Applied Mathematical Sciences, Vol. 5, 2011, no. 21, 1037 - 1049

Series Solutions with Convergence-Control Parameter for Three Highly Non-Linear PDEs: KdV, Kawahara and Gardner equations

Saeed Dinarvand [1,2*], Soroush Khosravi [3], Hasan Khoosheh [4] and Mohsen Nasrollahzadeh [5]

1. Mechanical Engineering Department, Engineering Faculty of Bu-Ali Sina University, Hamedan, Iran

2. Islamic Azad University, Central Tehran Branch Young Researchers Club, Tehran, Iran

3. Information Technology Management Department Management Faculty of Mehralborz University, Tehran, Iran

4. Islamic Azad University, Arak Branch Young Researchers Club, Arak, Iran

5. Electrical Engineering Department Islamic Azad University, South Tehran Branch, Tehran, Iran

Abstract

Here, an analytic method, namely the homotopy analysis method (shortly HAM), is applied to solve the KdV, Kawahara and Gardner equations. The HAM is a strong and easy-to-use analytic tool for nonlinear problems and dose not need small parameters in the equations. This method contains the auxiliary parameter h, which provides us with a simple way to adjust and control the convergence region of solution series. The obtained solutions, in comparison with the exact solutions admit a remarkable accuracy.

* E-mail address: saeed_dinarvand@yahoo.com

Keywords: Series solution, Non-linear PDEs, KdV equation; Kawahara equation; Gardner equation

1. Introduction

Finding explicit analytic solutions of nonlinear differential equations is extremely important in mathematical physics. In recent years, many powerful methods have been developed to construct explicit analytic solution of nonlinear differential equations. In 1992, Liao [1] employed the basic ideas of the homotopy in topology to propose method for nonlinear problems, namely homotopy analysis method (HAM), [2–6]. This method has many advantages over the classical methods, mainly, it is independent of any small or large quantities. So, the HAM can be applied no matter if governing equations and boundary/initial conditions contain small or large quantities or not. The HAM also avoids discretization and provides an efficient numerical solution with high accuracy, minimal calculation, and avoidance of physically unrealistic assumptions. Furthermore, the HAM always provides us with a family of solution expressions in the auxiliary parameter h, the convergence region and rate of each solution might be determined conveniently by the auxiliary parameter h. This method has been successfully applied to solving many types of nonlinear problems [7–11].

In this Letter, we extend the application of the homotopy analysis method to construct approximate solutions for the KdV, Kawahara and Gardner equations.

A substantial amount of research work has been directed for the study of the nonlinear KdV, Kawahara and Gardner equations given by

$$u_t + 6uu_x + u_{xxx} = 0, \quad (1)$$

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0 \quad (2)$$

and

$$u_t + 6uu_x - 6u^2u_x + u_{xxx} = 0, \quad (3)$$

respectively.

The well known KdV equation had been derived in 1885 by the two scientists Korteweg and de Vries to describe long wave propagation on shallow water. Although this equation was known since the last century, but its physical behaviour is still mysterious. This equation has many other direct physical applications to solids, liquids, gases and plasma: magnetohydrodynamic waves in a cold plasma [12], longitudinal waves propagating in a one dimensional lattice of equal masses coupled by nonlinear springs, the Fermi et al. problem [13,14], ion acoustic waves in a cold plasma [15], rotating flow in tube [16] and longitudinal dispersive waves in elastic rods [17].

The Kawahara equation is a fifth order KdV equation, which occurs in the theory of magneto-acoustic waves in a plasmas [18] and in the theory of shallow water waves with surface tension [19].

The Gardner equation (combined KdV-mKdV or eKdV equation) is a useful model for the description of internal solitary waves in shallow water [20] and have been widely studied by the various methods. The competition among dispersion, quadratic and cubic nonlinearities constitutes the main interest.

This Letter has been organized as follows. In Section 2, the basic idea of the HAM is introduced. In Section 3, we extend the application of the HAM to construct approximate solutions for the KdV, Kawahara and Gardner equations. Results are presented in Section 4.

2. Basic idea of the HAM

Let us consider the following differential equation

$$\mathrm{N}\left[u(\tau)\right]=0, \tag{4}$$

where N is a nonlinear operator, τ denotes independent variable, $u(\tau)$ is an unknown function, respectively. For simplicity, we ignore all boundary or initial conditions, which can be treated in the similar way. By means of generalizing the traditional homotopy method, Liao [5] constructs the so-called *zero-order deformation equation*

$$(1-p)\mathrm{L}\left[\varphi(\tau;p)-u_0(\tau)\right]=phH(\tau)\mathrm{N}\left[\varphi(\tau;p)\right], \tag{5}$$

where $p\in[0,1]$ is the embedding parameter, $h\neq 0$ is a non-zero auxiliary parameter, $H(\tau)\neq 0$ is an auxiliary function, L is an auxiliary linear operator, $u_0(\tau)$ is an initial guess of $u(\tau)$, $\varphi(\tau;p)$ is a unknown function, respectively. It is important, that one has great freedom to choose auxiliary things in HAM. Obviously, when $p=0$ and $p=1$, it holds

$$\varphi(\tau;0)=u_0(\tau), \qquad \varphi(\tau;1)=u(\tau),$$

respectively. Thus as *p* increases from 0 to 1, the solution $\varphi(\tau;p)$ varies from the initial guess $u_0(\tau)$ to the solution $u(\tau)$. Expanding $\varphi(\tau;p)$ in Taylor series with respect to *p*, we have

$$\varphi(\tau;p)=u_0(\tau)+\sum_{m=1}^{+\infty}u_m(\tau)p^m, \tag{6}$$

where

$$u_m(\tau)=\frac{1}{m!}\frac{\partial^m\varphi(\tau;p)}{\partial p^m}\bigg|_{p=0}. \tag{7}$$

If the auxiliary linear operator, the initial guess, the auxiliary parameter *h*, and the auxiliary function are so properly chosen, the series (4) converges at $p=1$, then we have

$$u(\tau) = u_0(\tau) + \sum_{m=1}^{+\infty} u_m(\tau), \tag{8}$$

which must be one of solutions of original nonlinear equation, as proved by Liao [5]. As $h = -1$ and $H(\tau) = 1$, equation (3) becomes

$$(1-p)\mathrm{L}\left[\varphi(\tau;p) - u_0(\tau)\right] + p\,\mathrm{N}\left[\varphi(\tau;p)\right] = 0, \tag{9}$$

which is used mostly in the homotopy perturbation method, where as the solution obtained directly, without using Taylor series [21,22].

According to the definition (5), the governing equation can be deduced from the *zero-order deformation equation* (3). Define the vector

$$\vec{u}_n = \{u_0(\tau), u_1(\tau), \ldots, u_n(\tau)\}.$$

Differentiating equation (3) *m* times with respect to the embedding parameter *p* and then setting $p = 0$ and finally dividing them by *m*!, we have the so-called *mth-order deformation equation*

$$\mathrm{L}\left[u_m(\tau) - \chi_m u_{m-1}(\tau)\right] = hH(\tau)R_m(\vec{u}_{m-1}), \tag{10}$$

where

$$R_m(\vec{u}_{m-1}) = \frac{1}{(m-1)!}\frac{\partial^{m-1}\mathrm{N}\left[\varphi(\tau;p)\right]}{\partial p^{m-1}}\bigg|_{p=0}, \tag{11}$$

and

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

It should be emphasized that $u_m(\tau)$ for $m \ge 1$ is governed by the linear equation (8) with the linear boundary conditions that come from original problem, which can be easily solved by symbolic computation software such as Maple and Mathematica.

3. Application

3.1. The solution with HAM

In this Section, we apply the homotopy analysis method for the KdV, Kawahara and Gardner equations. We start with initial approximation $u_0(x,t) = u(x,0)$ and the linear operator

$$\mathrm{L}\left[\varphi(x,t;p)\right] = \frac{\partial\varphi(x,t;p)}{\partial t}, \tag{12}$$

possesses the property

$$\mathrm{L}(c_1) = 0, \tag{13}$$

where c_1 is an integral constant to be determined by initial condition. Furthermore, equations (1), (2) and (3) suggest to define the nonlinear operators

$$\mathrm{N}_{KdV}\left[\varphi(x,t;p)\right]=\frac{\partial\varphi(x,t;p)}{\partial t}+6\varphi(x,t;p)\frac{\partial\varphi(x,t;p)}{\partial x}+\frac{\partial^3\varphi(x,t;p)}{\partial x^3}, \tag{14}$$

$$\mathrm{N}_{Kawahara}\left[\varphi(x,t;p)\right]=\frac{\partial\varphi(x,t;p)}{\partial t}+\varphi(x,t;p)\frac{\partial\varphi(x,t;p)}{\partial x}+\frac{\partial^3\varphi(x,t;p)}{\partial x^3}-\frac{\partial^5\varphi(x,t;p)}{\partial x^5} \tag{15}$$

and

$$\begin{aligned}\mathrm{N}_{Gardner}\left[\varphi(x,t;p)\right]=&\frac{\partial\varphi(x,t;p)}{\partial t}+6\varphi(x,t;p)\frac{\partial\varphi(x,t;p)}{\partial x}\\&-6(\varphi(x,t;p))^2\frac{\partial\varphi(x,t;p)}{\partial x}+\frac{\partial^3\varphi(x,t;p)}{\partial x^3},\end{aligned} \tag{16}$$

respectively. Using the above definition, with assumption $H(\tau)=1$, we construct the *zero-order deformation equation*

$$(1-p)\mathrm{L}\left[\varphi(x,t;p)-u_0(x,t)\right]=ph\mathrm{N}\left[\varphi(x,t;p)\right]. \tag{17}$$

Obviously, when $p=0$ and $p=1$,

$$\varphi(x,t;0)=u_0(x,t), \qquad \varphi(x,t;1)=u(x,t). \tag{18}$$

Differentiating the *zero-order deformation equation* (17) *m* times with respect to *p*, and finally dividing by *m*!, we have the *mth-order deformation equation*

$$\mathrm{L}\left[u_m(x,t)-\chi_m u_{m-1}(x,t)\right]=hR_m(\vec{u}_{m-1}), \tag{19}$$

subject to initial condition

$$u_m(x,0)=0, \tag{20}$$

where

$$R_{m_{KdV}}(\vec{u}_{m-1})=\frac{\partial u_{m-1}(x,t)}{\partial t}+6\sum_{n=0}^{m-1}u_n(x,t)\,\frac{\partial u_{m-1-n}(x,t)}{\partial x}+\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}, \tag{21}$$

$$\begin{aligned}R_{m_{Kawahara}}(\vec{u}_{m-1})=&\frac{\partial u_{m-1}(x,t)}{\partial t}+\sum_{n=0}^{m-1}u_n(x,t)\,\frac{\partial u_{m-1-n}(x,t)}{\partial x}\\&+\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}-\frac{\partial^5 u_{m-1}(x,t)}{\partial x^5},\end{aligned} \tag{22}$$

$$\begin{aligned}R_{m_{Gardner}}(\vec{u}_{m-1})=&\frac{\partial u_{m-1}(x,t)}{\partial t}+6\sum_{n=0}^{m-1}u_n(x,t)\,\frac{\partial u_{m-1-n}(x,t)}{\partial x}\\&-6\sum_{n=0}^{m-1}\left(\sum_{j=0}^{n}u_j(x,t)u_{n-j}(x,t)\right)\frac{\partial u_{m-1-n}(x,t)}{\partial x}+\frac{\partial^3 u_{m-1}(x,t)}{\partial x^3}\end{aligned} \tag{23}$$

And

$$\chi_m = \begin{cases} 0, & m \le 1, \\ 1, & m > 1. \end{cases}$$

Obviously, the solution of the *mth-order deformation equation* (19) for $m \ge 1$ becomes

$$u_m(x,t) = \chi_m u_{m-1}(x,t) + h\mathrm{L}^{-1}\left[R_m(\vec{u}_{m-1})\right]. \tag{24}$$

In the following parts, we consider the initial approximations and determine other components of the solution series for the KdV, Kawahara and Gardner equations.

3.1.1. The KdV equation

For the KdV equation, we choose the initial approximation

$$u_0(x,t) = u(x,0) = 2k^2\,\mathrm{sech}^2[kx], \tag{25}$$

where k is arbitrary constant and $k \neq 0$. Using previous formulae to determine other components of the solution series. From (21), (24) and (25), by the Mathematica package, we have

$$u_1(x,t) = -16hk^5 t\,\mathrm{sech}^2[kx]\tanh[kx], \tag{26}$$

$$u_2(x,t) = 8hk^5 t\,\mathrm{sech}^4[kx]\left(4hk^3 t(-2+\cosh[2kx]) - (1+h)\sinh[2kx]\right), \tag{27}$$

$$u_3(x,t) = -\frac{8}{3}hk^5 t\,\mathrm{sech}^4[kx]\left(24h(1+h)k^3 t\cosh[2kx] + \left(3+h(6+3h+32hk^6t^2)\right)\sinh[2kx] + 48hk^3 t(1+h-4hk^3 t\tanh[kx])\right), \tag{28}$$

$$\vdots$$

We used 15 terms in evaluating the approximate solution $u_{app_{KdV}} = \sum_{i=0}^{14} u_i$.

3.1.2. The Kawahara equation

For the Kawahara equation, we choose the initial approximation

$$u_0(x,t) = u(x,0) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4\left[\frac{x}{2\sqrt{13}}\right]. \tag{29}$$

From (22), (24) and (29), we can obtain the following components

$$u_1(x,t) = \frac{7560ht}{28561\sqrt{13}}\operatorname{sech}^4\left[\frac{x}{2\sqrt{13}}\right]\tanh\left[\frac{x}{2\sqrt{13}}\right], \tag{30}$$

$$u_2(x,t) = \frac{3780ht}{62748517}\operatorname{sech}^6\left[\frac{x}{2\sqrt{13}}\right]\left(-54ht + 38ht\cosh\left[\frac{x}{\sqrt{13}}\right] + 169\sqrt{13}(1+h)\sinh\left[\frac{x}{\sqrt{13}}\right]\right), \tag{31}$$

$$u_3(x,t) = \frac{7560ht}{137858491849}\operatorname{sech}^4\left[\frac{x}{2\sqrt{13}}\right]\left(158184h(1+h)t + \sqrt{13}(371293(1+h)^2 + 864h^2t^2)\tanh\left[\frac{x}{2\sqrt{13}}\right]\right. \tag{32}$$
$$\left. -90ht\operatorname{sech}^2\left[\frac{x}{2\sqrt{13}}\right]\left(2197(1+h) + 18\sqrt{13}ht\tanh\left[\frac{x}{2\sqrt{13}}\right]\right)\right),$$

$\vdots$

We used 10 terms in evaluating the approximate solution $u_{app_{Kawahara}} = \sum_{i=0}^{9} u_i$.

3.1.3. *The Gardner equation*

For the Gardner equation, we choose the initial approximation

$$u_0(x,t) = u(x,0) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{x}{2}\right]. \tag{33}$$

Therefore, from (23), (24) and (33), we can obtain the following components

$$u_1(x,t) = \frac{ht}{2(1+\cosh[x])}, \tag{34}$$

$$u_2(x,t) = \frac{ht}{4(1+\cosh[x])}\left(2 + 2h - ht\tanh\left[\frac{x}{2}\right]\right), \tag{35}$$

$$u_3(x,t) = \frac{1}{48}ht\operatorname{sech}^4\left[\frac{x}{2}\right]\left(6(1+h)^2 - 2h^2t^2 + (6 + h(12 + h(6+t^2)))\cosh[x] - 6h(1+h)t\sinh[x]\right), \tag{36}$$

$\vdots$

We used 10 terms in evaluating the approximate solution $u_{app_{Gardner}} = \sum_{i=0}^{9} u_i$.

4. Results and discussion

The series solutions contain the auxiliary parameter *h*. The validity of the method is based on such an assumption that the series (6) converges at $p = 1$. It is the auxiliary parameter *h* which ensures that this assumption can be satisfied. In general, by means of the so-called *h*-curve, it is straightforward to choose a proper value of *h* which ensures that the solution series is convergent. Figs. 1–3 show the *h*-curves obtained from the 14th-order approximate solution of the KdV equation and 9th-order approximate solutions of the Kawahara and Gardner equations. From these figures, the valid regions of *h* correspond to the line segments nearly parallel to the horizontal axis.

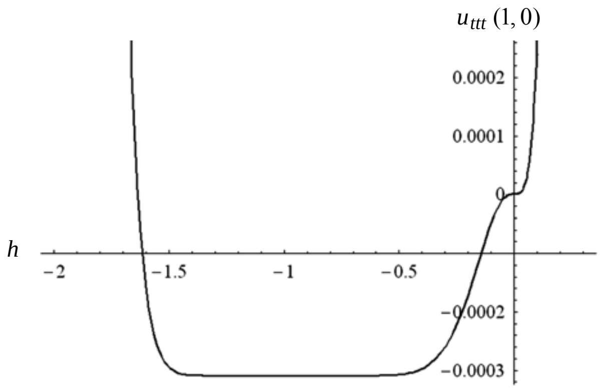

Fig. 1. The *h*-curve of $u_{ttt}(1,0)$ given by the 14th-order approximate solution of KdV equation, when $k = 0.275$.

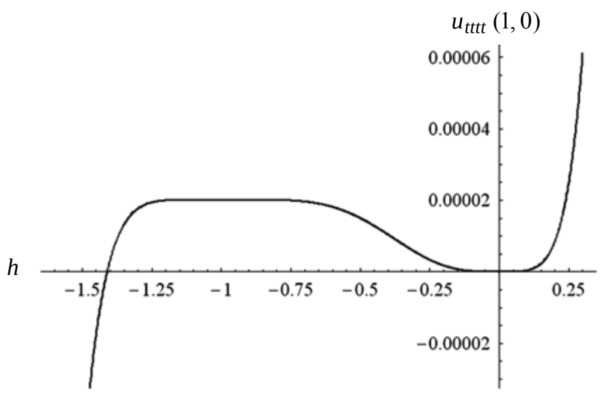

Fig. 2. The *h*-curve of $u_{tttt}(1,0)$ given by the 9th-order approximate solution of Kawahara equation.

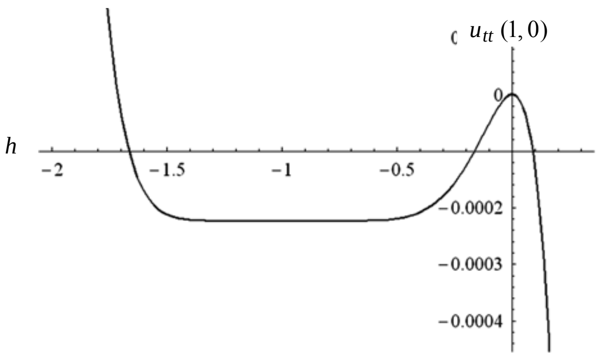

Fig. 3. The *h*-curve of $u_{tt}\,(1,0)$ given by the 9th-order approximate solution of Gardner equation.

For approximate solution of the KdV equation, we choose the auxiliary parameter $h = -0.96$ and for both approximate solutions of the Kawahara and Gardner equations, we choose the auxiliary parameter $h = -1$ by *h*-curves. In continuation, we compare approximate solutions of the KdV, Kawahara and Gardner equations, with exact solutions

$$u_{ex_{KdV}}(x,t) = 2k^2 \operatorname{sech}^2\left[k\left(x - 4k^2 t\right)\right], \qquad k \neq 0, \tag{37}$$

$$u_{ex_{kawahara}}(x,t) = -\frac{72}{169} + \frac{105}{169}\operatorname{sech}^4\left[\frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)\right] \tag{38}$$

and

$$u_{ex_{Gardner}}(x,t) = \frac{1}{2} + \frac{1}{2}\tanh\left[\frac{1}{2}(x - t)\right], \tag{39}$$

respectively. Tables 1–3 show the absolute errors for differences between the exact solutions and the approximate solutions obtained by HAM at some points. Besides, the behavior of the exact and approximate solutions are illustrated in Figs. 4–6. Note that when $h = -1$, the solution series obtained by the HAM is as the same solution series obtained by the homotopy perturbation method (shortly HPM), which proposed in 1998 by Dr. He [23].

Table 1

Absolute errors for the 14th-order approximate solution of the KdV equation given by HAM for $h = -0.96$, when $k = 0.2$.

x	t			
	2	4	6	8
1		2.72005×10^{-15}	1.02696×10^{-15}	1.57035×10^{-12}
5	3.81639×10^{-16}	9.22873×10^{-16}	1.02904×10^{-15}	1.12695×10^{-13}
15		1.04734×10^{-16}	1.69136×10^{-16}	7.34872×10^{-16}
25		9.69006×10^{-19}	7.72494×10^{-19}	3.20187×10^{-18}
35		2.64698×10^{-21}	1.93229×10^{-20}	9.26972×10^{-20}
45	2.41537×10^{-22}	4.30131×10^{-22}	1.33838×10^{-21}	1.57164×10^{-21}

Table 2
Absolute errors for the 9th-order approximate solution of the Kawahara equation given by HAM for $h = -1$.

x	t			
	2	4	6	8
1		8.50289×10^{-11}	4.00492×10^{-11}	5.55001×10^{-10}
5	8.73468×10^{-14}	8.92500×10^{-11}	5.12724×10^{-11}	9.04022×10^{-10}
15		2.53684×10^{-13}	1.42462×10^{-12}	2.46425×10^{-11}
25		1.00499×10^{-15}	6.20427×10^{-14}	1.08097×10^{-12}
35		2.92943×10^{-17}	2.28627×10^{-16}	5.27545×10^{-15}
45	1.72588×10^{-18}	5.60337×10^{-18}	9.15700×10^{-18}	2.86802×10^{-17}

Table 3
Absolute errors for the 9th-order approximate solution of the Gardner equation given by HAM for $h = -1$.

x	t			
	2	4	6	8
20	7.08100×10^{-13}	9.15160×10^{-10}	6.97847×10^{-8}	1.74108×10^{-7}
25	4.77396×10^{-15}	6.16629×10^{-12}	4.70209×10^{-10}	1.17316×10^{-8}
30	1.11022×10^{-16}	4.14113×10^{-14}	3.16835×10^{-12}	7.90468×10^{-11}
35	1.11022×10^{-16}	3.33067×10^{-16}	2.13163×10^{-14}	5.32574×10^{-13}
40	0	1.11022×10^{-16}	2.22045×10^{-16}	3.66374×10^{-15}
45	0	0	0	1.11022×10^{-16}

5. Conclusions

In this Letter we solved some problems by the homotopy analysis method. It can be concluded:

1. The HAM is very powerful and efficient method in finding analytical solutions for wide classes of nonlinear problems.
2. The HAM provides us with a convenient way to control the convergence of approximation series, which is a fundamental qualitative difference in analysis between the HAM and other methods.
3. The HAM can be applied no matter if governing equations and boundary/initial conditions contain small or large quantities or not.
4. The HAM provides an efficient numerical solution with high accuracy, minimal calculation, avoidance of physically unrealistic assumptions.

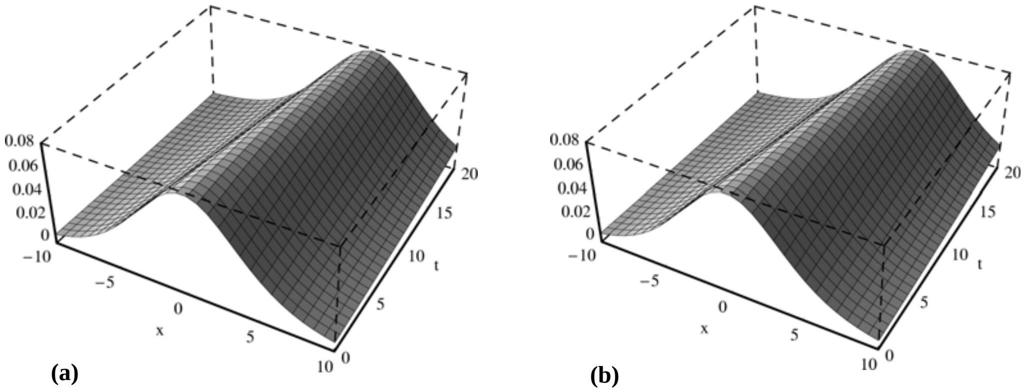

Fig. 4. The surfaces show the solutions obtained by: (a) HAM for $h = -0.96$; (b) exact solution, when $k = 0.2$, for the KdV equation.

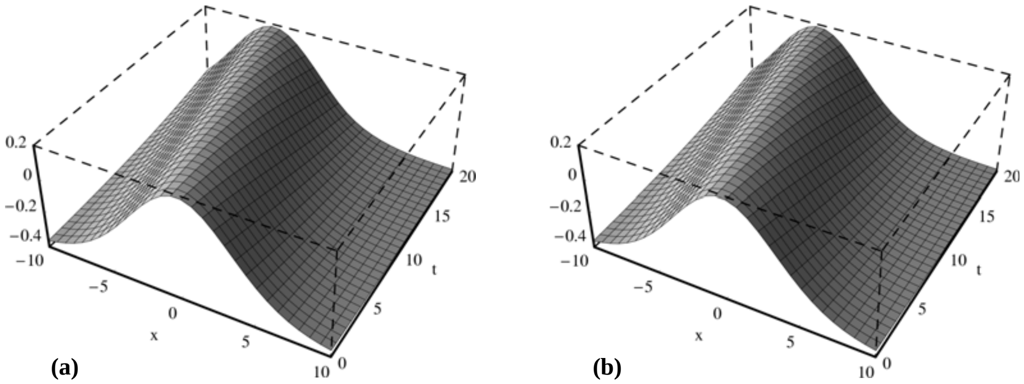

Fig. 5. The surfaces show the solutions obtained by: (a) HAM for $h = -1$; (b) exact solution, for the Kawahara equation.

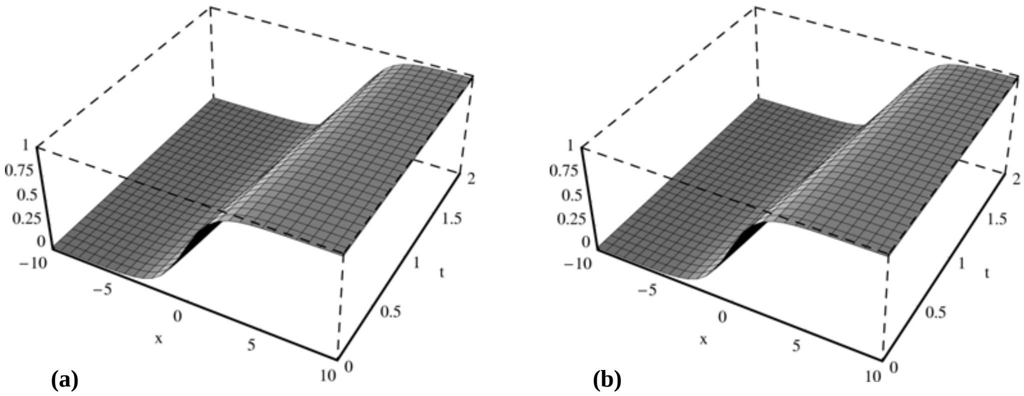

Fig. 6. The surfaces show the solutions obtained by: (a) HAM for $h = -1$; (b) exact solution, for the Gardner equation.

References

[1] S.J. Liao, The proposed homotopy analysis technique for the solution of non-linear problems, Ph.D. Thesis, Shanghai Jiao Tong University, 1992.
[2] S.J. Liao, A uniformly valid analytic solution of 2D viscous flow past a semi-infinite flat plate, J. Fluid Mech. 385 (1999) 101–128.
[3] S.J. Liao, An explicit, totally analytic approximation of Blasius viscous flow problems, Int. J. Nonlinear Mech. 34 (1999) 759–778.
[4] S.J. Liao, On the analytic solution of magnetohydrodynamic flows of non-Newtonian fluids over a stretching sheet, J. Fluid Mech. 488 (2003) 189– 212.
[5] S.J. Liao, Beyond perturbation: Introduction to the Homotopy Analysis Method, CRC Press/chapman Hall, Boca Raton, 2003.
[6] S.J. Liao, On the homotopy analysis method for nonlinear problems, Appl. Math. Comput. 147 (2004) 499–513.
[7] S. Dinarvand, A reliable treatment of the homotopy analysis method for viscous flow over a non-linearly stretching sheet in presence of a chemical reaction and under influence of a magnetic field, Cent. Eur. J. Phys. 7(1) (2009) 114–122.
[8] S. Dinarvand, On explicit, purely analytic solutions of off-centered stagnation flow towards a rotating disc by means of HAM, Nonlinear Anal. Real World Appl. 11 (2010) 3389–3398.
[9] S. Dinarvand, M. M. Rashidi , A reliable treatment of homotopy analysis method for two-dimensional viscous flow in a rectangular domain bounded by two moving porous walls, Nonlinear Anal. Real World Appl. 11 (2010) 1502–1512.
[10] S. Dinarvand, The laminar free-convection boundary-layer flow about a heated and rotating down-pointing vertical cone in the presence of a transverse magnetic field, Int. J. Numer. Meth. Fluids, In Press (2010).
[11] S. Dinarvand, A. Doosthoseini, E. Doosthoseini, M. M. Rashidi , Series solutions for unsteady laminar MHD flow near forward stagnation point of an impulsively rotating and translating sphere in presence of buoyancy forces, Nonlinear Anal. Real World Appl. 11 (2010) 1159–1169.
[12] C.S. Gardner, G.K. Morikawa, Similarity in the asymptotic behavior of collision-free hydromagnetic waves and water waves. New York Univ, Courant Inst. Math. Sci. Res. Rep. NYO-9082, 1960.
[13] E. Fermi, J. Pasta, S. Ulam, Studies of nonlinear problems, I. Los Alamos Report LA 1940; 1955.
[14] M.D. Kruskal, N.J. Zabusky, Progress on the Fermi–Pasta–Ulam nonlinear string problem. Princeton Plasma Physics Laboratory Annual Rep. MATT-Q-21, Princeton, NJ, (1963) 301–308.
[15] T. Taniuti, H. Washimi, Propagation of ion-acoustic solitary waves of small amplitude, Phys. Rev. Lett. 17 (1966) 996–1004.
[16] S. Leibovich, Weakly non-linear waves in rotating fluids, J Fluid Mech. 42 (1970) 803–825.
[17] G.A. Nariboli, Nonlinear longitudinal dispersive waves in elastic rods. Iowa State Univ. Engineering Res. Inst Preprint, 1969. p. 442.

[18] T.J. Kawahara, Oscillatory solitary waves in dispersive media, Phys. Soc. Jpn. 33 (1972) 260–264.
[19] J.K. Hunter, J. Scheurle, Existence of perturbed solitary wave solutions to a model equation for water waves, Physica D 32 (1988) 253–268.
[20] R. Grimshaw, D. Pelinovsky, T. Talipova, A. Kurkin, Simulation of the transformation of internal solitary waves on oceanic shelves, J. Phys. Oceanogr. 34 (2004) 2774–2779.
[21] J.H. He, A coupling method for homotopy technique and perturbation technique for non-linear problem, Int. J. Nonlinear. Mech. 35 (2000) 37–43.
[22] J.H. He, Homotopy perturbation method for solving boundary value problems, Phys. Lett. A 350 (2006) 87–88.

Received: September, 2010