{i} r_l}$$

where $0 \leq r_i \leq n - m - \sum_{l=1}^{i-1} r_l \qquad i = 2, \cdots, m-1$.
Suppose that R_i is independent of $X_{i:m:n}$, so we can write:

$$\begin{aligned} P(R=r) &= P(R_m = r_m | R_{m-1} = r_{m-1}, \cdots R_1 = r_1) \cdots P(R_2 = r_2 | R_1 = r_1) P(R_1 = r_1) \\ &= \frac{(n-m)!}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!} p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i} \end{aligned} \tag{3.2}$$

hence the likelihood function is as follows:

$$L(\lambda_1, \lambda_2, \alpha, R) = c^* \alpha^m \lambda_1^{n_1 \alpha} \lambda_2^{n_2 \alpha} (\lambda_1^\alpha + \lambda_2^\alpha)^{n^*} \prod_{i=1}^m x_{i:m:n}^{\alpha-1} \exp\{-\sum_{i=1}^m (r_i+1) x_{i:m:n}^\alpha (\lambda_1^\alpha + \lambda_2^\alpha)\}$$

$$p^{\sum_{i=1}^{m-1} r_i} (1-p)^{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i) r_i}$$

where $C^* = \frac{(n-m)!c}{\prod_{i=1}^{m-1} r_i!(n-m-\sum_{i=1}^{m-1} r_i)!}$ so the logarithm of this likelihood function can be written as follows:

$$l(\lambda_1, \lambda_2, \alpha, R) = constant + m \ln \alpha + n_1 \alpha \ln \lambda_1 + n_2 \alpha \ln \lambda_2 + n^* \ln(\lambda_1^\alpha + \lambda_2^\alpha) + (\alpha - 1) \sum_{i=1}^m \ln x_{i:m:n}$$

$$- \sum_{i=1}^m (r_i+1) x_{i:m:n}^\alpha (\lambda_1^\alpha + \lambda_2^\alpha) + \sum_{i=1}^{m-1} r_i \ln p + [(m-1)(n-m) - \sum_{i=1}^{m-1} (m-i) r_i] \ln(1-p).$$

Since the probability $P(R=r)$ is free of the parameters $\alpha, \lambda_1, \lambda_2$, the estimations of these parameters can be directly driven from equation 3.2.
Hence, for a fixed α, the differentiation with respect to the parameters yields:

$$\hat{\lambda}_1(\alpha) = \left(\frac{n_1}{n_1+n_2} \frac{m}{\sum_{i=1}^m (r_i+1) x_{i:m:n}^\alpha}\right)^{\frac{1}{\alpha}} \tag{3.3}$$

and

$$\hat{\lambda}_2(\alpha) = \left(\frac{n_2}{n_1+n_2} \frac{m}{\sum_{i=1}^m (r_i+1) x_{i:m:n}^\alpha}\right)^{\frac{1}{\alpha}} \tag{3.4}$$

putting these estimators in the likelihood function and differentiation with respect to the α, we will have the estimator of α as $\alpha = h(\alpha)$ where:

$$h(\alpha) = \left(\frac{\sum_{i=1}^m (r_i+1) x_{i:m:n}^\alpha \ln x_{i:m:n}^\alpha}{\sum_{i=1}^m (r_i+1) x_{i:m:n}^\alpha} - \frac{\sum_{i=1}^m \ln x_{i:m:n}}{m}\right)^{-1}$$

In order to find the solution of the equation $\alpha = h(\alpha)$, we use a simple iterative method as follows. We employ an initial value α^0, and write the next solution as $\alpha^1 = h(\alpha^0)$ and

so on. This process ends as soon as $|\alpha^{n+1} - \alpha^n| < \varepsilon$ for preassigned value of ε. Once we obtain $\hat{\alpha}$, the MLE λ_1 and λ_2 can be obtained as $\hat{\lambda}_1(\hat{\alpha})$ and $\hat{\lambda}_2(\hat{\alpha})$.
The MLE parameter p is given by

$$\hat{p} = \frac{\sum_{i=1}^{m-1} r_i}{(m-1)(n-m) - \sum_{i=1}^{m-1}(m-i-1)r_i}$$

Here n_1 is a $Bin(n_1+n_2, \frac{\lambda_1^\alpha}{\lambda_1^\alpha+\lambda_2^\alpha})$ random variable and n_2 is $Bin(n_1+n_2, \frac{\lambda_2^\alpha}{\lambda_1^\alpha+\lambda_2^\alpha})$ random variable.
When the MLE's exist but are not in compact forms, we can used the AMLE's where they have explicit expressions.

4 Approximative Maximum Likelihood Estimations

Since the MLE p is the compact form, we ignoring the AMLE it. For obtain the AMLE's $\alpha, \lambda_1, \lambda_2$ we can using the same kind of arguments used by Pareek et.al (2009). Now, ignoring the cause of failures. Then the likelihood function of progressive type-II censoring $X_{1:m:n}, \dots, X_{m:m:n}$ is given by

$$L(\lambda, \alpha) = L(\lambda, \alpha|R = r)P(R = r)$$

Since the probability $P(R = r)$ is free of the parameters, we ignoring it.

$$L(\lambda, \alpha|R = r) = c\prod_{i=1}^{m} \alpha\lambda^\alpha x_{i:m:n}^{\alpha-1} e^{-(\lambda x_{i:m:n})^\alpha}(e^{-(\lambda x_{i:m:n})^\alpha})^{r_i} \tag{4.5}$$

where $c = n(n - r_1 - 1)\cdots(n - r_1 - r_2 - \cdots - r_{m-1} - m + 1)$.
Let $\sigma = \frac{1}{\alpha}$ and $\mu = -\ln\lambda$ and $V_{i:m:n} = \ln X_{i:m:n}$. The likelihood function $V_{i:m:n}, i = 1, \dots, m$ is:

$$L(\mu, \sigma|R = r) = c\prod_{i=1}^{m} \frac{1}{\sigma} e^{\frac{v_{i:m:n}-\mu}{\sigma} - e^{\frac{v_{i:m:n}-\mu}{\sigma}}} (e^{\frac{v_{i:m:n}-\mu}{\sigma}})^{r_i}$$

Now, let $g(z_{i:m:n}) = e^{z_{i:m:n} - e^{z_{i:m:n}}}, \bar{G}(z_{i:m:n}) = e^{-e^{z_{i:m:n}}}$ and $z_{i:m:n} = \frac{v_{i:m:n}-\mu}{\sigma}$. We can rewrite the likelihood function as follow,

$$L(\mu, \sigma|R = r) = c\prod_{i=1}^{m} \frac{1}{\sigma} g(z_{i:m:n})(\bar{G}(z_{i:m:n}))^{r_i}$$

Taking the log-likelihood, we obtain,

$$l(\mu,\sigma|R=r)=\ln c-m\ln\sigma+\sum_{i=1}^{m}g(z_{i:m:n})+\sum_{i=1}^{m}r_i\bar{G}(z_{i:m:n})$$

Taking derivative with respect to μ and σ,

$$\frac{\partial l(\mu,\sigma|R=r)}{\partial\mu}=-\frac{1}{\sigma}\sum_{i=1}^{m}\frac{g'(z_{i:m:n})}{g(z_{i:m:n})}+\sum_{i=1}^{m}\frac{r_i}{\sigma}\times\frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})}=0 \tag{4.6}$$

$$\frac{\partial l(\mu,\sigma|R=r)}{\partial\sigma}=-\frac{m}{\sigma}-\sum_{i=1}^{m}\frac{g'(z_{i:m:n})}{g(z_{i:m:n})}\times\frac{z_{i:m:n}}{\sigma}+\sum_{i=1}^{m}r_i\frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})}\times\frac{z_{i:m:n}}{\sigma}=0 \tag{4.7}$$

Relations 4.6 and 4.7 do not have explicit solutions. We expand the function $\frac{g'(z_{i:m:n})}{g(z_{i:m:n})}$ and $\frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})}$ in Taylor series around the $G^{-1}(p_i)=\ln(-\ln q_i)=\mu_i$ where $p_i=\frac{i}{n+1}$, $q_i=1-p_i$ similarly as Balasooriya and balakrishnan(2000) or Pareek et.al(2009) and compute the AMLE's μ and σ. We get

$$\frac{g'(z_{i:m:n})}{g(z_{i:m:n})}\approx\alpha_i-\beta_i z_{i:m:n} \tag{4.8}$$

$$\frac{g(z_{i:m:n})}{\bar{G}(z_{i:m:n})}\approx 1-\alpha_i+\beta_i z_{i:m:n} \tag{4.9}$$

where $\alpha_i=1+\ln q_i(1-\ln(-\ln q_i))$ *and* $\beta_i=-\ln q_i$. Using approximation 4.8 and 4.9 in 4.6 and 4.7, we obtain

$$-\sum_{i=1}^{m}(\alpha_i-\beta_i z_{i:m:n})+\sum_{i=1}^{m}r_i(1-\alpha_i+\beta_i z_{i:m:n})=0 \tag{4.10}$$

$$-m-\sum_{i=1}^{m}(\alpha_i-\beta_i z_{i:m:n})z_{i:m:n}+\sum_{i=1}^{m}r_i(1-\alpha_i+\beta_i z_{i:m:n})z_{i:m:n}=0 \tag{4.11}$$

From 4.10 we obtain $\tilde{\mu}$ as

$$\tilde{\mu}=A-B\tilde{\sigma}$$

where

$$A=\frac{\sum_{i=1}^{m}(r_i+1)\beta_i v_{i:m:n}}{\sum_{i=1}^{m}(r_i+1)\beta_i}\qquad and\qquad B=\frac{\sum_{i=1}^{m}\alpha_i-\sum_{i=1}^{m}r_i(1-\alpha_i)}{\sum_{i=1}^{m}(r_i+1)\beta_i}.$$

From 4.11, we obtain $\tilde{\sigma}$ as a solution of the quadratic equation

$$m\sigma^2+D\sigma-E=0$$

where

$$D=\sum_{i=1}^{m}\alpha_i(y_{i:m:n}-A)-\sum_{i=1}^{m}r_i(1-\alpha_i)(y_{i:m:n}-A)-2B(\sum_{i=1}^{m}(r_i+1)\beta_i(y_{i:m:n}-A)),$$

$$E=\sum_{i=1}^{m}(r_i+1)\beta_i(y_{i:m:n}-A)^2.$$

Therefore,

$$\tilde{\sigma}=\frac{-D+\sqrt{D^2+4Em}}{2m},$$

then $\tilde{\alpha}=\frac{1}{\tilde{\sigma}}$ is a $AMLE$ of α. Now compute $\tilde{\lambda}_1=\hat{\lambda}_1(\tilde{\alpha})$ and $\tilde{\lambda}_2=\hat{\lambda}_2(\tilde{\alpha})$ from 3.3 and 3.4 as the AMLE's of λ_1 and λ_2 respectively.

5 Asymptotic Confidence Interval

To find the confidence interval for the estimators we have to determine the exact distribution of the estimators which is in fact difficult, hence for this objective we try with the asymptotic distribution of the maximum likelihood estimator, which produce an approximation of the confidence interval.

The asymptotic distribution of the maximum likelihood can be written as follows (Miller 1981):

$$[(\hat{\lambda}_1-\lambda_1),(\hat{\lambda}_2-\lambda_2),(\hat{\alpha}-\alpha),(\hat{p}-p)]\sim N_4(0,\mathbf{I}^{-1}(\lambda_1,\lambda_2,\alpha,p)),$$

where $I^{-1}(\lambda_1,\lambda_2,\alpha,p)$ denotes the variance-covariance matrix in term of the unknown parameters. The elements of this matrix denoted by

$$I_{jl}(\lambda_1,\lambda_2,\alpha,p)\equiv I_{jl}(\theta_1,\theta_2,\theta_3,\theta_4),\qquad j=1,\cdots,4,\ \ l=1,\cdots,4.$$

It can be approximated by:

$$I_{jl}(\hat{\lambda_1},\hat{\lambda_2},\hat{\alpha},\hat{p})=-E[\frac{\partial^2 l(\theta)}{\partial\theta_j\partial\theta_l}]$$

where θ is the vector of unknown parameters. The probability density function $X_{i:m:n}$ for $i=1,\cdots,m$ is

$$f_{X_{i:m:n}|R}(x_i|r)=\sum_{k=1}^{i}(-1)^{k+1}c_k(1-F(x_i))^{b_k}.f(x_i),\qquad 0<x_i<\infty$$

where $r_0 = 0$ and

$$c_k = \frac{\prod_{t^*=0}^{i-1}(n - t^* - \sum_{j=1}^{t^*} r_j)}{\prod_{t=1}^{k-1}[(\sum_{j=1}^{t} r_{i-k+j}) + t]\prod_{z=1}^{i-k}[(\sum_{j=1}^{z} r_{i-k-j+1}) + z]},$$

$$b_k = (\sum_{t=1}^{k-1} r_{i-t}) + n + k - (\sum_{t'=1}^{i-1}(r_{t'} + 1)) - 2.$$

The Fisher information matrix can be obtained as follow:

$$I = \begin{pmatrix} I_{11} & I_{12} & I_{13} & I_{14} \\ I_{21} & I_{22} & I_{23} & I_{24} \\ I_{31} & I_{32} & I_{33} & I_{34} \\ I_{41} & I_{42} & I_{43} & I_{44} \end{pmatrix}.$$

Where,

$$\begin{cases} I_{11} = \frac{n_1\alpha}{\lambda_1^2} - \frac{n^*\alpha\lambda_1^{\alpha-2}[(\alpha-1)\lambda_2^\alpha - \lambda_1^\alpha]}{(\lambda_1^\alpha+\lambda_2^\alpha)^2} + [(\alpha(\alpha-1)\lambda_1^{\alpha-2})M_1] \\ I_{12} = I_{21} = n^*\frac{\alpha^2(\lambda_1\lambda_2)^{(\alpha-1)}}{(\lambda_1^\alpha+\lambda_2^\alpha)^2} \\ I_{13} = I_{31} = -\frac{n_1}{\lambda_1} - \frac{n^*\lambda_1^{\alpha-1}[\lambda_1^\alpha+\lambda_2^\alpha+\alpha\lambda_2^\alpha \ln\frac{\lambda_1}{\lambda_2}]}{(\lambda_1^\alpha+\lambda_2^\alpha)^2} + \{\lambda_1^{\alpha-1}[(1+\alpha\ln\lambda_1)M_1 + M_2]\} \\ I_{14} = I_{41} = I_{24} = I_{42} = I_{34} = I_{43} = 0 \\ I_{22} = \frac{n_2\alpha}{\lambda_2^2} - \frac{n^*\alpha\lambda_2^{\alpha-2}[(\alpha-1)\lambda_2^\alpha - \lambda_1^\alpha]}{(\lambda_1^\alpha+\lambda_2^\alpha)^2} + [\alpha(\alpha-1)\lambda_2^{\alpha-2}M_1] \\ I_{23} = I_{32} = -\frac{n_2}{\lambda_2} - \frac{n^*\lambda_2^{\alpha-1}[\lambda_1^\alpha+\lambda_2^\alpha+\alpha\lambda_1^\alpha \ln\frac{\lambda_2}{\lambda_1}]}{(\lambda_1^\alpha+\lambda_2^\alpha)^2} + \{\lambda_2^{\alpha-1}[(1+\alpha\ln\lambda_2)M_1 + M_2]\} \\ I_{33} = \frac{m}{\alpha^2} - \frac{n^*(\lambda_1\lambda_2)^\alpha(\ln\frac{\lambda_1}{\lambda_2})^2}{(\lambda_1^\alpha+\lambda_2^\alpha)^2} + \{[\lambda_1^\alpha(\ln\lambda_1)^2 + \lambda_2^\alpha(\ln\lambda_2)^2]M_1\} \\ \quad +[\frac{2(\lambda_1^\alpha\ln\lambda_1+\lambda_2^\alpha\ln\lambda_2)M_2}{\alpha}] + [\frac{(\lambda_1^\alpha+\lambda_2^\alpha)}{\alpha^2}M_3] \\ I_{44} = \frac{\sum_{i=1}^{m-1} r_i}{p^2} + \frac{[(n-m)(m-1)-\sum_{i=1}^{m-1}(m-i)r_i]}{(1-p)^2} \end{cases}$$

Now, we compute M_1, M_2 and M_3 as follows:
let $Y_{i:m:n} = X_{i:m:n}^\alpha$ then $Y_{i:m:n} \sim \exp(\lambda_1^\alpha + \lambda_2^\alpha)$ and

$$M_1 = E(\sum_{i=1}^{m}(r_i + 1)Y_{i:m:n}) = \frac{m}{\lambda_1^\alpha + \lambda_2^\alpha}.$$

Since,

$$E(Y_{i:m:n}) = E_R(E(Y_{i:m:n}|R=r)) = \sum_{r_1=0}^{g(r_1)} \sum_{r_2=0}^{g(r_2)} \cdots \sum_{r_{m-1}=0}^{g(r_{m-1})} E(Y_{i:m:n}|R=r).P(R=r)$$

where $g(r_1) = n-m, g(r_i) = n-m-r_1-\cdots-r_{i-1}$, $i = 2, \cdots, m-1$, and $P(R=r)$ is defined in 3.2. Then we get,

$$\begin{cases} M_2 = E[\sum_{i=1}^{m}(r_i+1)Y_{(i:m:n)}\ln Y_{(i:m:n)}] = E_R[\sum_{i=1}^{m}(r_i+1)E(Y_{(i:m:n)}\ln Y_{(i:m:n)}|R=r)] \\ \\ M_3 = E[\sum_{i=1}^{m}(r_i+1)Y_{(i:m:n)}(\ln Y_{(i:m:n)})^2] = E_R[\sum_{i=1}^{m}(r_i+1)E(Y_{(i:m:n)}(\ln Y_{(i:m:n)})^2|R=r)] \end{cases}$$

where, $E(Y_{(i:m:n)}lnY_{(i:m:n)}|R=r) = \sum_{k=1}^{i}\frac{c_k(-1)^{k+1}}{(b_k+1)^2(\lambda_1^\alpha+\lambda_2^\alpha)}[\psi(2) - ln[(b_k+1)(\lambda_1^\alpha+\lambda_2^\alpha)]]$ and

$$E(Y_{(i:m:n)}(lnY_{(i:m:n)})^2|R=r) = \sum_{k=1}^{i}\frac{c_k(-1)^{k+1}}{(b_k+1)^2(\lambda_1^\alpha+\lambda_2^\alpha)^2}\{\psi'(2) + (\psi(2) - ln[(b_k+1)(\lambda_1^\alpha+\lambda_2^\alpha)])^2\}.$$

where $\psi(.)$ and $\psi'(.)$ are the digamma and trigamma functions, respectively.

6 Simulation Study

To illustrate the precision of the estimators we have designed 5 schemes for simulation as below:

Scheme 1: $n = 50,\ m = 20$ and $p = 0.1,\ 0.5,\ 0.8$

Scheme 2: $n = 75,\ m = 20$ and $p = 0.1,\ 0.5,\ 0.8$

Scheme 3: $n = 75,\ m = 30$ and $p = 0.1,\ 0.5,\ 0.8$

Scheme 4: $n = 75,\ m = 45$ and $p = 0.1,\ 0.5,\ 0.8$

Scheme 5: $n = 75,\ m = 60$ and $p = 0.1,\ 0.5,\ 0.8$

In each of them we have simulated 1000 replications, based on initial (true) values of parameters in the model, we estimate the parameters and obtain the confidence interval, the results for these simulations is summarized in the tables 1,2 and 3.

We observed that the MLE's and AMLE's behave in a similar manner in all the cases considered. Therefore, AMLE's can used in place of MLE's. For p and m fixed, as n increases, the biases, mean square error, and the average confidence lengths increase for all the estimators. For p and n fixed, as m increases, the biases, mean square error, and the average confidence lengths decrease for all the estimators.

Table 1: The average relative estimates of $\lambda_1, \lambda_2, \alpha$ and their mean square errors(within bracket){when true values are $\lambda_1 = 0.6$, $\lambda_2 = 0.4$ and $\alpha = 3$} for each scheme.

		p=0.1			p=0.5			p=0.8		
		λ_1	λ_2	α	λ_1	λ_2	α	λ_1	λ_2	α
1	MLE	0.523	0.375	2.914	0.518	0.344	2.910	0.534	0.349	2.964
	MSE	(0.053)	(0.063)	(0.074)	(0.048)	(0.063)	(0.081)	(0.045)	(0.062)	(0.026)
	AMLE	0.543	0.421	2.917	0.525	0.423	2.924	0.584	0.355	2.975
	MSE	(0.053)	(0.050)	(0.069)	(0.063)	(0.053)	(0.776	(0.046)	(0.052)	(0.022)
2	MLE	0.450	0.480	2.906	0.451	0.487	2.910	0.499	0.439	2.924
	MSE	(0.026)	(0.054)	(0.088)	(0.122)	(0.098)	(0.1006)	(0.036)	(0.067)	(0.058)
	AMLE	0.446	0.469	2.909	0.494	0.452	2.919	0.489	0.443	2.968
	MSE	(0.024)	(0.035)	(0.082)	(0.11)	(0.040)	(0.079)	(0.027)	(0.056)	(0.052)
3	MLE	0.486	0.464	2.926	0.465	0.4378	2.912	0.475	0.4234	2.933
	MSE	(0.019)	(0.047)	(0.105)	(0.108)	(0.014)	(0.074)	(0.074)	(0.055)	(0.039)
	AMLE	0.491	0.439	2.935	0.479	0.434	2.923	0.485	0.381	2.944
	MSE	(0.017)	(0.045)	(0.083)	(0.146)	(0.077)	(0.079)	(0.055)	(0.035)	(0.036)
4	MLE	0.534	0.459	2.968	0.514	0.464	2.956	0.567	0.426	2.969
	MSE	(0.012)	(0.0348)	(0.040)	(0.086)	(0.056)	(0.961)	(0.019)	(0.048)	(0.012)
	AMLE	0.538	0.417	2.998	0.526	0.457	2.959	0.575	0.386	2.975
	MSE	(0.0104)	(0.033)	(0.004)	(0.089)	(0.032)	(0.056)	(0.016)	(0.043)	(0.031)
5	MLE	0.589	0.409	2.976	0.634	0.402	2.968	0.623	0.401	2.995
	MSE	(0.031)	(0.008)	(0.045)	(0.063)	(0.009)	(0.026)	(0.018)	(0.011)	(0.047)
	AMLE	0.598	0.402	2.998	0.599	0.389	2.969	0.591	0.399	2.996
	MSE	(0.036)	(0.003)	(0.033)	(0.059)	(0.014)	(0.009)	(0.010)	(0.013)	(0.041)

Table 2: The average relative estimates of λ_1, λ_2,α and their mean square errors(within bracket){when true values are $\lambda_1 = 0.9$, $\lambda_2 = 0.1$ and $\alpha = 2$}for each scheme.

		p=0.1			p=0.5			p=0.8		
		λ_1	λ_2	α	λ_1	λ_2	α	λ_1	λ_2	α
1	MLE	0.763	0.094	1.654	0.987	0.134	1.756	0.874	0.0994	1.9224
	MSE	(0.163)	(0.033)	(0.084)	(0.109)	(0.182)	(0.102)	(0.072)	(0.046))	(0.113)
	AMLE	0.873	0.105	1.737	0.918	0.109	1.839	0.884	0.1002	1.934
	MSE	(0.161)	(0.036)	(0.071)	(0.117)	(0.191)	(0.113)	(0.073)	(0.053)	(0.101)
2	MLE	0.678	0.145	1.344	1.006	0.187	1.454	0.769	0.074	1.235
	MSE	(0.217)	(0.055)	(0.114)	(0.125)	(0.213)	(0.115)	(0.102)	(0.053)	(0.120)
	AMLE	0.667	0.120	1.437	0.987	0.134	1.567	0.789	0.129	1.234
	MSE	(0.209)	(0.055)	(0.106)	(0.115)	(0.220)	(0.116)	(0.109)	(0.049)	(0.117)
3	MLE	0.667	0.128	1.934	0.678	0.156	1.822	0.875	0.094	1.812
	MSE	(0.059)	(0.023)	(0.049)	(0.036)	(0.092)	(0.054)	(0.050)	(0.010)	(0.039)
	AMLE	0.679	0.123	1.93	0.77	0.09	1.88	0.878	0.097	1.898
	MSE	(0.051)	(0.021)	(0.046)	(0.021)	(0.077)	(0.055)	(0.041)	(0.016)	(0.036)
4	MLE	0.805	0.095	1.966	0.778	0.145	1.911	0.889	0.098	1.969
	MSE	(0.042)	(0.007)	(0.022)	(0.041)	(0.061)	(0.021)	(0.037)	(0.0088)	(0.031)
	AMLE	0.819	0.1007	1.998	0.785	0.106	1.946	0.893	0.099	1.973
	MSE	(0.034)	(0.006)	(0.004)	(0.024)	(0.061)	(0.019)	(0.033)	(0.008)	(0.031)
5	MLE	0.853	0.098	1.987	0.889	0.097	1.976	0.911	0.101	1.944
	MSE	(0.028)	(0.002)	(0.007)	(0.020)	(0.057)	(0.001)	(0.029)	(0.006)	(0.010)
	AMLE	0.876	0.102	1.964	0.904	0.102	1.920	0.909	0.099	1.998
	MSE	(0.021)	(0.003)	(0.015)	(0.021)	(0.054)	(0.013)	(0.011)	(0.006)	(0.001)

Table 3: The average 95% confidence lengths and the corresponding coverage percentage(within brackets) {when true values are $\lambda_1 = 0.6$, $\lambda_2 = 0.4$ and $\alpha = 3$} for each scheme.

		p=0.1			p=0.5			p=0.8		
		λ_1	λ_2	α	λ_1	λ_2	α	λ_1	λ_2	α
1	MLE	0.839	0.7201	2.9024	0.894	0.6891	2.9001	0.856	0.6941	2.9542
		(0.97)	(0.96)	(0.97)	(0.95)	(0.95)	(0.96)	(0.97)	(0.96)	(0.96)
	AMLE	0.85	0.7661	2.9072	1.941	0.7681	2.9142	0.91	0.7001	0.9652
		(0.97)	(0.96)	(0.97)	(0.95)	(0.95)	(0.96)	(0.96)	(0.96)	(0.96)
2	MLE	0.8526	0.8193	2.8446	0.8536	0.8063	2.8485	0.8122	0.5791	2.8626
		(0.96)	(0.95)	(0.96)	(0.95)	(0.95)	(0.96)	(0.96)	(0.95)	(0.95)
	AMLE	0.8686	0.7983	2.8476	0.8966	0.69134	2.85844	0.8921	0.9016	2.9075
		(0.96)	(0.95)	(0.97)	(0.95)	(0.94)	(0.96)	(0.96)	(0.95)	(0.95)
3	MLE	1.1714	0.891	2.8889	1.1764	0.8618	2.8749	1.1459	0.8474	2.8761
		(0.96)	(0.95)	(0.97)	(0.95)	(0.94)	(0.95)	(0.96)	(0.94)	(0.95)
	AMLE	1.202	0.876	2.9397	1.1873	0.8482	2.9359	1.1869	0.8053	2.9065
		(0.97)	(0.96)	(0.97)	(0.92)	(0.94)	(0.96)	(0.95)	(0.93)	(0.95)
4	MLE	1.3112	0.946	2.8953	1.3752	0.921	2.8837	1.3882	0.913	2.9264
		(0.97)	(0.96)	(0.96)	(0.95)	(0.94)	(0.95)	(0.96)	(0.94)	(0.95)
	AMLE	1.3192	0.904	2.9257	1.3554	0.944	2.9167	1.3962	0.879	2.8727
		(0.95)	(0.96)	(0.96)	(0.95)	(0.92)	(0.93)	(0.92)	(0.94)	(0.95)
5	MLE	1.515	0.925	2.9596	1.5454	0.918	2.9811	1.5344	0.9169	2.9275
		(0.97)	(0.96)	(0.95)	(0.95)	(0.93)	(0.94)	(0.95)	(0.95)	(0.94)
	AMLE	1.509	0.918	2.9682	1.5097	0.9053	2.9827	1.5015	0.9147	2.9286
		(0.97)	(0.95)	(0.96)	(0.96)	(0.93)	(0.92)	(0.96)	(0.94)	(0.95)

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